Enumerative Algebraic Geometry

via

Techniques of Symplectic Topology

and

Analysis of Local Obstructions

Enumerative Geometry

Subject Matter

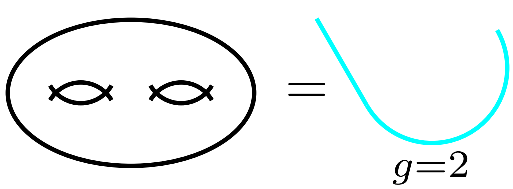
determine # of *geometric* objects
that satisfy given *geometric* conditions

Example

of lines through 2 points in Euclidian space is 1

Typical Setting

Objects:	(complex) curves/Riemann surfaces
	in algebraic manifolds (e.g. $\mathbb{P}^n$)
Conditions:	genus/complex structure
	homology class
	singularities
	pass thr. submanifolds (e.g. pts)



simple node
$x^2{=}y^2$

simple cusp
$x^2{=}y^3$

Classical Example

Formulation

n_d=# of rat. deg.-d curves thr. $3d-1$ pts. in $\mathbb{P}^2$

What is n_d?

rational	genus=zero (S^2)
degree-d	$[\mathcal{C}]=d[\ell]\in H_2(\mathbb{P}^2;\mathbb{Z})$

Classical Results (by 1870s)

$$n_1=1, \quad n_2=1, \quad n_3=12, \quad n_4=620$$

Recent Results (1993)

(Kontsevich-Manin, Ruan-Tian)

$$n_d=\frac{1}{6(d-1)}\sum_{d_1+d_2=d}\left(d_1d_2-2\frac{(d_1-d_2)^2}{3d-2}\right)\binom{3d-2}{3d_1-1}d_1d_2n_{d_1}n_{d_2}$$

d	1	2	3	4	5	6
n_d	1	1	12	620	87,304	26,312,976

also recursion for $n_d(\mu)$, μ=submanifolds in $\mathbb{P}^n$

Symplectic Topology

General Question

When are two symplectic manifolds equivalent?

Pseudoholomorphic Curves (Gromov'85)

(V,ω)=symplectic manifold, $A\in H_2(V;\mathbb{Z})$

J=compatible (almost) complex structure

$$\mathfrak{M}_{0,0}(V,A)=\big\{u\in C^\infty(S^2,V)\colon u_*[S^2]=A,\; \bar{\partial}_J u=0\big\}/PSL_2$$

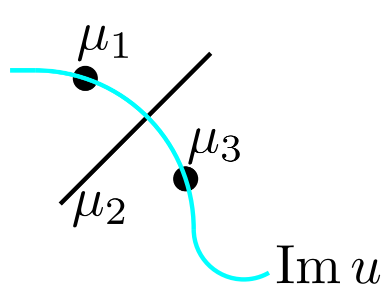
Symplectic Invariants

Fact: $\exists\ \overline{\mathfrak{M}}_{0,0}(V,A)$ in good cases

Corollary: if $\mu_1,\ldots,\mu_N$=submanifolds in V,

$$\#\big\{[u]\in\overline{\mathfrak{M}}_{0,0}(V,A)\colon \mathrm{Im}\,u\cap\mu_l\neq\emptyset,\ l=1,\ldots,N\big\}$$

depends only on ω, A, and $[\mu_l]\in H_*(V;\mathbb{Q})$



Example

$(\mathbb{P}^2, \omega, J)$=Fubini-Study structure

$$\mathfrak{M}_{0,0}(\mathbb{P}^2, d) = \big\{ u \in C^\infty(S^2, \mathbb{P}^2) \colon u_*[S^2] = d[\ell], \; \bar{\partial}_J u = 0 \big\} / PSL_2$$

$p_1, \ldots, p_{3d-1}$=points in $\mathbb{P}^2$

$$\#\big\{ [u] \in \overline{\mathfrak{M}}_{0,0}(\mathbb{P}^2, d) \colon p_l \in \operatorname{Im} u \big\} = n_d$$

More Generally

$$\mathfrak{M}_{0,N}(\mathbb{P}^n, d) = \big\{ (u; y_1, \ldots, y_N) \colon y_l \in S^2 \big\} / PSL_2$$

Fact: $\exists$ "nice" $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d)$

AG: Kontsevich'93, Fulton-Pandharipande'97

SG: McDuff-Salamon'93, Ruan-Tian'93

Fact (Pandharipande'95)

intersections of *tautological* classes in $H^*\big(\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d); \mathbb{Q}\big)$ are computable

$$\big\{ \textit{tautological} \text{ classes } \big\} \supset \big\{ \text{relevant classes} \big\}$$

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Conditions:	genus/complex structure
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	singularities
	pass thr. submanifolds (e.g. pts)

Two Types of Problems

Problem 1

Determine # of rational curves with
the given uni-pointed singularities
(e.g. cusp of specified form)
Goal: answer in terms of ITC

Problem 2

Determine $n_{g,d}(\mu)=$ # of genus-g curves with
the given complex structure
Goal: answer in terms of ITC and
genus-g symplectic invariants

Problem 1

Example

$\left|\mathcal{S}_1(\mu)\right| = \#$ deg.-d rat. curves with a cusp thr. $3d-2$ pts in $\mathbb{P}^2$

What is $\left|\mathcal{S}_1(\mu)\right|$*?*

Contribution to the Euler Class

Setup

$$\begin{array}{c} V^n \\ s \uparrow \downarrow \\ X^{2n} \end{array}$$

X cmpt

$s \in \Gamma(X;V)$

$Z \subset X$

What is $\mathcal{C}_Z(s)$?

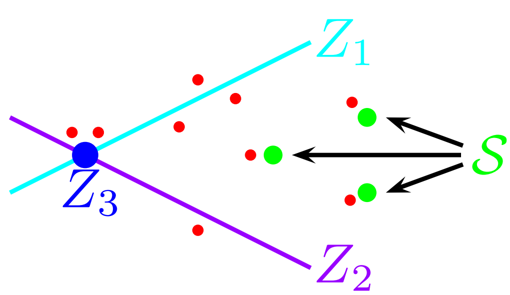
If $s \pitchfork 0$, $\quad \mathcal{C}_Z(s) = {}^{\pm}\big|s^{-1}(0) \cap Z\big|$

More Generally

$$\mathcal{C}_Z(s) \equiv {}^{\pm}\big|\{s+\varepsilon\}^{-1}(0) \cap W_Z\big| \text{ if}$$

$\varepsilon \in \Gamma(X;V)$ small & generic

W_Z=small neighborhood of Z



$\mathcal{C}_{Z_1}(s) = 3$

$\mathcal{C}_{Z_2}(s) = 1$

$\mathcal{C}_{Z_3}(s) = 2$

$$|\mathcal{S}| = \langle e(V), [X] \rangle - \sum \mathcal{C}_{Z_i}(s)$$

Computation of $\mathcal{C}_Z(s)$

Setup

$$V^n \longrightarrow X^{2n}, \ \ \text{X cmpt}, \ \ s \in \Gamma(X;V), \ \ Z \subset X$$

Definition of $\mathcal{C}_Z(s)$

$$\mathcal{C}_Z(s) \equiv^{\pm} \big|\{s+\varepsilon\}^{-1}(0) \cap W_Z\big| \ \ \text{if}$$

$\varepsilon \in \Gamma(X;V)$ small & generic

W_Z=small neighborhood of Z

Easy

$\mathcal{C}_Z(s)$ is well-defined if

$s^{-1}(0) \cap Z$ & $s^{-1}(0) - Z$ are closed

Proposition

$\mathcal{C}_Z(s)$ is well-defined if

(i) Z is smooth & W_Z is modellable on $F \longrightarrow Z$

(ii) $s|W_Z \approx \big(\text{polynomial } \alpha\colon F \longrightarrow V\big)$

$$\boxed{\mathcal{C}_Z(s) =^{\pm} \big|\big\{v \in F : \bar{\nu}_v + \alpha(v) = 0\big\}\big|}$$

for generic $\bar{\nu} \in \Gamma(Z;V)$

Zeros of Polynomial Maps

Setup

$$\begin{array}{ccc} F^k & \xrightarrow{\psi_{\alpha,\bar\nu}\equiv\bar\nu+\alpha} & \mathcal{O}^{k+m} \\ & \searrow \quad \swarrow & \\ & X^{2m} & \end{array}$$

X cmpt

$F=\bigoplus F_i, \quad \alpha=\sum\alpha_i$

$\alpha_i\in\Gamma\big(X;\mathrm{Hom}(F_i^{\otimes d_i},\mathcal{O})\big)$

$\bar\nu\in\Gamma(X;F)$ generic w.r.t. α

Facts

(1) ${}^{\pm}\big|\psi_{\alpha,\bar\nu}^{-1}(0)\big|$ depends on α, but not $\bar\nu$

(2) if $\alpha|F_x$ is injective $\forall x\in X$,

$${}^{\pm}\big|\psi_{\alpha,\bar\nu}^{-1}(0)\big|=\big\langle e\big(\mathcal{O}/\alpha(F)\big),[X]\big\rangle=\big\langle c(\mathcal{O})c(F)^{-1};[X]\big\rangle$$

Example

$$X=\mathbb{P}^1=\big\{\ell=[u,v]:(u,v)\in\mathbb{C}^2-\{0\}\big\}$$

$$F=\mathbb{C}, \qquad \mathcal{O}=\mathbb{C}\oplus\gamma^*$$

(1) if $\alpha(\ell;c)=(\ell;c,0), \quad {}^{\pm}\big|\psi_{\alpha,\bar\nu}^{-1}(0)\big|=1$

(2) if $\alpha(\ell;1)=(\ell;0,c\cdot u), \quad {}^{\pm}\big|\psi_{\alpha,\bar\nu}^{-1}(0)\big|=0$

Computation of ${}^{\pm}\left|\psi_{\alpha,\bar{\nu}}^{-1}(0)\right|$

$$(X,F,\mathcal{O},\alpha)$$

$$X\longrightarrow\mathbb{P}F_i \quad \Big\Downarrow \quad F_i\longrightarrow\gamma^{d_i}$$

$$(X,F,\mathcal{O},\alpha),\ \alpha \text{ is linear}$$

$$X\longrightarrow\mathbb{P}F \quad \Big\Downarrow \quad F\longrightarrow\gamma$$

$$(X,F,\mathcal{O},\alpha),\ \alpha \text{ is linear \& } \operatorname{rk} F=1$$

$$ {}^{\pm}\left|\psi_{\alpha,\bar{\nu}}^{-1}(0)\right| = \left\langle c(\mathcal{O})c(F)^{-1},[X]\right\rangle - \mathcal{C}_{\alpha^{-1}(0)}(\alpha^{\perp}),$$

$$\alpha^{\perp}\in\Gamma\big(X;\operatorname{Hom}(F,\mathcal{O}/\mathbb{C}\bar{\nu})\big)$$

$$\Big\Downarrow$$

$$(X_j,F_j,\mathcal{O}_j,\alpha_j),\ \operatorname{rk}\mathcal{O}_j<\operatorname{rk}\mathcal{O}$$

Example 1

$\big|\mathcal{S}_1(\mu)\big|$=# deg.-$d$ rat. curves with a cusp thr. $3d-2$ pts in $\mathbb{P}^2$

$$\big|\mathcal{S}_1(\mu)\big| = \big\langle 3a^2 + 3ac_1(\mathcal{L}^*) + c_1^2(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle - \big|\mathcal{V}_2(\mu)\big|$$

d	1	2	3	4	5	6
$\vert\mathcal{S}_1(\mu)\vert$	0	0	12	2,304	435,168	156,153,600

Example 2

$\big|\mathcal{S}_2(\mu)\big|$=# deg.-$d$ rat. curves with a $(3,4)$-cusp thr. $3d-4$ pts in $\mathbb{P}^2$

$$\begin{aligned}\big|\mathcal{S}_2(\mu)\big| &= \big\langle 33a^2c_1^2(\mathcal{L}^*) + 18ac_1^3(\mathcal{L}^*) + 4c_1^4(\mathcal{L}^*), [\bar{\mathcal{V}}_1(\mu)]\big\rangle \\ &\quad - \big\langle 21a^2 + 9a\big(c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*)\big) \\ &\qquad + 2\big(c_1^2(\mathcal{L}_1^*) + c_1^2(\mathcal{L}_2^*)\big) + c_1(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*), [\bar{\mathcal{V}}_2(\mu)]\big\rangle \\ &\quad + 3\big|\mathcal{V}_3(\mu)\big|\end{aligned}$$

d	2	3	4	5	6	7
$\vert\mathcal{S}_2(\mu)\vert$	0	0	147	54,612	23,177,124	14,617,373,280

Extent of Applications

(1) count rat. curves w. specified cusp in $\mathbb{P}^n$

(2) should apply to G/P (e.g. $Gr_k\mathbb{C}^n$)
to get ITC's

Two Types of Problems

Problem 1

Determine # of *rational* curves with
the given uni-pointed singularities
(e.g. cusp of specified form)
Goal: answer in terms of ITC

Problem 2

Determine $n_{g,d}(\mu)$ = # of *genus-g* curves with
the given complex structure
Goal: answer in terms of ITC and
genus-g symplectic invariants

Problem 2

Genus One, $\mathbb{P}^n$ (Ionel'96)

If $\mu_1, \ldots, \mu_N$ are submanifolds in $\mathbb{P}^n$,

$$CR_1(\mu) \equiv \mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - 2n_{1,d}(\mu),$$

(1) is expressible in terms of $\{n_{d'}(\mu')\}$

(2) is # of zeros of an affine map between vector bundles over $\bar{\mathcal{V}}_1(\mu)$

$\mathrm{RT}_{g,d}(\cdot;\cdot)$=sympl. genus-$g$ invariant of $(\mathbb{P}^n, \omega_{FS})$ as defined in Ruan-Tian'95

Genus $g \geq 2$

If $g=2$ & $n=2,3$ or $g=3$ & $n=2$,

$$CR_g(\mu) \equiv \mathrm{RT}_{g,d}(\mu_1;\mu_2,\ldots,\mu_N) - m(g)\cdot n_{g,d}(\mu),$$

(1) is # of zeros of affine maps between vector bundles over $\Sigma^j \times \bar{\mathcal{S}}_j(\mu)$, with $\bar{\mathcal{S}}_j(\mu) \subset \bar{\mathcal{V}}_{k_j}(\mu)$

(2) is expressible in terms of ITCs.

$\mathbf{g=2,\ n=2}$

$$n_{2,d} = 3(d^2-1)n_d + \frac{1}{2}\sum_{d_1+d_2=d}\Big(d_1^2 d_2^2 + 28 - 16\frac{9d_1d_2-1}{3d-2}\Big)\binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}$$

d	1	2	3	4	5	6
n_d	0	0	0	14,400	6,350,400	3,931,128,000

Symplectic & Enumerative Invariants

$$\mathcal{H}_d(\mu) = \Big\{ (y_1, \ldots, y_N; u) \Big|\; u\colon \Sigma \longrightarrow \mathbb{P}^n,\; u(y_l) \in \mu_l,\; u_*[\Sigma] = [u(\Sigma)] = d\ell,\; \boxed{\bar{\partial}_{J,j} u|_z = 0} \Big\}$$

$$\boxed{m(g) \cdot n_{g,d}(\mu) = |\mathcal{H}_d(\mu)|}$$

If $\nu \in \Gamma(\Sigma \times \mathbb{P}^n; \pi_\Sigma^* T\Sigma \otimes \pi_{\mathbb{P}^n}^* T\mathbb{P}^n)$, let

$$\mathcal{M}_{\nu,d}(\mu) = \Big\{ (y_1, \ldots, y_N; u) \Big|\; u\colon \Sigma \longrightarrow \mathbb{P}^n,\; u(y_l) \in \mu_l,\; u_*[\Sigma] = d\ell,\; \boxed{\bar{\partial}_{J,j} u|_z = \nu|_{(z,u(z))}} \Big\}$$

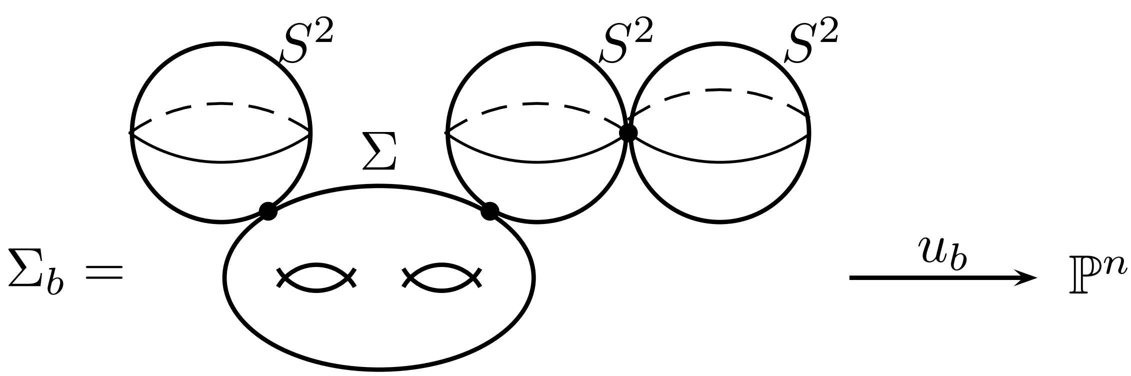
For a generic ν, $\mathrm{RT}_{g,d}(;\mu) \equiv {}^{\pm}|\mathcal{M}_{\nu,d}(\mu)|$

Symplectic vs. Enumerative Invariants

If $\nu_i \longrightarrow 0$ & $(\underline{y}_i, u_i) \in \mathcal{M}_{\nu_i,d}(\mu)$, $\lim_{i\longrightarrow\infty} (\underline{y}_i, u_i) =$

(1) $b \in \mathcal{H}_d(\mu)$, OR

(2) $b = (\Sigma_b, \underline{y}, u)$, $\Sigma_b = \Sigma \cup \bigcup S_h^2$, $u_b : \Sigma_b \longrightarrow \mathbb{P}^n$,
$y_l \in \Sigma_b$, $u_b(y_l) \in \mu_l$, $\boxed{\bar{\partial} u_b = 0}$



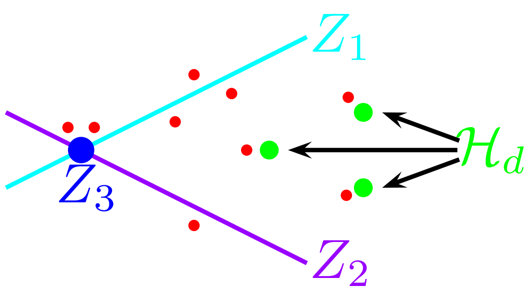
Symplectic vs. Enumerative Invariants

$$\Gamma^{0,1} \underset{\bar\partial}{\overset{\longleftarrow}{\longrightarrow}} \bar{C}^\infty$$

$$\mathcal{H}_d = \bar\partial^{-1}(0) \cap C^\infty$$

$$\mathcal{M}_{\nu,d} = \{\bar\partial - \nu\}^{-1}(0)$$

$\nu \in \Gamma(\bar{C}^\infty; F)$ small & generic



$$|\mathcal{H}_d| = \mathrm{RT}_{g,d}(\cdot;\mu) - \sum \mathcal{C}_{Z_i}(\bar\partial)$$

Computation of $\mathcal{C}_{Z_i}(\bar\partial)$

Goal: reduce to counting zeros of a polynomial map between *finite*-rank vector bundles over $\bar{Z}_i$

Method: obstruction-bundle approach (Taubes'84)

Norms: as in Li-Tian'96

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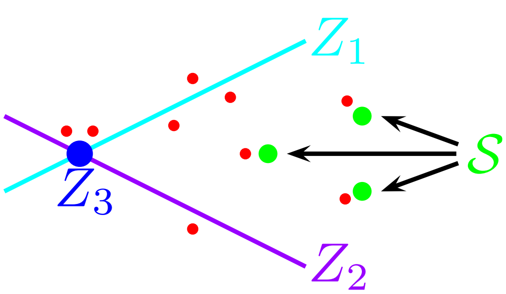

$$|\mathcal{S}| = \langle e(V),[X]\rangle - \sum \mathcal{C}_{Z_i}(s)$$

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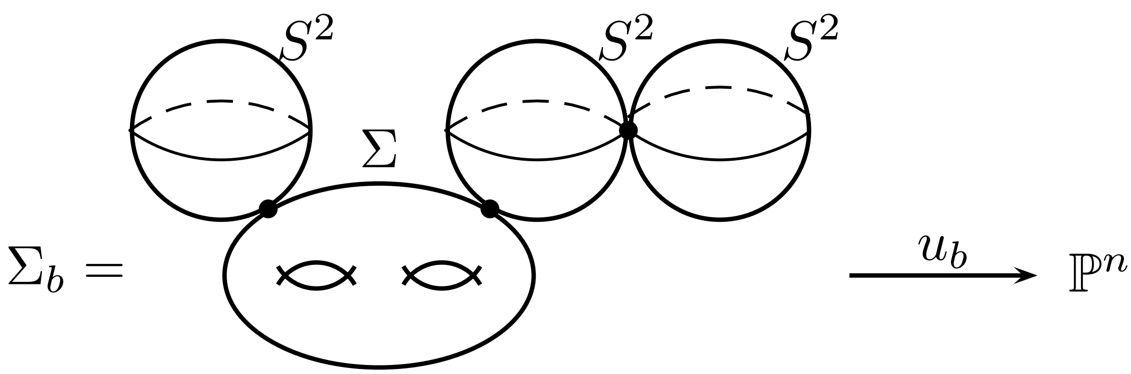
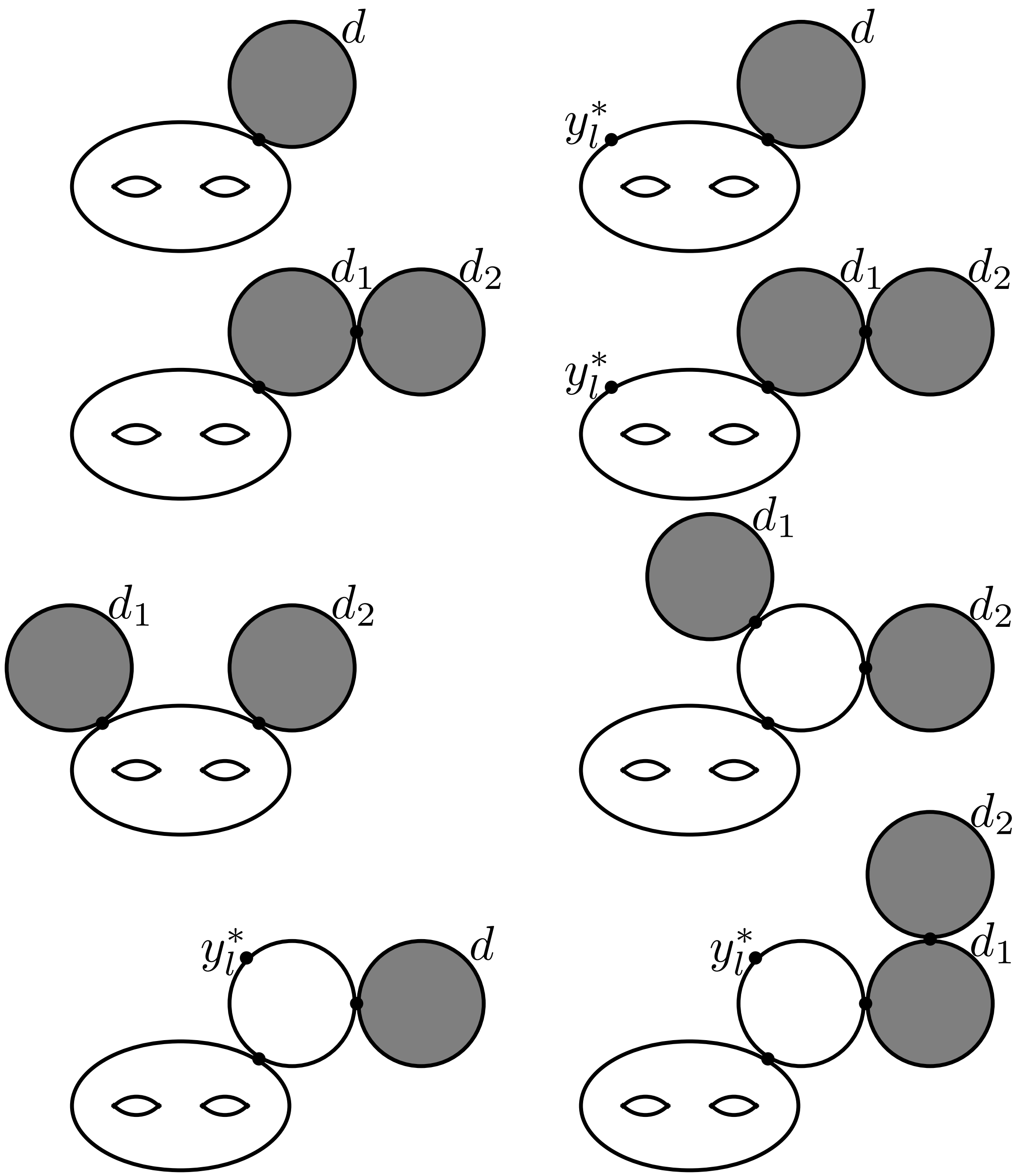


(2a) $u_b|\Sigma$ is simple & $\Sigma_b \supset S^2$, or

(2b) $u_b|\Sigma$ is multiply-covered, or

(2c) $u_b|\Sigma$ is constant.

Potential Strata Z_i in the $\mathbf{n=2}$ Case



$$d_1, d_2 > 0, \qquad d_1 + d_2 = d$$

Extent of Applications

(1) $g \leq 7$ for $n=2$; $g=3$ for $n=3$; $g=2$ for $n=4$

(2) might apply to G/P to get ITC's