Counting Rational Curves

of Arbitrary Shape

in Projective Spaces

Enumerative Geometry

Subject Matter

determine # of *geometric* objects
that satisfy given *geometric* conditions

Example

of lines thr. 2 pts in $\mathbb{R}^n$ is 1

Typical Setting

Objects: (complex) curves

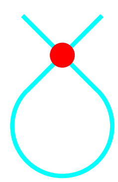
in algebraic manifolds (e.g. $\mathbb{P}^n$)

Conditions: genus/complex structure

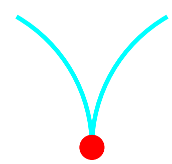
homology class

singularities

pass thr. submanifolds (e.g. pts)



simple node
$x^2 = y^2$



simple cusp
$x^2 = y^3$

Enumerative Geometry

Classical Example

n_d=# of rat. deg.-d curves thr. $3d-1$ pts. in $\mathbb{P}^2$

What is n_d?

rational	geom. genus=zero (S^2)
degree-d	$[\mathcal{C}] = d[\ell] \in H_2(\mathbb{P}^2; \mathbb{Z})$

Classical Results (by 1870s)

$n_1 = 1, \quad n_2 = 1, \quad n_3 = 12, \quad n_4 = 620$

Recent Results (1993)

(Kontsevich-Manin, Ruan-Tian)

$$n_d = \frac{1}{6(d-1)} \sum_{d_1+d_2=d} \left(d_1 d_2 - 2\frac{(d_1-d_2)^2}{3d-2} \right) \binom{3d-2}{3d_1-1} d_1 d_2 n_{d_1} n_{d_2}$$

d	1	2	3	4	5	6
n_d	1	1	12	620	87,304	26,312,976

also recursion for $n_d(\mu)$, μ=submanifolds in $\mathbb{P}^n$

Modern Approach

Summary

Count stable maps into $\mathbb{P}^n$, not curves in $\mathbb{P}^n$

Pseudoholomorphic Maps (Gromov'85)

(V,ω)=sympl. manifold, $A \in H_2(V;\mathbb{Z})$

J=compatible (almost) complex str.

$$\mathfrak{M}_{0,0}(V,A) = \big\{ u \in C^\infty(S^2,V) : u_*[S^2] = A, \\ \bar{\partial}_J u = 0 \big\} / PSL_2$$

Fact: $\exists\ \overline{\mathfrak{M}}_{0,0}(V,A)$

Example

$(\mathbb{P}^2, \omega, J)$=Fubini-Study str.

$$\mathfrak{M}_{0,0}(\mathbb{P}^2, d) = \big\{ u \in C^\infty(S^2, \mathbb{P}^2) \colon u_*[S^2] = d[\ell], \\ \bar{\partial}_J u = 0 \big\} / PSL_2$$

$p_1, \ldots, p_{3d-1}$=points in $\mathbb{P}^2$

$$\#\big\{ [u] \in \mathfrak{M}_{0,0}(\mathbb{P}^2, d) \colon p_l \in \operatorname{Im} u \big\} = n_d$$

More Generally

$$\mathfrak{M}_{0,N}(\mathbb{P}^n, d) = \big\{ (u; y_1, \ldots, y_N) \colon y_l \in S^2 \big\} / PSL_2$$

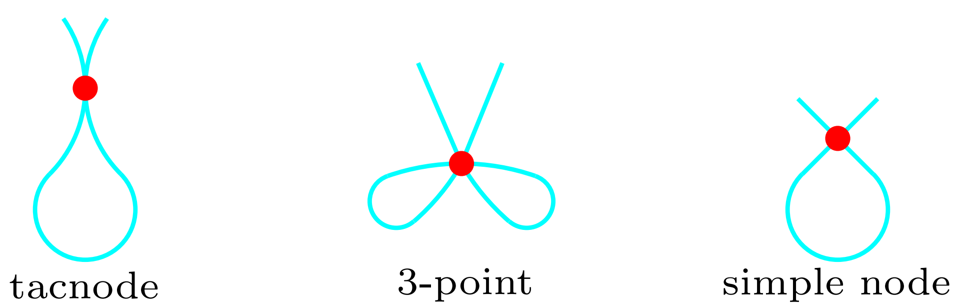
Fact: $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d)$ is "nice"

AG: Kontsevich'93, Fulton-Pandharipande'97

ST: McDuff-Salamon'93, Ruan-Tian'93

Counting Curves vs. Stable Maps

type	*via curves*	*via stable maps*
tacnode in $\mathbb{P}^2$	Ran'02	DH'88+Vakil'98
3-point in $\mathbb{P}^2$	Ran'02	KQR'96, Vakil'98
node	$\mathbb{P}^3$: Ran'02	$\mathbb{P}^n$: Z.'02



Subject of the Talk

Method for solving a large class of
enum. problems involving rational curves in $\mathbb{P}^n$

Highlights

- uses only topology of $\overline{\mathfrak{M}}_{0,N}(\mathbb{P}^n, d)$
- appl. in each specific case is mechanical

Applications

- determine # of rational curves in $\mathbb{P}^n$ with one singular point of given type not involving tacnode-like conditions
- describe stable-map closure of the space of normalizations of such curves

Example

Determine # of rational curves in $\mathbb{P}^n$

that have a point with four branches,

one of which is a cusp and another is a 2-cusp

Problematic

Curves with 2 nodes, a tacnode, etc.

if $n \geq 4$ if $n \geq 5$

One-Component 3-Pointed Rational Curves in $\mathbb{P}^3$

d	4	4	4	4	4	5	5	6
(p, ℓ)	$(6, 1)$	$(5, 3)$	$(4, 5)$	$(3, 7)$	(2,9)	$(8, 1)$	$(7, 3)$	$(10, 1)$
$\frac{1}{6}\lvert\mathcal{V}_1^{(2)}(\mu)\rvert$	0	0	0	60	$1,280$	8	264	$4,680$

One-Component Tacnodal Rational Curves in $\mathbb{P}^3$

d	4	4	4	4	4	5	5
(p, ℓ)	$(6, 1)$	$(5, 3)$	$(4, 5)$	$(3, 7)$	(2,9)	(8,1)	(7,3)
$\frac{1}{2}\lvert\mathcal{S}_1^{(1)}(\mu)\rvert$	0	0	0	$1,296$	$27,648$	960	9,792

Ingredients

(1) topology:

boundary contributions to euler class

boundary contributions to pseudocycles

zeros of affine maps between vector bundles

(2) analysis:

construction of "good" bundle sections

behavior of these near the boundary

Step 0 (trivial)

Relate the desired set of curves to

a subset $\mathcal{Z}$ of $\mathfrak{M}_{0,N}(\mathbb{P}^n, d)$ (or related space)

Eventual Goal

Express $|\mathcal{Z}|$ in terms of "Level 0" numbers:

(1) tautological classes in $\overline{\mathfrak{M}}_{0,*}(\mathbb{P}^n, *)$

(2) equivalently, the numbers $n_*(*)$ (Pand.'95)

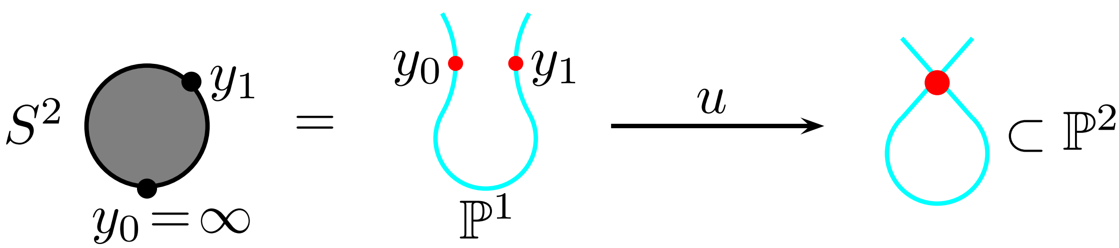
Example

$\mathcal{Z}^* =$ rat. degree-d curves thr. $3d-1$ pts, counted with a choice of a node

Question: What is $|\mathcal{Z}^*|$?

$$\mathcal{S} = \big\{[u, y_0, y_1] \in \mathfrak{M}_{0,2}(\mathbb{P}^2, d) : p_i \in \operatorname{Im} u\big\}$$

$$\begin{aligned}\mathcal{Z} &= \big\{b \in \mathcal{S} : \operatorname{ev}_0(b) = \operatorname{ev}_1(b)\big\} \\ &= \big\{b \in \mathcal{S} : \{\operatorname{ev}_0 \times \operatorname{ev}_1\}(b) \in \Delta_{\mathbb{P}^2 \times \mathbb{P}^2}\big\}\end{aligned}$$



$$\boxed{|\mathcal{Z}^*| = \tfrac{1}{2}|\mathcal{Z}|}$$

Step 1 (easy)

Describe subset $\mathcal{S}$ of $\mathfrak{M}_{0,N}(\mathbb{P}^n, d)$ s.t.

(1a) $\mathcal{Z} = s^{-1}(0)$ for a "good" $s \in \Gamma(\mathcal{S}; V)$

(1b) or "pseudocycle equivalent" of (1a)

(2) Level of $\mathcal{S}$ < Level of $\mathcal{Z}$

Goal

Express $|\mathcal{Z}|$ in terms of *lower-level numbers*

Topology, Part I

Motivation

$$\begin{array}{c} V^n \\ s \uparrow\downarrow \\ \bar{\mathcal{S}}^n \end{array}$$

$\bar{\mathcal{S}}^n$ cmpt mnfd

$s \in \Gamma(\bar{\mathcal{S}}; V)$ $\qquad \mathcal{Z} \equiv s|_{\mathcal{S}}^{-1}(0)$

$s|_{\mathcal{S}} \pitchfork 0_V$

What is $|\mathcal{Z}|$?

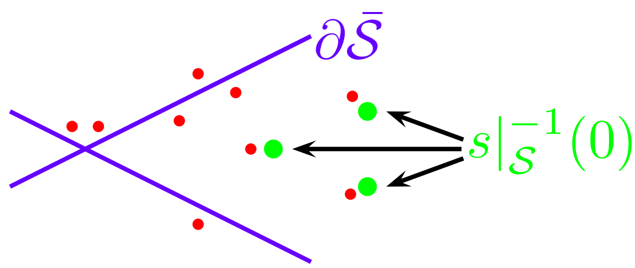
$$^{\pm}|\mathcal{Z}| = \langle e(V), \bar{\mathcal{S}} \rangle - \mathcal{C}_{\partial \bar{\mathcal{S}}}(s)$$

Formally

$$\mathcal{C}_{\partial \bar{\mathcal{S}}}(s) \equiv {}^{\pm}\big|\{s+\varepsilon\}^{-1}(0) \cap W\big| \quad \text{if}$$

$\varepsilon \in \Gamma(\bar{\mathcal{S}}; V)$ small & generic

W=small neighborhood of $\partial \bar{\mathcal{S}}$



Computation of $\mathcal{C}_{\partial\bar{\mathcal{S}}}(s)$

Assumptions on s

$$\partial\bar{\mathcal{S}} = \bigsqcup \mathcal{Z}_i$$

$$\exists W_i, \text{ small nhbd of } \mathcal{Z}_i, \text{ s.t. } s|_{W_i} \approx \alpha_i \circ \rho_i$$

$$\begin{array}{ccccc} \mathcal{N}\mathcal{Z}_i & \xrightarrow{\rho_i} & F_i & \xrightarrow{\alpha_i} & V \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{Z}_i & & \end{array}$$

$$\text{rk } F_i \le \text{rk } \mathcal{N}\mathcal{Z}_i$$

$$\alpha_i \in \Gamma(\mathcal{Z}_i; \text{Hom}(F_i; V)) \text{ non-degen.}$$

$$\rho_i = \text{polyn. if rk } F_i = \text{rk } \mathcal{N}\mathcal{Z}_i$$

$$\boxed{\mathcal{C}_{\mathcal{Z}_i}(s) = \deg \rho_i \cdot N(\alpha_i)}$$

$$N(\alpha_i) = \left|\{(x; w) \in F_i : \alpha_i(x; w) + \nu(x) = 0\}\right|$$

$$\nu \in \Gamma(\bar{\mathcal{Z}}_i; V)$$

$$\deg \rho_i = \begin{cases} 0, & \text{if } \operatorname{rk} F_i < \operatorname{rk} \mathcal{N}\mathcal{Z}_i \\ \operatorname{sgn} \rho_i \cdot |\det \rho_i|, & \text{if } \operatorname{rk} F_i = \operatorname{rk} \mathcal{N}\mathcal{Z}_i \end{cases}$$

$$\rho_i \longleftrightarrow \mathcal{A} = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} \qquad \det \rho_i = \det \mathcal{A}$$

$$\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \equiv \mathcal{A}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\operatorname{sgn} \rho_i = \begin{cases} 1, & c_i > 0 \ \forall i \\ -1, & c_i < 0 \ \forall i \\ 0, & \text{otherwise} \end{cases}$$

Summary

Setup

$$\begin{array}{c} V^n \\ s \uparrow \downarrow \\ \bar{\mathcal{S}}^n \end{array}$$

$\bar{\mathcal{S}}^n$ cmpt mnfd

$s \in \Gamma(\bar{\mathcal{S}}; V)$ $\qquad$ $\mathcal{Z} \equiv s|_{\mathcal{S}}^{-1}(0)$

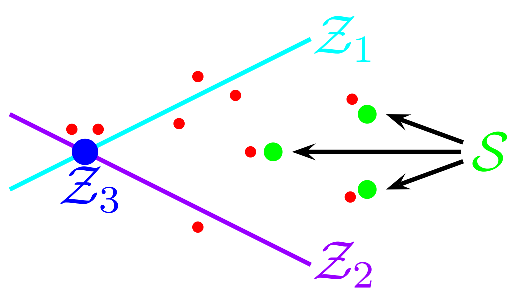
$s|_{\mathcal{S}} \pitchfork 0_V$

$$\partial\bar{\mathcal{S}} = \bigsqcup \mathcal{Z}_i, \qquad s|_{W_i} \approx \alpha_i \circ \rho_i$$

Conclusion

$$\boxed{{}^{\pm}|\mathcal{Z}| = \langle e(V), \bar{\mathcal{S}} \rangle - \mathcal{C}_{\partial\bar{\mathcal{S}}}(s)}$$

$$\boxed{\mathcal{C}_{\partial\bar{\mathcal{S}}}(s) = \sum \mathcal{C}_{\mathcal{Z}_i}(s)} \qquad \boxed{\mathcal{C}_{\mathcal{Z}_i}(s) = \deg \rho_i \cdot N(\alpha_i)}$$



$\mathcal{C}_{\mathcal{Z}_1}(s) = 3$

$\mathcal{C}_{\mathcal{Z}_2}(s) = 1$

$\mathcal{C}_{\mathcal{Z}_3}(s) = 2$

Topology, Part II

Counting Zeros of Affine Maps

$$\begin{array}{ccc} F^k & \xrightarrow{\psi_{\alpha,\nu} \equiv \nu+\alpha} & \mathcal{O}^{k+m} \\ & \searrow \quad \swarrow & \\ & \bar{\mathcal{S}}^m & \end{array}$$

$\bar{\mathcal{S}}$ cmpt

$\alpha \in \Gamma\big(\bar{\mathcal{S}};\mathrm{Hom}(F,\mathcal{O})\big)$

$\nu \in \Gamma(\bar{\mathcal{S}};\mathcal{O})$ generic w.r.t. α

Facts

(1) $N(\alpha) \equiv {}^{\pm}\big|\psi_{\alpha,\nu}^{-1}(0)\big|$ depends on α, but not ν

(2) if $\alpha|F_x$ is injective $\forall x \in \bar{\mathcal{S}}$,

$${}^{\pm}\big|\psi_{\alpha,\nu}^{-1}(0)\big| = \big\langle e\big(\mathcal{O}/\alpha(F)\big), \bar{\mathcal{S}}\big\rangle = \big\langle c(\mathcal{O})c(F)^{-1}, \bar{\mathcal{S}}\big\rangle$$

Example

$$\bar{\mathcal{S}} = \mathbb{P}^1 = \left\{\ell = [u,v] : (u,v) \in \mathbb{C}^2 - \{0\}\right\}$$
$$F = \mathbb{C}, \qquad \mathcal{O} = \mathbb{C} \oplus \gamma^*$$

$$(1) \quad \text{if } \alpha(\ell;c) = (\ell;c,0), \qquad N(\alpha) = 1$$

$$(2) \quad \text{if } \alpha(\ell;c) = (\ell;0,c\cdot u), \quad N(\alpha) = 0$$

Computation of $N(\alpha)$

$$(\bar{\mathcal{S}}, F, \mathcal{O}, \alpha)$$

$$\bar{\mathcal{S}} \longrightarrow \mathbb{P}F \quad \Big\Downarrow \quad F \longrightarrow \gamma$$

$$(\bar{\mathcal{S}}, F, \mathcal{O}, \alpha),\ \mathrm{rk}\ F = 1$$

$$\boxed{N(\alpha) =^{\pm} \left|\alpha^{\perp}|^{-1}_{\mathcal{S}-\alpha^{-1}(0)}(0)\right|}$$

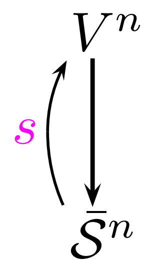
$$\alpha^{\perp} \in \Gamma\big(\bar{\mathcal{S}}; \mathrm{Hom}(F, \mathcal{O}/\mathbb{C}\nu)\big)$$

$$\Big\Downarrow$$

$$(\bar{\mathcal{S}}_j, F_j, \mathcal{O}_j, \alpha_j),\ \mathrm{rk}\,\mathcal{O}_j < \mathrm{rk}\,\mathcal{O}$$

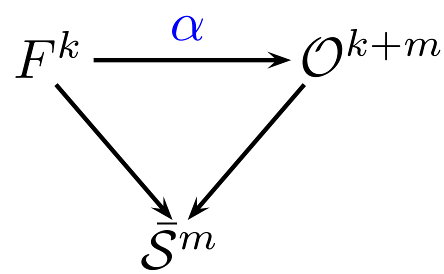
Summary of Topology

Setup



$s \in \Gamma(\bar{\mathcal{S}}; V)$

$s|_{\mathcal{S}} \pitchfork 0_V$



$\partial\bar{\mathcal{S}} = \bigsqcup \mathcal{Z}_i, \quad s|_{W_i} \approx \alpha_i \circ \rho_i$

Conclusion

$$\pm\left|s|_{\mathcal{S}}^{-1}(0)\right| = \langle e(V), \bar{\mathcal{S}}\rangle - \sum \deg \rho_i \cdot N(\alpha_i)$$

$$N(\alpha) = \pm\left|\alpha^{\perp}|_{\mathcal{S}-\alpha^{-1}(0)}^{-1}(0)\right|$$

Important Generalizations

(i) $\mathcal{S} \subset \overline{\mathfrak{M}}$, $\overline{\mathfrak{M}}$=strat. top.orbi., $\dim \partial \bar{\mathcal{S}} < \dim \mathcal{S}$

(ii) $\partial \bar{\mathcal{S}} \subset \sqcup \mathcal{Z}_i$

(1) $s \in \Gamma(\mathcal{S}, V)$, $\alpha \in \Gamma\big(\mathcal{S}, \mathrm{Hom}(F, \mathcal{O})\big)$

(2) extension to pseudocycles

Example

$\mathcal{Z}^* =$ rat. degree-d curves thr. $3d-1$ pts, counted with a choice of a node

Question: What is $|\mathcal{Z}^*|$?

$$\mathcal{S} = \big\{[u, y_0, y_1] \in \mathfrak{M}_{0,2}(\mathbb{P}^2, d) \colon p_i \in \operatorname{Im} u\big\}$$

$$\mathcal{Z} = \big\{\{\mathrm{ev}_0 \times \mathrm{ev}_1\}|_{\mathcal{S}}\big\}^{-1}\big(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}\big).$$

$\mathrm{ev}_0 \times \mathrm{ev}_1 \colon \mathcal{S} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is a pseudocycle

Reasons:

(1) $\dim \partial \bar{\mathcal{S}} < \dim \mathcal{S}$

(2) $\mathrm{ev}_0 \times \mathrm{ev}_1 \colon \bar{\mathcal{S}} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ is continuous

(3) $\mathrm{ev}_0 \times \mathrm{ev}_1$ is smooth on all strata of $\bar{\mathcal{S}}$

Consequences

(1) get $\left[\mathrm{ev}_0 \times \mathrm{ev}_1\right] \in H_4(\mathbb{P}^2 \times \mathbb{P}^2; \mathbb{Z})$

(2) $$\left\langle\!\left\langle \{\mathrm{ev}_0 \times \mathrm{ev}_1\}^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}), \bar{\mathcal{S}} \right\rangle\!\right\rangle \equiv \left[\mathrm{ev}_0 \times \mathrm{ev}_1\right] \cdot \left[\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}\right] \in \mathbb{Z}$$

$$|\mathcal{Z}| = \left\langle\!\left\langle \{\mathrm{ev}_0 \times \mathrm{ev}_1\}^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}), \bar{\mathcal{S}} \right\rangle\!\right\rangle - \mathcal{C}_{\partial \bar{\mathcal{S}}}\left(\mathrm{ev}_0 \times \mathrm{ev}_1; \Delta_{\mathbb{P}^2 \times \mathbb{P}^2}\right)$$

$$\left[\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}\right] = [p \times \mathbb{P}^2] + [\ell \times \ell] + [\mathbb{P}^2 \times p]$$

Conclusion

$$\left\langle\!\left\langle \{\mathrm{ev}_0 \times \mathrm{ev}_1\}^{-1}(\Delta_{\mathbb{P}^2 \times \mathbb{P}^2}), \bar{\mathcal{S}} \right\rangle\!\right\rangle = 0 + d^2 \cdot n_d + 0$$

Ingredients of the Method

(1) topology:

boundary contributions to euler class

boundary contributions to pseudocycles

zeros of affine maps between vector bundles

(2) analysis:

construction of "good" bundle sections

behavior of these near the boundary

Analysis, Part I

Construction of "Good Bundles"

$$\mathfrak{M}_{0,*}(\mathbb{P}^n, d) = \big\{(u; y_0, \ldots) \colon u \in \mathcal{H}(S^2; \mathbb{P}^n)\big\}/PSL_2$$
$$= \mathcal{B}/S^1$$
$$\overline{\mathfrak{M}}_{0,*}(\mathbb{P}^n, d) = \bar{\mathcal{B}}/S^1$$

Key Property

$$\mathcal{B} \subset \big\{(u; y_0, \ldots) \colon u \in \mathcal{H}(S^2; \mathbb{P}^n)\big\}$$

$L_0 \equiv \bar{\mathcal{B}} \times_{S^1} \mathbb{C}$ is the "universal tangent bundle at y_0"

Construction of "Good" Bundle Sections

$$\mathcal{D}_0^{(m)} \in \Gamma\big(\overline{\mathfrak{M}}_{0,*}(\mathbb{P}^n,d),\ \mathrm{Hom}(L_0^{\otimes m}, \mathrm{ev}_0^* T\mathbb{P}^n)\big)$$
$$\mathcal{D}_0^{(1)}\big[(u,y_0,\ldots),c\big] = c\cdot du|_{y_0=\infty} e_\infty$$
$$\mathcal{D}_0^{(2)}\big[(u,y_0,\ldots),c\big] = \tfrac{1}{2!}c^2\cdot \tfrac{D}{ds}du|_\infty e_\infty$$
$$e_\infty = (1,0,0) \in T_\infty S^2$$

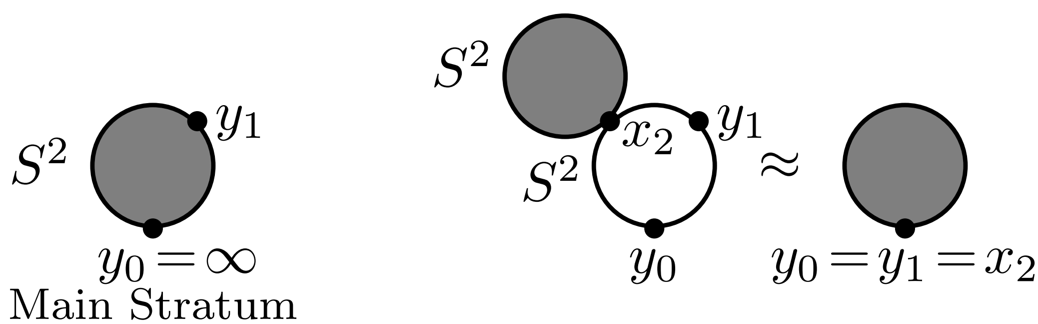
$$\mathcal{D}_l^{(m)} \in \Gamma\big(\mathfrak{M}_{0,*}(\mathbb{P}^n,d),\ \mathrm{Hom}(L_0^{*\otimes m}, \mathrm{ev}_l^* T\mathbb{P}^n)\big)$$
$$\mathcal{D}_l^{(1)}\big[(u,y_0,\ldots),c\big] = c\cdot du|_{y_l} e_l$$
$$\mathcal{D}_l^{(2)}\big[(u,y_0,\ldots),c\big] = \tfrac{1}{2!}c^2\cdot \tfrac{D}{ds}du|_{y_l} e_l$$
$$e_l = \tfrac{\partial}{\partial x} \in T_{y_l}\mathbb{C} \approx T_{y_l}S^2$$

Analysis, Part II

Theorem

Near each boundary stratum of $\overline{\mathfrak{M}}_{0,*}(\mathbb{P}^n, d)$, each section $\mathcal{D}_l^{(m)}$ admits a power-series type of expansion involving the "derivatives" at the marked and singular points.

Example



$$\text{Normal Bundle} \approx L_0^* \otimes L_2 \approx L_2$$

$$\mathcal{D}_0^{(1)}\phi(\upsilon) = \big\{\mathcal{D}_2^{(1)} + \varepsilon_1(\upsilon)\big\}\upsilon$$

$$\mathcal{D}_0^{(2)}\phi(\upsilon) = x_2 \otimes \big\{\mathcal{D}_2^{(1)} + \varepsilon_1(\upsilon)\big\}\upsilon + \big\{\mathcal{D}_2^{(2)} + \varepsilon_2(\upsilon)\big\}\upsilon^{\otimes 2}$$

$$\vdots$$

Example

$\mathcal{Z}^* =$ rat. degree-d curves thr. $3d-1$ pts, counted with a choice of a node

Question: What is $|\mathcal{Z}^*|$?

$$\mathcal{S} = \big\{[u,y_0,y_1] \in \mathfrak{M}_{0,2}(\mathbb{P}^2,d) : p_i \in \operatorname{Im} u\big\}$$

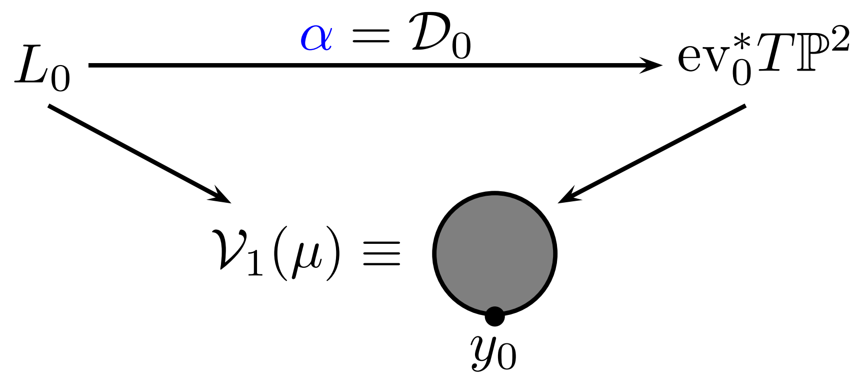
$$\mathcal{Z} = \big\{\{\mathrm{ev}_0 \times \mathrm{ev}_1\}|_{\mathcal{S}}\big\}^{-1}\big(\Delta_{\mathbb{P}^2\times\mathbb{P}^2}\big).$$

$$\boxed{|\mathcal{Z}| = d^2 \cdot n_d - \mathcal{C}_{\partial\bar{\mathcal{S}}}\big(\mathrm{ev}_0 \times \mathrm{ev}_1; \Delta_{\mathbb{P}^2\times\mathbb{P}^2}\big)}$$

$$\begin{aligned}
\mathrm{ev}_1(\phi(\upsilon)) - \mathrm{ev}_0(\phi(\upsilon)) &= \sum_{k=1}^{\infty} y_1^{-k} \mathcal{D}_0^{(k)} \phi(\upsilon) \\
&= \sum_{k=1}^{\infty} (y_1 - x_2)^{-k} \big\{\mathcal{D}_2^{(k)} + \varepsilon_k(\upsilon)\big\} \upsilon^{\otimes k} \\
&= (y_1 - x_2)^{-1} \big\{\mathcal{D}_k^{(1)} + \varepsilon(\upsilon)\big\} \upsilon
\end{aligned}$$

Consequences

$$\mathcal{C}_{\partial\bar{\mathcal{S}}}\big(\mathrm{ev}_0\times\mathrm{ev}_1;\Delta_{\mathbb{P}^2\times\mathbb{P}^2}\big) = N(\alpha)$$



α does not vanish:

$$\begin{aligned} N(\alpha) &= \big\langle c(\mathrm{ev}_0^* T\mathbb{P}^n)c(L_0)^{-1}, \bar{\mathcal{V}}_1(\mu)\big\rangle \\ &= \big\langle 3\mathrm{ev}_0^* c_1(\gamma^*) + c_1(L_0^*), \bar{\mathcal{V}}_1(\mu)\big\rangle \\ &= 3d\cdot n_d - 2\cdot n_d \end{aligned}$$

Conclusion

$$|\mathcal{Z}| = d^2\cdot n_d - (3d-2)n_d = 2\binom{d-1}{2}n_d$$

$$\boxed{|\mathcal{Z}^*| = \binom{d-1}{2}n_d}$$

Rational Tacnodal Curves in $\mathbb{P}^n$

Question

What is the number $|\mathcal{Z}^*|$ of rat. tacnodal curves that pass thr. constraints $\mu_1, \ldots, \mu_N$ in $\mathbb{P}^n$?

Constraints

$3d-2$ points in $\mathbb{P}^2$

p pts and q lines in $\mathbb{P}^3$, $2p+q=4d-3$

$\vdots$

Setup

$$\mathcal{S}_0 = \big\{[u, y_0, y_1] \in \mathfrak{M}_{0,2}(\mathbb{P}^n, d) \colon \mu_l \cap \mathrm{Im}\, u \neq \emptyset\big\}$$

$$\mathcal{S}_1 = \big\{b \in \mathcal{S}_0 \colon \mathrm{ev}_0(b) = \mathrm{ev}_1(b)\big\}$$

$$\mathcal{S}_1' = \mathbb{P}(L_0 \oplus L_0^*)\big|_{\mathcal{S}_1}$$

$$\mathcal{Z} = \big\{(b, \ell) \in \mathcal{S}_1' \colon \mathcal{D}_{0,1}(b, \ell) = 0\big\}$$

$$\mathcal{D}_{0,1}(b, \ell) \in \Gamma(\mathcal{S}_1'; \gamma^* \otimes \mathrm{ev}_0^* T\mathbb{P}^n)$$

$$\mathcal{D}_{0,1}(b; \upsilon_0, \upsilon_1) = \mathcal{D}_0^{(1)} \upsilon_0 + \mathcal{D}_1^{(1)} \upsilon_1$$

Remark

$\mathcal{D}_{0,1}$ does *not* extend over $\partial \bar{\mathcal{S}}_1'$

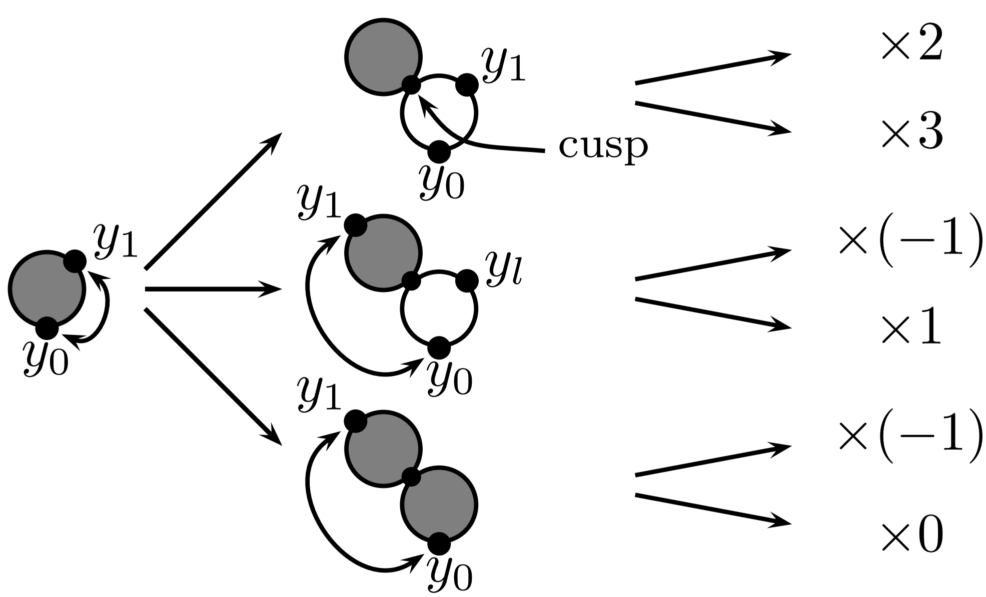
$$\boxed{|\mathcal{Z}| = \left\langle e(\gamma^* \otimes \mathrm{ev}_0^* T\mathbb{P}^n), \bar{\mathcal{S}}_1' \right\rangle - \mathcal{C}_{\partial \bar{\mathcal{S}}_1'}(\mathcal{D}_{0,1})}$$

Step 1

- Check that $\dim \partial \bar{\mathcal{S}}_1 < \dim \mathcal{S}_1$
- Describe the strata of $\partial \bar{\mathcal{S}}_1$

Step 2

Figure out the boundary contributions



Infinite-Dimensional Analogue

$$\begin{array}{c} \Gamma^{0,1} \\ \bar\partial \uparrow \downarrow \\ \bar{C}^\infty_d(\Sigma;\mathbb{P}^n) \end{array}$$

$$\Sigma = (\Sigma, j) = \text{Riemann surface}$$

$$\Gamma^{0,1}(f) = \Gamma(\Sigma; \Lambda^{0,1}_{J,j} T^*\Sigma \otimes f^*T\mathbb{P}^n)$$

$$\left\langle e(\Gamma^{0,1}; \text{ind } \bar\partial), \bar{C}^\infty_d(\Sigma;\mathbb{P}^n) \right\rangle \equiv {}^{\pm}\left|\{\bar\partial + \varepsilon\}^{-1}(0)\right|$$

$$\equiv \mathrm{RT}_{g,d}$$

Li-Tian, Fukaya-Ono, Seibert

Fact

$\mathrm{RT}_{g,d}$ for $\mathbb{P}^n$ is easily computable

Applications

$$\begin{aligned} n_{g,d} &= \Big|\big\{ f\colon (\Sigma, j) \longrightarrow \mathbb{P}^n \text{ passes thr.} \\ &\qquad \text{appropriate \# of pts, lines, etc.}\big\}\Big| \\ &= \Big|\big\{\bar\partial|_{C^\infty_d(\Sigma;\mathbb{P}^n)}\big\}^{-1}(0)\Big| \\ &\backsimeq \Big|\big\{\text{genus-}g \text{ fixed complex str. curves} \\ &\qquad \text{thr. appropriate constraints in } \mathbb{P}^n\big\} \end{aligned}$$

$$\begin{aligned} n_{g,d} &= \big\langle e(\Gamma^{0,1}; \text{ind}\ \bar\partial), \bar C^\infty_d(\Sigma;\mathbb{P}^n)\big\rangle - \mathcal{C}_{\partial \bar C^\infty_d(\Sigma;\mathbb{P}^n)}(\bar\partial) \\ &= \mathrm{RT}_{g,d} - \mathcal{C}_{\partial \bar C^\infty_d(\Sigma;\mathbb{P}^n)}(\bar\partial) \end{aligned}$$

“Near Fact”

$\exists$ partition $\bar{\partial}^{-1}(0)\cap\partial\bar{C}_d^\infty(\Sigma;\mathbb{P}^n)=\bigsqcup_{i\in I}\mathcal{Z}_i$ s.t.

- $\mathcal{C}_{\partial\bar{C}_d^\infty(\Sigma;\mathbb{P}^n)}(\bar{\partial})=\sum_{i\in I} m_i\cdot N(\alpha_i)$
- I depends only on n and g (or j if $g\geq 3$)
- m_i depends only on i
- $\alpha_i\in\Gamma\big(\mathcal{Z}_i;\mathrm{Hom}(F_i;\mathcal{O}_i)\big)$ is good and “minimally” dependent on d and constraints

True if

$g=0$	$m_i=0$	easy
$g=1$	$m_1=1$; $m_i=0$ if $i\neq 1$	$\sim$Ionel’96
$g=2$, $n=2,3$	$m_i\in\{0,1,2,3\}$	Z.’02
$g=3$, $n=3$ j non-hyperell.	$m_i\in\{0,1,2,3,4\}$ or $m_i\in\{0,1,2,3,4,10\}$	Z.’02

Also $g=2$, $n=4$; $g=3$, $n=3$; $g=4,5,6,7$, $n=2$.

Final Remark

$$n^j_{g,d} = \Big|\Big\{ f\colon (\Sigma, j) \longrightarrow \mathbb{P}^n \text{ passes thr. appropriate \# of pts, lines, etc.} \Big\}\Big|$$

$g=0$	$n^j_{g,d} = \mathrm{RT}_{g,d}$
$g=1$	$n^j_{g,d} \neq \mathrm{RT}_{g,d}$ $n^j_{g,d}$ is independent of $j \in \overline{\mathfrak{M}}_{1,1}$
$g=2$	$n^j_{g,d} \neq \mathrm{RT}_{g,d}$ $n^j_{g,d}$ is independent of $j \in \mathfrak{M}_2$ $n^j_{g,d}$ is dependent on $j \in \overline{\mathfrak{M}}_2$
$g=3$	$n^j_{g,d} \neq \mathrm{RT}_{g,d}$ If j is not hyperelliptic, $n^j_{g,d}$ depends on # of hyperflexes