Enumeration of One-Nodal Rational Curves in Projective Spaces

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June 6, 2004

Abstract

We give a formula computing the number of one-nodal rational curves that pass through an appropriate collection of constraints in a complex projective space. The formula involves intersections of tautological classes on moduli spaces of stable rational maps. We combine the methods and results from three different papers.

Contents

1 Introduction

Enumerative algebraic geometry is a field of mathematics that dates back to the nineteenth century. However, many of its most fundamental problems remained unsolved until the early 1990s. For

*Partially supported by NSF grant DMS-9803166

example, let d be a positive integer and $\mu=(\mu_1,\ldots,\mu_N)$ an N-tuple of linear subspaces of $\mathbb{P}^n$ of codimension at least two such that

$$\mathrm{codim}_{\mathbb{C}}\mu \equiv \sum_{l=1}^{l=N}\mathrm{codim}_{\mathbb{C}}\mu_l - N = d(n+1)+n-3.$$

If the constraints μ are in general position, denote by $n_d(\mu)$ the number of rational degree-d curves that pass through $\mu_1,\ldots,\mu_N$. This number is finite and depends only on the homology classes of the constraints. If $d=1$, it can be computed using Schubert calculus; see [GH]. All but very-low-degree numbers $n_d(\mu)$ remained unknown until [KM] and [RT] derived a recursive formula for these numbers. In this paper, we prove

Theorem 1.1 *Suppose $n\geq 3$, $d\geq 1$, and $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that*

$$\mathit{codim}_{\mathbb{C}}\mu \equiv \sum_{l=1}^{l=N}\mathit{codim}_{\mathbb{C}}\mu_l - N = d(n+1)-1. \tag{1.1}$$

Then the number of degree-d rational curves that have a simple node and pass through the constraints μ is given by

$$n_d^{(1)}(\mu) = \frac{1}{2}\big(RT_{1,d}(\mu_1;\mu_2,\ldots,\mu_N) - CR_1(\mu)\big), \qquad \textit{where}$$

$$CR_1(\mu) = \sum_{k=1}^{2k\leq n+1}(-1)^{k-1}(k-1)!\sum_{l=0}^{n+1-2k}\binom{n+1}{l}\big\langle a^l\eta_{n+1-2k-l},\big[\bar{\mathcal{V}}_k(\mu)\big]\big\rangle.$$

The symplectic invariant $RT_{1,d}(\cdot;\cdot)$ and the top intersections $\big\langle a^l\eta_{n+1-2k-l},\big[\bar{\mathcal{V}}_k(\mu)\big]\big\rangle$ are computable via algorithms described elsewhere.

n	3	4	5	5	6
d	4	4	4	6	6
μ	(5,5)	(5,1,4)	(5,1,0,4)	(2,1,1,7)	(2,1,1,1,6)
$n_d^{(1)}(\mu)$	1,800	1,800	1,800	20,340	20,340

Table 1: The Number $n_d^{(1)}(\mu)$ of One-Nodal Degree-d Rational Curves in $\mathbb{P}^n$

For the purposes of this table, we assume that the constraints $\mu_1,\ldots,\mu_N$ are linear subspaces of $\mathbb{P}^n$ of codimension at least two. We describe such a tuple μ of constraints by listing the number of linear subspaces of codimension $2,\ldots,n$ among $\mu_1,\ldots,\mu_N$. For example, the triple $(5,1,4)$ in the third column indicates that the tuple μ consists of 5 two-planes, 1 line, and 4 points in general position in $\mathbb{P}^4$.

In the statement of Theorem 1.1, $\mathrm{RT}_{1,d}(\cdot;\cdot)$ denotes the genus-one degree-d symplectic invariant of $\mathbb{P}^n$ defined in [RT]. This invariant can be expressed in terms of the numbers $n_d(\cdot)$; see [RT]. In particular, it is computable. Brief remarks concerning the meaning of $\mathrm{RT}_{1,d}(\cdot;\cdot)$ can be found at the beginning of Section 3.

The compact oriented topological manifold $\bar{\mathcal{V}}_k(\mu)$ consists of unordered k-tuples of stable rational maps of total degree d. Each map comes with a special marked point (i,∞). All these marked points are mapped to the same point in $\mathbb{P}^n$. In particular, there is a well-defined evaluation map

$$\mathrm{ev}\colon \bar{\mathcal{V}}_k(\mu) \longrightarrow \mathbb{P}^n,$$

which sends each tuple of stable maps to the value at one of the special marked points. We also require that the union of the images of the maps in each tuple intersect each of the constraints $\mu_1,\ldots,\mu_N$. In fact, the elements in the tuple carry a total of N marked points, $y_1,\ldots,y_N$, in addition to the k special marked points. These marked points are mapped to the constraints $\mu_1,\ldots,\mu_N$, respectively. Roughly speaking, each element of $\bar{\mathcal{V}}_k(\mu)$ corresponds to a degree-d rational curve in $\mathbb{P}^n$, which has at least k irreducible components, and k of the components meet at the same point in $\mathbb{P}^n$. The precise definition of the spaces $\bar{\mathcal{V}}_k(\mu)$ can be found in Subsection 2.2.

The cohomology classes a and η_l are tautological classes in $\bar{\mathcal{V}}_k(\mu)$. In fact,

$$a = \mathrm{ev}^* c_1\big(\mathcal{O}_{\mathbb{P}^n}(1)\big).$$

Let $\bar{\mathcal{V}}_k'(\mu)$ be the oriented topological manifold defined as $\bar{\mathcal{V}}_k(\mu)$, except without specifying the marked points $y_1,\ldots,y_N$ mapped to the constraints $\mu_1,\ldots,\mu_N$. Then, there is well-defined forgetful map,

$$\pi\colon \bar{\mathcal{V}}_k(\mu) \longrightarrow \bar{\mathcal{V}}_k'(\mu),$$

which drops the marked points $y_1,\ldots,y_N$ and contracts the unstable components. The cohomology class $\eta_l \in H^{2l}(\bar{\mathcal{V}}_k(\mu))$ is the sum of all degree-l monomials in the elements of the set

$$\big\{\pi^*\psi_{(1,\infty)},\ldots,\pi^*\psi_{(k,\infty)}\big\} \subset H^2\big(\bar{\mathcal{V}}_k(\mu)\big).$$

As common in algebraic geometry, $\psi_{(i,\infty)}$ denotes the first chern class of the universal cotangent line bundle for the marked point (i,∞). In Subsection 2.2, we give a definition of η_l that does not involve the projection map π. An algorithm for computing the intersection numbers involved in the statement of Theorem 1.1 is given in Subsection 5.7 of [Z2]. It is closely related to the algorithm of [P2] for computing intersections of tautological classes in moduli spaces of stable rational maps into $\mathbb{P}^n$.

If $n=2$, we denote by $n_d^{(1)}(\mu)$ the number of rational degree-d curves passing through the constraints counted with a choice of the node on each curve. The formula of Theorem 1.1 gives

$$n_d^{(1)}(\mu) = \binom{d-1}{2} n_d(\mu). \tag{1.2}$$

This identity is clear, since the arithmetic genus of every degree-d curve in $\mathbb{P}^2$ is $\binom{d-1}{2}$. Equation (1.2) is used in [P1] to count genus-one plane curves with complex structure fixed. More precisely, if μ is a tuple of constraints in $\mathbb{P}^n$ satisfying condition (1.1), let $n_{1,d}(\mu)$ denote the number of genus-one degree-d curves that pass through the constraints μ and have a fixed generic complex structure on the normalization, i.e. its j-invariant is different from 0 and 1728. The key step in [P1] is to show that

$$n_{1,d}(\mu) = n_d^{(1)}(\mu), \tag{1.3}$$

if μ is a tuple of $3d-1$ points in $\mathbb{P}^2$. One of the main ingredients in proving Theorem 1.1 is Proposition 4.1, which states that (1.3) is valid for any tuple μ that satisfies condition (1.1). Note that the numbers listed in Table 1 are consistent with (1.3) and facts of classical algebraic geometry. In particular, the image of every degree-4 map from a genus-one curve to $\mathbb{P}^n$ lies in a $\mathbb{P}^3$ and the image of every degree-6 map lies in a $\mathbb{P}^5$; see [ACGH, p116]. Thus, the first three numbers in the table should be the same, and the last two numbers should be the same. The proof of Proposition 4.1 extends the degeneration argument of [P1] and builds up on modifications described in [Z1]. We work with the moduli space $\overline{\mathfrak{M}}_{1,N}(\mathbb{P}^n, d)$ of stable degree-d maps from genus-one N-pointed curves into $\mathbb{P}^n$ and study what happens in the limit to the maps that pass through the constraints μ as the j-invariant of the domain tends infinity, i.e. the domain degenerates to a rational curve with two points identified.

Proposition 4.1 is not useful for determining the numbers $n_{1,d}(\mu)$ in $\mathbb{P}^n$ if $n \geq 3$, since the right-hand side of (1.3) is unknown. Computation of $n_{1,d}(\mu)$ for all projective spaces is the subject of [I], where an entirely different approach is taken. The main step in computing these numbers is showing that

$$2n_{1,d}(\mu) = \mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - CR_1(\mu),$$

where $CR_1(\mu)$ is the number of zeros of an explicit affine map between vector bundles over $\bar{\mathcal{V}}_1'(\mu)$; see Proposition 3.2. The remaining step is to express this number of zeros topologically. In general, if the linear part of an affine map ψ does not vanish, it is easy to determine the signed cardinality of $\psi^{-1}(0)$; see Lemma 2.5. The approach of [I] is to replace the linear part α of the affine map under consideration by a nonvanishing linear map over a space obtained from $\bar{\mathcal{V}}_1'(\mu)$ by a sequence of blowups and then to express the resulting intersection number in terms of intersection numbers on the spaces $\bar{\mathcal{V}}_k'(\mu)$. The main problem with this approach is that the new linear map is not described in [I] and it is not clear how to construct it in general. In addition, the normal bundles of certain spaces needed for the second part of this approach are given incorrectly; see Lemma 2.8 or equation (2.27) in [I] for example. Both of these statements can be corrected without affecting the computability of the intersection numbers, but presumably with a change in the final result. If $n=2$, no blowup is needed. If $n=3,4$, the zero set of α is a complex manifold and the "derivative" of α in the normal direction along $\alpha^{-1}(0)$ is nondegenerate. In such cases, only one blowup is needed and a linear map with the required properties can be constructed fairly easily. Furthermore, Lemma 2.8 of [I] requires no correction in the $n=2,3,4$ cases, while equation (2.27) is never used. If $n=2,3$, $CR_1(\mu)$ and $n_{1,d}(\mu)$ are then expressed in terms of the numbers $n_{d'}(\mu')$, with $d' \leq d$ and μ' related to μ. Several numbers $n_{1,d}(\mu)$ for $\mathbb{P}^4$ are given in [I] as well. However, no topological formula, like that of Theorem 1.1, is given for $CR_1(\mu)$ or $n_{1,d}(\mu)$ for $\mathbb{P}^n$ with $n \geq 4$ and no number $n_{1,d}(\mu)$ is given for $\mathbb{P}^n$ with $n \geq 5$.

We obtain the expression of Theorem 1.1 for the number $CR_1(\mu)$ in Section 3; see Proposition 3.1. Our approach involves no blowups and requires relatively little understanding of the global structure of the spaces $\bar{\mathcal{V}}_k(\mu)$. Instead we describe $CR_1(\mu)$ as the euler class of a bundle minus the sum of contributions to the euler class from smooth, but usually noncompact, strata of the zero set of the linear part $\alpha_{1,0}$ of the affine map. Computation of these contributions in good cases involves counting the zeros of affine maps again, but with the rank of the target bundle reduced by one; see Subsection 2.1. Of course, if we are to have any hope of computing these contributions, we need to understand the behavior of $\alpha_{1,0}$ near the smooth strata of its zero set. Proposition 2.7 describes the behavior of $\alpha_{1,0}$ and of related linear maps near the boundary strata of $\bar{\mathcal{V}}_k(\mu)$.

Theorem 1.1 follows immediately from Propositions 3.1 and 4.1. Their proofs are mutually independent. Section 4 uses some of the notation defined in Subsection 2.2. The topological tools of Subsection 2.1, the descriptive notation of Subsection 2.2, and the structure theorem of Subsection 2.3 are integral to the computations of Section 3.

In brief, we enumerate one-nodal rational curves from genus-one fixed-complex-structure invariants. Can a similar approach be used with higher-genera enumerative invariants? Let μ be an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that

$$\mathrm{codim}_{\mathbb{C}}\mu = d(n+1) - n.$$

Denote by $n_{2,d}(\mu)$ the number of genus-two degree-d curves that pass through the constraints μ and have a fixed generic complex structure on the normalization. Let $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ denote the number of rational two-component curves connected at three nodes, of rational curves with a triple point, and of rational curves with a tacnode, respectively. If $n=2$, we take $n_d^{(3)}(\mu)$ to be the number of two-component rational curves with a choice of three nodes common to both components. In all cases, the curves have degree-d and pass through the constraints μ. Completing the degeneration argument of [KQR], it is shown in [Z1] that

$$n_{2,d}(\mu) = 6\big(n_d^{(3)}(\mu) + \tau_d(\mu) + T_d(\mu)\big), \tag{1.4}$$

if μ is a tuple of $3d-2$ points in $\mathbb{P}^2$. The arguments of [KQR] and [Z1] should extend to show that equation (1.4) is valid for arbitrary constraints μ in all projective spaces. On the other hand, $n_{2,d}(\mu)$ for $\mathbb{P}^3$ is computed in [Z2] and the method extends at least to $\mathbb{P}^4$. Thus, in those two cases, we should be able to express the sum of the numbers $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ in terms of intersection numbers of the spaces $\bar{\mathcal{V}}_k(\mu)$. The relation (1.4) is obtained by considering a degeneration to a specific singular genus-two curve. Perhaps, different relations can be obtained by considering degeneration to other singular genus-two curves. With enough different relations, we would be able to compute the numbers $n_d^{(3)}(\mu)$, $\tau_d(\mu)$, and $T_d(\mu)$ at least for $\mathbb{P}^3$ and $\mathbb{P}^4$.

Since the initial submission of this paper, a formula for the numbers $n_d^{(1)}(\mu)$ in $\mathbb{P}^3$, i.e. the lowest-dimensional case of Theorem 1.1, has also appeared in [R]. The approach of [R] is completely unrelated to the one presented here; it uses more classical tools of algebraic geometry, instead of the moduli space of stable maps.

The author thanks T. Mrowka for many useful discussions and E. Ionel for comments on the original version of this paper.

2 Background

2.1 Topology

We begin by describing the topological tools used in the next section. In particular, we review the notion of contribution to the euler class of a vector bundle from a (not necessarily closed) subset of the zero set of a section. We also recall how one can enumerate the zeros of an affine map between vector bundles. These concepts are closely intertwined. Details can be found in Section 3 of [Z2], where these concepts are presented in a greater generality.

Throughout this paper, all vector bundles are assumed to be complex and normed. If $F\longrightarrow\mathcal{M}$ is a smooth vector bundle, closed subset Y of F is *small* if it contains no fiber of F and is preserved under scalar multiplication. If $\mathcal{Z}$ is a compact oriented zero-dimensional manifold, we denote the signed cardinality of $\mathcal{Z}$ by $^{\pm}|\mathcal{Z}|$. If k is an integer, we write $[k]$ for the set of positive integers not exceeding k.

Definition 2.1 *Suppose $F,\mathcal{O}\longrightarrow\mathcal{M}$ are smooth vector bundles, Ω is an open subset of F, and $\phi\colon\Omega\longrightarrow\mathcal{O}$ is a smooth bundle map.*
(1) Bundle map $\alpha\colon F\longrightarrow\mathcal{O}$ is a dominant term of ϕ if there exists $\varepsilon\in C^0(F;\mathbb{R})$ such that

$$\big|\phi(\upsilon)-\alpha(\upsilon)\big|\le\varepsilon(\upsilon)|\alpha(\upsilon)|\quad\forall\upsilon\in\Omega\qquad\textit{and}\qquad\lim_{\upsilon\longrightarrow0}\varepsilon(\upsilon)=0.$$

(2) The dominant term $\alpha\colon F\longrightarrow\mathcal{O}$ of ϕ is the resolvent of ϕ if $\alpha\colon F\longrightarrow\mathcal{O}$ is linear map which is injective on every fiber of F.
(3) The bundle map $\phi\colon\Omega\longrightarrow\mathcal{O}$ is hollow if there exist a vector bundle $\tilde{F}\longrightarrow\mathcal{M}$ of rank less than the rank of F, a smooth bundle map $\rho\colon F\longrightarrow\tilde{F}$, and a linear map $\alpha\colon\tilde{F}\longrightarrow\mathcal{O}$, which is injective on every fiber, such that $\alpha\circ\rho$ is a dominant term of ϕ.

If $F\longrightarrow\mathcal{M}$ is a vector bundle, we denote by $\gamma_F\longrightarrow\mathbb{P}F$ the tautological line bundle and by $\pi_{\mathbb{P}F}\colon\mathbb{P}F\longrightarrow\mathcal{M}$ the bundle projection map. If α is a section of the bundle $\mathrm{Hom}(F,\mathcal{O})$, let $\tilde{\alpha}$ be the section of $\mathrm{Hom}(\gamma_F,\pi_{\mathbb{P}F}^*\mathcal{O})$ induced by α.

The base spaces we work with in the next two sections are closely related to spaces of rational maps into $\mathbb{P}^n$ of total degree d that pass through the N constraints $\mu_1,\ldots,\mu_N$. From the algebraic geometry point of view, spaces of rational maps are algebraic stacks, but with a fairly obscure local structure. We view these spaces as *mostly smooth*, or *ms-*, manifolds: compact oriented topological manifolds stratified by smooth manifolds, such that the boundary strata have (real) codimension at least two. Subsection 2.3 gives explicit descriptions of neighborhoods of boundary strata and of the behavior of certain bundle sections near such strata. We call the main stratum $\mathcal{M}$ of ms-manifold $\bar{\mathcal{M}}$ the *smooth base* of $\bar{\mathcal{M}}$. Definition 3.7 in [Z2] also introduces the natural notions of *ms-maps* between ms-manifolds, *ms-bundles* over ms-manifolds, and *ms-sections* of ms-bundles.

Definition 2.2 *Let $\bar{\mathcal{M}}=\mathcal{M}_n\sqcup\bigsqcup_{i=0}^{n-2}\mathcal{M}_i=\mathcal{M}\sqcup\bigsqcup_{i=0}^{n-2}\mathcal{M}_i$ be an ms-manifold of dimension n.*
(1) If $\mathcal{Z}\subset\mathcal{M}_i$ is a smooth oriented submanifold, a normal-bundle model for $\mathcal{Z}$ is a tuple (F,Y,ϑ), where
(1a) $F\longrightarrow\mathcal{Z}$ is a smooth vector bundle and Y is a small subset of F;
(1b) for some $\delta\in C^\infty(\mathcal{Z};\mathbb{R}^+)$, $\vartheta\colon F_\delta-(Y-\mathcal{Z})\longrightarrow\bar{\mathcal{M}}$ is a continuous map such that

(1b-i) $\vartheta\colon F_\delta-(Y-\mathcal{Z})\longrightarrow\bar{\mathcal{M}}$ *is a homeomorphism onto an open neighborhood of* $\mathcal{Z}$ *in* $\mathcal{M}\cup\mathcal{Z}$*;*
(1b-ii) $\vartheta|_{\mathcal{Z}}$ *is the identity map, and* $\vartheta\colon F_\delta–Y–\mathcal{Z}\longrightarrow\mathcal{M}$ *is an orientation-preserving diffeomorphism on an open subset of* $\mathcal{M}$*.*
(2) A *closure* *of normal-bundle model* (F,Y,ϑ) *for* $\mathcal{Z}$ *is a tuple* $(\bar{\mathcal{Z}},F',\pi)$*, where*
(2a) $\bar{\mathcal{Z}}$ *is an ms-manifold with smooth base* $\mathcal{Z}$*;*
(2b) $\pi\colon\bar{\mathcal{Z}}\longrightarrow\bar{\mathcal{M}}$ *is an ms-map such that* $\pi|_{\mathcal{Z}}$ *is the identity map;*
(2c) $F'\longrightarrow\bar{\mathcal{Z}}$ *is an ms-bundle such that* $F'|_{\mathcal{Z}}=F$*.*

We use a normal-bundle model for $\mathcal{Z}$ to describe the behavior of bundle sections over $\bar{\mathcal{M}}$ near $\mathcal{Z}$. Each section we encounter in this paper exhibits one of the two kinds of behavior described by Definition 2.3.

Definition 2.3 *Suppose* $\bar{\mathcal{M}}$ *is an ms-manifold,* $V\longrightarrow\bar{\mathcal{M}}$ *is an ms-bundle,* $s\in\Gamma(\bar{\mathcal{M}};V)$*, and* $\mathcal{Z}\subset s^{-1}(0)$*.*
(1) $\mathcal{Z}$ *is* *s-hollow* *if there exist a normal-bundle model* (F,Y,ϑ) *for* $\mathcal{Z}$ *and a bundle isomorphism* $\vartheta_V\colon\vartheta^*V\longrightarrow\pi_F^*V$*, covering the identity on* $F_\delta-(Y-\mathcal{Z})$*, such that*
(1a) $\vartheta_V|_{F_\delta-Y-\mathcal{Z}}$ *is smooth and* $\vartheta_V|_{\mathcal{Z}}$ *is the identity;*
(1b) the map $\phi\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-\mathcal{Z})\longrightarrow V$ *is hollow.*
(2) $\mathcal{Z}$ *is* *s-regular* *if there exist a normal-bundle model* (F,Y,ϑ) *for* $\mathcal{Z}$ *with closure* $(\bar{\mathcal{Z}},F',\pi)$*, section* $\alpha\in\Gamma(\bar{\mathcal{Z}},Hom(F',\pi^*V))$*, and a bundle isomorphism* $\vartheta_V\colon\vartheta^*V\longrightarrow\pi_F^*V$ *covering the identity on* $F_\delta-(Y-\mathcal{Z})$*, such that*
(2a) $\vartheta_V|_{F_\delta-Y-\mathcal{Z}}$ *is smooth and* $\vartheta_V|_{\mathcal{Z}}$ *is the identity;*
(2b) $\alpha|_{\mathcal{Z}}$ *is nondegenerate and is the resolvent for* $\phi\equiv\vartheta_V\circ\vartheta^*s\colon F_\delta-(Y-\mathcal{Z})\longrightarrow V$*;*
(2c) the space $\mathbb{P}F'$ *admits a decomposition into subspaces* $\{\mathcal{Z}_i\}$ *such that each spaces* $\mathcal{Z}_i$ *is either* $\tilde{\alpha}$*-hollow or satisfies (2a) and (2b) with* s *replaced by* $\tilde{\alpha}$*.*

If $\bar{\mathcal{M}}$ is a smooth manifold and $\mathcal{Z}$ is a smooth compact submanifold of $\bar{\mathcal{M}}$ such that s vanishes along $\mathcal{Z}$, but the derivative of s in the normal direction along $\mathcal{Z}$ is nondegenerate, $\mathcal{Z}$ is s-regular. The full-rank linear map α is the derivative of the section s in the normal direction along $\mathcal{Z}$. However, if the derivative of s in the normal direction does not have full rank, $\mathcal{Z}$ may not be s-hollow. For example, if s is the section of the trivial line bundle over $\mathbb{C}$ given by $s(z)=z^2$, the submanifold $\{0\}$ is not s-hollow. In fact, $\{0\}$ is s-regular in the sense of Section 3 in [Z2]. On the other hand, if s is the section of the trivial rank-two bundle over $\mathbb{C}\times\mathbb{C}$ given by

$$\mathbb{C}\times\mathbb{C}\longrightarrow\mathbb{C}\times\mathbb{C},\qquad s(z,w)=(zw,zw^2),$$

$\{0\}$ is s-hollow, while the submanifold $\{0\}\times\mathbb{C}^*$ is s-regular. In contrast, the submanifold $\{0\}\times\mathbb{C}$ is not s-regular.

We call $s\in\Gamma(\bar{\mathcal{M}};V)$ a *regular section* if $\bar{\mathcal{M}}$ can be composed into s-hollow and s-regular subspaces. We call

$$\alpha\in\Gamma\big(\bar{\mathcal{Z}};\mathrm{Hom}(F',\mathcal{O})\big)$$

a *regular linear map* if α satisfies the requirements of (2c) of Definitions 2.3.

If $\alpha\in\Gamma(\bar{\mathcal{M}};\mathrm{Hom}(E,\mathcal{O}))$ is a linear map and $\mathrm{rk}\,E+\frac{1}{2}\dim\bar{\mathcal{M}}=\mathrm{rk}\,\mathcal{O}$, the zero set of the affine map

$$\psi_{\alpha,\bar{\nu}}\colon E\longrightarrow\mathcal{O},\qquad \psi_{\alpha,\bar{\nu}}(\upsilon)=\bar{\nu}_\upsilon+\alpha(\upsilon),$$

is a zero-dimensional oriented submanifold of $E|\mathcal{M}$, if $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ is a generic section; see Lemma 3.10 in [Z2]. If α is a regular linear map, $\psi_{\alpha,\bar{\nu}}^{-1}(0)$ is a finite set for a generic choice of $\bar{\nu}$, and the number

$$N(\alpha)\equiv^{\pm}\big|\psi_{\alpha,\bar{\nu}}^{-1}(0)\big|$$

is independent of such a choice of $\bar{\nu}$.

We are now ready to state the first part of the computational method of this paper, Proposition 2.4. The second part is Lemma 2.5.

Proposition 2.4 *Let $V\longrightarrow\bar{\mathcal{M}}$ be an ms-bundle of rank n over an ms-manifold of dimension $2n$. Suppose $\mathcal{U}$ is an open subset of $\mathcal{M}$ and $s\in\Gamma(\bar{\mathcal{M}};V)$ is such that $s|_{\mathcal{U}}$ is transversal to the zero set.*
(1) If $s^{-1}(0)\cap\mathcal{U}$ is a finite set, ${}^{\pm}|s^{-1}(0)\cap\mathcal{U}|=\langle e(V),[\bar{\mathcal{M}}]\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)$.
(2) If $\bar{\mathcal{M}}-\mathcal{U}=\bigsqcup_{i=1}^{i=k}\mathcal{Z}_i$, where each $\mathcal{Z}_i$ is s-regular or s-hollow, then $s^{-1}(0)\cap\mathcal{U}$ is finite, and

$${}^{\pm}\big|s^{-1}(0)\cap\mathcal{U}\big|=\big\langle e(V),[\bar{\mathcal{M}}]\big\rangle-\mathcal{C}_{\bar{\mathcal{M}}-\mathcal{U}}(s)=\big\langle e(V),[\bar{\mathcal{M}}]\big\rangle-\sum_{i=1}^{i=k}\mathcal{C}_{\mathcal{Z}_i}(s).$$

If $\mathcal{Z}_i$ is s-hollow, $\mathcal{C}_{\mathcal{Z}_i}(s)=0$. If $\mathcal{Z}_i$ is s-regular and $\alpha_i\colon F_i'\longrightarrow\pi_i^*V$ is the corresponding linear map,

$$\mathcal{C}_{\mathcal{Z}_i}(s)=N(\alpha_i).$$

Finally, if $\alpha_i\in\Gamma(\bar{\mathcal{Z}}_i;Hom(F_i',\pi_i^*V)$ has full-rank rank over all of $\bar{\mathcal{Z}}_i$,

$$\mathcal{C}_{\mathcal{Z}_i}(s)=\big\langle\pi_i^*c(V)\cdot c(F_i')^{-1},[\bar{\mathcal{Z}}_i]\big\rangle.$$

This proposition is a special case of Corollary 3.13 in [Z2]. Proposition 2.4 reduces the problem of computing $\mathcal{C}_{\mathcal{Z}_i}(s)$ for an s-regular manifold $\mathcal{Z}_i$ to counting the zeros of an affine map between two vector bundles. The general setting for the latter problem is the following. Suppose $E,\mathcal{O}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles, such that $\mathrm{rk}\,E+\frac{1}{2}\dim\bar{\mathcal{M}}=\mathrm{rk}\,\mathcal{O}$, and $\alpha\colon E\longrightarrow\mathcal{O}$ is a regular linear map. Let $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ be such that the map

$$\psi_{\alpha,\bar{\nu}}\equiv\bar{\nu}+\alpha\colon E\longrightarrow\mathcal{O}$$

is transversal to the zero set in $\mathcal{O}$ on $E|\mathcal{M}$, and all its zeros are contained in $E|\mathcal{M}$. Then $N(\alpha)\equiv^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ depends only on α. If the rank of E is zero, then clearly

$$N(\alpha)=^{\pm}\big|\psi_{\alpha,\bar{\nu}}^{-1}(0)\big|=\big\langle e(\mathcal{O}),[\bar{\mathcal{M}}]\big\rangle.$$

If the rank of E is positive and $\bar{\nu}$ is generic, the section $\bar{\nu}$ does not vanish and thus determines a trivial line subbundle $\mathbb{C}\bar{\nu}$ of $\mathcal{O}$. Let $\mathcal{O}^{\perp}=\mathcal{O}/\mathbb{C}\bar{\nu}$ and denote by $\alpha^{\perp}$ the composition of α with the quotient projection map. If E is a line bundle and α is a linear map,

$$N(\alpha)=^{\pm}\big|\psi_{\alpha,\bar{\nu}}^{-1}(0)\big|=\big\langle e(E^*\otimes\mathcal{O}^{\perp}),[\bar{\mathcal{M}}]\big\rangle-\mathcal{C}_{\alpha^{-1}(0)}(\alpha^{\perp}).$$

By Proposition 2.4, computation of $\mathcal{C}_{\alpha^{-1}(0)}(\alpha^{\perp})$ again involves counting the zeros of affine maps, but with the rank of the new target bundle, i.e. $E^*\otimes\mathcal{O}^{\perp}$, one less than the rank of the original one, i.e. $\mathcal{O}$. On the other hand, if the rank of E is bigger than one, $N(\alpha)=N(\tilde{\alpha})$; see Subsection 3.3 in [Z2]. Thus, at least in reasonably good cases, the number $N(\alpha)$ can be determined in finitely many steps.

The next lemma summarizes the results of Subsection 3.3 in [Z2]. Let $\lambda_E=c_1(\gamma_E^*)\in H^2(\mathbb{P}E)$.

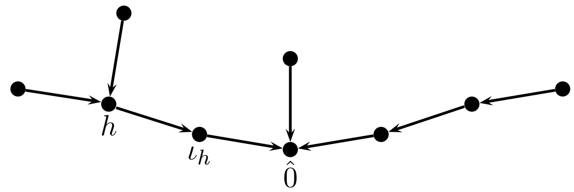


Figure 1: A Rooted Tree

Lemma 2.5 *Suppose $\bar{\mathcal{M}}$ is an ms-manifold and $E,\mathcal{O}\longrightarrow\bar{\mathcal{M}}$ are ms-bundles such that*

$$rk\,E+\frac{1}{2}\dim\bar{\mathcal{M}}=rk\,\mathcal{O}.$$

If $\alpha\in\Gamma(\bar{\mathcal{M}};Hom(E,\mathcal{O}))$ and $\bar{\nu}\in\Gamma(\bar{\mathcal{M}};\mathcal{O})$ are such that α is regular, $\bar{\nu}$ has no zeros, the map

$$\psi_{\alpha,\bar{\nu}}\equiv\bar{\nu}+\alpha\colon E\longrightarrow\mathcal{O}$$

is transversal to the zero set on $E|\mathcal{M}$, and all its zeros are contained in $E|\mathcal{M}$, then $\psi_{\alpha,\bar{\nu}}^{-1}(0)$ is a finite set, ${}^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|$ depends only on α, and

$$N(\alpha)\equiv^{\pm}|\psi_{\alpha,\bar{\nu}}^{-1}(0)|=\big\langle c(\mathcal{O})c(E)^{-1},[\bar{\mathcal{M}})]\big\rangle-\mathcal{C}_{\tilde{\alpha}^{-1}(0)}(\tilde{\alpha}^{\perp}).$$

Furthermore, if $n=rk\,E$,

$$\lambda_E^n+\sum_{k=1}^{k=n}c_k(E)\lambda_E^{n-k}=0\in H^{2n}(\mathbb{P}E)\quad\textit{and}\quad\big\langle\mu\lambda_E^{n-1},[\mathbb{P}E]\big\rangle=\big\langle\mu,[\bar{\mathcal{M}}]\big\rangle\quad\forall\mu\in H^{2m-2n}(\bar{\mathcal{M}}).\quad(2.1)$$

2.2 Notation

In this subsection, we describe the most important notation used in this paper. Some of the notation is only sketched; see Section 2 in [Z3] for more details.

Let $q_N:\mathbb{C}\longrightarrow S^2\subset\mathbb{R}^3$ be the stereographic projection mapping the origin in $\mathbb{C}$ to the north pole. We identify $\mathbb{C}$ with $S^2-\{\infty\}$ via the map q_N, where

$$\infty=(0,0,-1)\in S^2\subset\mathbb{R}^3.$$

Let $e_\infty=(1,0,0)\in T_\infty S^2$.

Definition 2.6 *A finite partially ordered set I is a* <u>linearly ordered set</u> *if for all $i_1,i_2,h\in I$ such that $i_1,i_2<h$, either $i_1\le i_2$ or $i_2\le i_1$.*
A linearly ordered set I is a <u>rooted tree</u> *if I has a unique minimal element, i.e. there exists $\hat{0}\in I$ such that $\hat{0}\le i$ for all $i\in I$.*

In Figure 1, the dots denote the elements of a rooted tree I and the arrows describe the partial ordering. If I is a linearly ordered set, let $\hat{I}$ be the subset of the non-minimal elements of I. For every $h\in\hat{I}$, denote by $\iota_h\in I$ the largest element of I which is smaller than h; see Figure 1. Suppose $I=\bigsqcup_{k\in K}I_k$ is the splitting of I into rooted trees such that k is the minimal element of I_k. If $\hat{1}\notin I$,

Figure 2: Linearly Ordered Sets I and $I+_{k_1}\hat{1}$

we define the linearly ordered set $I+_k\hat{1}$ to be the set $I+\hat{1}$ with all partial-order relations of I along with the relations

$$k<\hat{1} \qquad \text{and} \qquad \hat{1}<h \quad \text{if} \quad h\in\hat{I}_k;$$

see Figure 2.

If S is a (possibly singular) complex curve and M is a finite set, a $\mathbb{P}^n$*-valued bubble map with* M*-marked points* is a tuple

$$b=\big(S,M,I;x,(j,y),u\big),$$

where I is a linearly ordered set, and

$$x\colon \hat{I}\longrightarrow S\cup S^2, \quad j\colon M\longrightarrow I, \quad y\colon M\longrightarrow S\cup S^2, \quad \text{and} \quad u\colon I\longrightarrow C^\infty(S;\mathbb{P}^n)\cup C^\infty(S^2;\mathbb{P}^n)$$

are maps such that

$$x_h\in\begin{cases} S^2-\{\infty\}, & \text{if } \iota_h\in\hat{I};\\ S, & \text{if } \iota_h\notin\hat{I};\end{cases} \qquad y_l\in\begin{cases} S^2-\{\infty\}, & \text{if } j_l\in\hat{I};\\ S, & \text{if } j_l\notin\hat{I};\end{cases} \qquad u_i\in\begin{cases} C^\infty(S^2;\mathbb{P}^n), & \text{if } i\in\hat{I};\\ C^\infty(S;\mathbb{P}^n), & \text{if } i\notin\hat{I};\end{cases}$$

and $u_h(\infty)=u_{\iota_h}(x_h)$ for all $h\in\hat{I}$. We associate such a tuple with Riemann surface

$$\Sigma_b=\Big(\bigsqcup_{i\in I}\Sigma_{b,i}\Big)\Big/\!\sim, \quad \text{where} \qquad \Sigma_{b,i}=\begin{cases}\{i\}\times S^2, & \text{if } i\in\hat{I};\\ \{i\}\times S, & \text{if } i\notin\hat{I},\end{cases} \quad \text{and} \quad (h,\infty)\sim(\iota_h,x_h) \;\; \forall h\in\hat{I},$$

with marked points $(j_l,y_l)\in\Sigma_{b,j_l}$, and continuous map $u_b:\Sigma_b\longrightarrow\mathbb{P}^n$, given by $u_b|\Sigma_{b,i}=u_i$ for all $i\in I$. We require that all the singular points of Σ_b and all the marked points be distinct. Furthermore, if $S=S^2$, all these points are to be different from each of the special marked points $(i,\infty)\in\Sigma_{b,i}$, where i is a minimal element of I, i.e. one of the elements of the set $I-\hat{I}$. In addition, if $\Sigma_{b,i}=S^2$ and $u_{i*}[S^2]=0\in H_2(\mathbb{P}^n;\mathbb{Z})$, then $\Sigma_{b,i}$ must contain at least two singular and/or marked points of Σ_b other than (i,∞). If $S\neq S^2$, but S is unstable, u_i must satisfy a similar stability condition whenever $\Sigma_{b,i}=S$. In particular, if S is a torus or a circle of spheres and the restriction of u_i to a component S_h of S is homologically zero, S_h contains at least one marked point of Σ_b. Two bubble maps b and b' are *equivalent* if there exists a homeomorphism $\phi\colon\Sigma_b\longrightarrow\Sigma_{b'}$ such that $u_b=u_{b'}\circ\phi$, $\phi(j_l,y_l)=(j_l',y_l')$ for all $l\in M$, $\phi|_{\Sigma_{b,i}}$ is holomorphic for all $i\in I$, and $\phi(\Sigma_{b,i})\subset\Sigma_{b,i'}$ for some $i'\in I'-\hat{I}'$ if $i'\in I-\hat{I}$.

The general structure of bubble maps is described by tuples $\mathcal{T}=(S,M,I;j,\underline{d})$, with $d_i\in\mathbb{Z}$ specifying the degree of the map u_b on $\Sigma_{b,i}$. We call such tuples *bubble types*. Bubble type $\mathcal{T}$ is *simple* if I is a rooted tree; $\mathcal{T}$ is *basic* if $\hat{I}=\emptyset$ and $d_i\neq 0$ for all $i\in I$; $\mathcal{T}$ is *semiprimitive* if $\iota_h\notin\hat{I}$, $d_{\iota_h}=0$, and $d_h\neq 0$

for all $h\in\hat{I}$. The above equivalence relation on the set of bubble maps induces an equivalence relation on the set of bubble types. For each $h,i\in I$, let

$$D_i\mathcal{T}=\{h\in\hat{I}\colon i<h\},\quad \bar{D}_i\mathcal{T}=D_i\mathcal{T}\cup\{i\},\quad H_i\mathcal{T}=\{h\in\hat{I}\colon \iota_h=i\},\quad M_i\mathcal{T}=\{l\in M\colon j_l=i\},$$

$$\chi_{\mathcal{T}}h=\begin{cases}0, & \text{if } \forall i\in I \text{ s.t. } h\in\bar{D}_i\mathcal{T},\ d_i=0;\\ 1, & \text{if } \forall i\in I \text{ s.t. } h\in D_i\mathcal{T},\ d_i=0,\ \text{but } d_h\neq 0;\\ 2, & \text{otherwise;}\end{cases}\qquad \chi(\mathcal{T})=\big\{h\in I\colon \chi_{\mathcal{T}}h=1\big\}.$$

Denote by $\mathcal{H}_{\mathcal{T}}$ the space of all holomorphic bubble maps with structure $\mathcal{T}$.

The automorphism group of every bubble type $\mathcal{T}$ we encounter in the next two sections is trivial. Thus, every bubble type discussed below is presumed to be automorphism-free.

If S is a circle of spheres, we denote by $\mathcal{M}_{\mathcal{T}}$ the set of equivalence classes of bubble maps in $\mathcal{H}_{\mathcal{T}}$. For each bubble type $\mathcal{T}=(S^2,M,I;j,\underline{d})$, let

$$\mathcal{U}_{\mathcal{T}}=\big\{[b]\colon b=\big(S^2,M,I;x,(j,y),u\big)\in\mathcal{H}_{\mathcal{T}},\ u_{i_1}(\infty)=u_{i_2}(\infty)\ \forall i_1,i_2\in I-\hat{I}\big\}.$$

Then there exists $\mathcal{B}_{\mathcal{T}}\subset\mathcal{H}_{\mathcal{T}}$ such that $\mathcal{U}_{\mathcal{T}}$ is the quotient of a subset $\mathcal{B}_{\mathcal{T}}$ of $\mathcal{H}_{\mathcal{T}}$ by a $\tilde{G}_{\mathcal{T}}\equiv(S^1)^I$-action. Denote by $\mathcal{U}_{\mathcal{T}}^{(0)}$ the quotient of $\mathcal{B}_{\mathcal{T}}$ by $G_{\mathcal{T}}\equiv(S^1)^{\hat{I}}\subset\tilde{G}_{\mathcal{T}}$. Then $\mathcal{U}_{\mathcal{T}}$ is the quotient of $\mathcal{U}_{\mathcal{T}}^{(0)}$ by the residual $G^*_{\mathcal{T}}\equiv(S^1)^{I-\hat{I}}\subset\tilde{G}_{\mathcal{T}}$ action. Corresponding to these quotients, we obtain line orbi-bundles $\big\{L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}\colon i\in I\big\}$. Let

$$\mathcal{F}\mathcal{T}=\bigoplus_{h\in\hat{I}}\mathcal{F}_h\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}},\qquad\text{where}\qquad \mathcal{F}_h\mathcal{T}=L_h\mathcal{T}\otimes L^*_{\iota_h}\mathcal{T}.$$

Denote by $\mathcal{F}^{\emptyset}\mathcal{T}$ the open subset of $\mathcal{F}\mathcal{T}$ consisting of vectors with all components nonzero.

The Gromov-convergence topology on the space of equivalence classes of bubble maps induces a partial ordering on the set of bubble types and their equivalence classes such that the spaces

$$\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}=\bigcup_{\mathcal{T}'\le\mathcal{T}}\mathcal{U}_{\mathcal{T}'}^{(0)}\qquad\text{and}\qquad \bar{\mathcal{U}}_{\mathcal{T}}=\bigcup_{\mathcal{T}'\le\mathcal{T}}\mathcal{U}_{\mathcal{T}'}$$

are compact and Hausdorff. The $G^*_{\mathcal{T}}$-action on $\mathcal{U}_{\mathcal{T}}^{(0)}$ extends to an action on $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$, and thus the line orbi-bundles $L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}$ with $i\in I-\hat{I}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}$. These bundles can be identified with the universal tangent line bundles for appropriate sections of the universal bundle over $\bar{\mathcal{U}}_{\mathcal{T}}$. The evaluation maps

$$\mathrm{ev}_l\colon\mathcal{H}_{\mathcal{T}}\longrightarrow\mathbb{P}^n,\quad \mathrm{ev}_l\big((S,M,I;x,(j,y),u)\big)=u_{j_l}(y_l),$$

descend to all the quotients and induce continuous maps on $\bar{\mathcal{U}}_{\mathcal{T}}$ and $\bar{\mathcal{U}}_{\mathcal{T}}^{(0)}$. If $\mu=\mu_M$ is an M-tuple of subvarieties of $\mathbb{P}^n$, let

$$\mathcal{M}_{\mathcal{T}}(\mu)=\big\{b\in\mathcal{M}_{\mathcal{T}}\colon \mathrm{ev}_l(b)\in\mu_l\ \forall l\in M\big\}$$

and define spaces $\mathcal{U}_{\mathcal{T}}(\mu),\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$, etc. in a similar way. If $S=S^2$, we define another evaluation map,

$$\mathrm{ev}\colon\mathcal{B}_{\mathcal{T}}\longrightarrow\mathbb{P}^n\quad\text{by}\quad \mathrm{ev}\big((S^2,M,I;x,(j,y),u)\big)=u_{\hat{0}}(\infty),$$

Figure 3: The Domains of Elements of $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\mathcal{T}(M_0)}$

where $\hat{0}$ is any minimal element of I. This map descends to $\mathcal{U}_{\mathcal{T}}^{(0)}$ and $\mathcal{U}_{\mathcal{T}}$. If $\mu=\mu_{\tilde{M}}$ is an $\tilde{M}$-tuple of constraints, let

$$\mathcal{U}_{\mathcal{T}}(\mu)=\big\{b\in\mathcal{U}_{\mathcal{T}}\colon \mathrm{ev}_l(b)\in\mu_l\ \forall l\in M\cap\tilde{M},\ \mathrm{ev}(b)\in\mu_l\ \forall l\in M-\tilde{M}\big\}$$

and define $\mathcal{U}_{\mathcal{T}}^{(0)}(\mu)$, etc. similarly.

Suppose $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a bubble type, $k\in I-\hat{I}$, and M_0 is nonempty subset of $M_k\mathcal{T}$. Let

$$\mathcal{T}/M_0=\big(S^2,I,M-M_0;j|(M-M_0),\underline{d}\big).$$

Define $\mathcal{T}(M_0)\equiv(S^2,M,I+_k\hat{1};j',\underline{d}')$ by

$$j'_l=\begin{cases}k, & \text{if } l\in M_0;\\ \hat{1}, & \text{if } l\in M_k\mathcal{T}-M_0;\\ j_l, & \text{otherwise;}\end{cases}\qquad d'_i=\begin{cases}0, & \text{if } i=k;\\ d_k, & \text{if } i=\hat{1};\\ d_i, & \text{otherwise.}\end{cases}$$

The tuples $\mathcal{T}/M_0$ and $\mathcal{T}(M_0)$ are bubble types as long as $d_k\neq0$ or $M_0\neq M_k\mathcal{T}$. In Figure 3, we show the domain of an element of the space $\mathcal{U}_{\mathcal{T}}$, where $I=\{k\}$ is a single-element set, and the domain of an element of the space $\mathcal{U}_{\mathcal{T}(M_0)}$, where $M_0=\{l_1,l_2\}$ is a two-element set. In this and later figures, we denote each component of the domain by a disk and shade the component(s) on which the map into $\mathbb{P}^n$ is nonconstant. We indicate marked points on the ghost components, i.e. the components on which the map is constant, by putting small dots on the boundary of the corresponding disk. The point labeled by k, i.e. the same way as the component, is the special marked point (k,∞). Proposition 2.7 and Lemma 2.8, as well as the decomposition (2.4), show that it is crucial to clearly distinguish between ghost and non-ghost components.

Note that

$$\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)=\overline{\mathfrak{M}}_{\{\hat{1}\}\sqcup M_0}\times\bar{\mathcal{U}}_{\mathcal{T}/M_0}(\mu), \tag{2.2}$$

where $\overline{\mathfrak{M}}_{\{\hat{1}\}\sqcup M_0}$ denotes the Deligne-Mumford moduli space of rational curves with $(\{\hat{0},\hat{1}\}\sqcup M_0)$-marked points. If $\mathcal{T}$ is a basic bubble type, let

$$c_1(\mathcal{L}_k^*\mathcal{T})\equiv c_1(L_k^*\mathcal{T})-\sum_{\emptyset\neq M_0\subset M_k\mathcal{T}}PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\big[\bar{\mathcal{U}}_{\mathcal{T}(M_0)}(\mu)\big]\in H^2\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big). \tag{2.3}$$

This cohomology class is well-defined; see Subsection 5.2 in [Z2]. Whenever the bubble type $\mathcal{T}$ is clear from context, we will write $c_1(L_k^*)$ and $c_1(\mathcal{L}_k^*)$ for $c_1(L_k^*\mathcal{T})$ and $c_1(\mathcal{L}_k^*\mathcal{T})$, respectively. We illustrate definition (2.3) in Figure 4 in the case $I=\{k\}$ is a single-element set. In this figure, as well in the future ones, we denote spaces of tuples of stable maps by drawing a picture of the

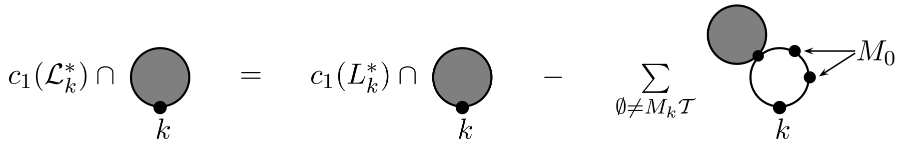


Figure 4: An Example of Definition (2.3)

domain of a typical element of such a space.

We are now ready to explain the claim of Theorem 1.1. Let n, d, N, and μ be as in the statement of the theorem. If $k\geq 1$ and $m\geq 1$, denote by $\bar{\mathcal{V}}_{k,m}(\mu)$ the disjoint union of the spaces $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ taken over equivalence classes of basic bubble types $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})$ with $|M_0|=m$, $|I|=k$, $d_i>0$, and $\sum d_i=d$. Let $\bar{\mathcal{V}}_k(\mu)=\bar{\mathcal{V}}_{k,0}(\mu)$. We define the spaces $\mathcal{V}_{k,m}(\mu)$ similarly. Let

$$\big\{c_1(\mathcal{L}_i^*)\colon i\in[k]\big\},\big\{c_1(L_i^*)\colon i\in[k]\big\}\subset H^2\big(\bar{\mathcal{V}}_{k,m}(\mu);\mathbb{Z}\big)$$

be given by

$$\big\{c_1(\mathcal{L}_i^*)\big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\colon i\in[k]\big\}=\big\{c_1(\mathcal{L}_i^*\mathcal{T})\colon i\in I\big\},\quad \big\{c_1(L_i^*)\big|\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\colon i\in[k]\big\}=\big\{c_1(L_i^*\mathcal{T}):i\in I\big\},$$

where $\mathcal{T}$ is as above and $[k]=\{1,\ldots,k\}$. We denote by $\eta_l,\tilde{\eta}_l\in H^{2l}\big(\bar{\mathcal{V}}_{k,m}(\mu);\mathbb{Z}\big)$ the sum of all degree-l monomials in $\big\{c_1(\mathcal{L}_i^*)\colon i\in[k]\big\}$ and in $\big\{c_1(L_i^*)\colon i\in[k]\big\}$, respectively. For example,

$$\eta_3=c_1^3(\mathcal{L}_1^*)+c_1^2(\mathcal{L}_1^*)c_1(\mathcal{L}_2^*)+c_1(\mathcal{L}_1^*)c_1^2(\mathcal{L}_2^*)+c_1^3(\mathcal{L}_2^*)\in H^6\big(\bar{\mathcal{V}}_{2,m}(\mu);\mathbb{Z}\big).$$

Finally, let $a=\mathrm{ev}^*c_1(\gamma_{\mathbb{P}^n}^*)\in H^2\big(\bar{\mathcal{V}}_{k,m}(\mu);\mathbb{Z}\big)$, where $\gamma_{\mathbb{P}^n}\longrightarrow\mathbb{P}^n$ denotes the tautological line bundle.

We next describe a generalization of the splitting (2.2) which is used in computations in Section 3. If $\mathcal{T}=(S^2,I,[N]-M_0;j,\underline{d})$ is a bubble type, let

$$\bar{\mathcal{T}}=\big(S^2,\bar{I},[N]-\bar{M}_0;j|([N]-\bar{M}_0),\underline{d}|\bar{I}\big),\quad\text{where}\quad \bar{I}=I-\big\{i\in I-\hat{I}\colon d_i=0\big\},\quad \bar{M}_0=M_0\cup\bigcup_{i\in I-\bar{I}}M_i\mathcal{T}.$$

Note that if $\mathcal{T}$ is semiprimitive, $\bar{\mathcal{T}}$ is basic. Furthermore,

$$\mathcal{U}_{\mathcal{T}}(\mu)=\prod_{i\in I-\bar{I}}\mathfrak{M}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}\times\mathcal{U}_{\bar{\mathcal{T}}}(\mu),\tag{2.4}$$

$$\bar{\mathcal{U}}_{\mathcal{T}}(\mu)=\prod_{i\in I-\bar{I}}\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu),\tag{2.5}$$

where $\mathfrak{M}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}$ denotes the main stratum of $\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}$. If $i\in I-\bar{I}$, by definition, the bundle $L_i\mathcal{T}\longrightarrow\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ is the pullback by the projection map of the bundle

$$L_{\hat{0}}\mathcal{T}_i^{(0)}\longrightarrow\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}=\bar{\mathcal{U}}_{\mathcal{T}_i^{(0)}},\quad\text{where}\quad \mathcal{T}_i^{(0)}=\big(S^2,H_i\mathcal{T}+M_i\mathcal{T},\{\hat{0}\};\hat{0},0\big).$$

We call the latter bundle the *tautological line bundle* over $\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}$. This is the universal tangent line at the marked point $\hat{0}\in\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}$. The decomposition (2.4) for the bubble $\mathcal{T}(M_0)$ of Figure 3

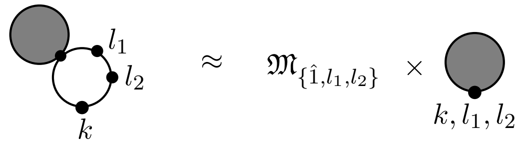


Figure 5: An Example of the Decomposition (2.4)

is illustrated in Figure 5.

Finally, if X is any space, $F \longrightarrow X$ is a normed vector bundle, and $\delta\colon X \longrightarrow \mathbb{R}$ is any function, let

$$F_\delta = \big\{(b,v)\in F\colon |v|_b < \delta(b)\big\}.$$

Similarly, if Ω is a subset of F, let $\Omega_\delta = F_\delta \cap \Omega$. If $\upsilon=(b,v)\in F$, denote by b_υ the image of υ under the bundle projection map, i.e. b in this case.

2.3 A Structural Description

We now describe the structure of the spaces $\bar{\mathcal{V}}_{k,m}(\mu)$ and the behavior of certain bundle sections over $\bar{\mathcal{V}}_{k,m}(\mu)$ near the boundary strata.

If $b=\big(S^2,M,I;x,(j,y),u\big)\in\mathcal{B}_{\mathcal{T}}$ and $k\in I$, let

$$\mathcal{D}_{\mathcal{T},k}b = du_k\big|_\infty e_\infty.$$

If $\tilde{\mathcal{T}}$ is a basic bubble type, the maps $\mathcal{D}_{\mathcal{T},k}$ with $\mathcal{T}\le\tilde{\mathcal{T}}$ and $k\in I-\hat{I}$ induce a continuous section of $\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{U}}^{(0)}_{\tilde{\mathcal{T}}}$ and a continuous section of the bundle $L_k^*\tilde{\mathcal{T}}\otimes\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}$, described by

$$\mathcal{D}_{\tilde{\mathcal{T}},k}[b,c_k] = c_k\mathcal{D}_{\mathcal{T},k}b, \quad \text{if} \quad b\in\mathcal{U}^{(0)}_{\mathcal{T}},\ c_k\in\mathbb{C}.$$

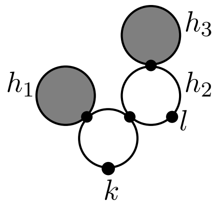
Proposition 2.7 *Suppose $p>2$, $n\ge 2$, $d\ge 1$, $N\ge 1$, $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position, such that*

$$codim_{\mathbb{C}}\mu \equiv \sum_{l=1}^{l=N} codim_{\mathbb{C}}\mu_l - N = d(n+1)-1,$$

and M_0 is a subset of $[N]$. If $\tilde{\mathcal{T}}=(S^2,[N]-M_0,\tilde{I};\tilde{j},\tilde{d})$ is a basic bubble type such that $\sum\tilde{d}_i=d$, the space $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ is an ms-manifold of (real) dimension $2\big(n+1-2|\tilde{I}|-|M_0|\big)$ and $L_k\tilde{\mathcal{T}}$ for $k\in\tilde{I}$ and $ev^*T\mathbb{P}^n$ are ms-bundles over $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$. If $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})<\tilde{\mathcal{T}}$, there exist $\delta,C\in C^\infty\big(\mathcal{U}_{\mathcal{T}}(\mu);\mathbb{R}^+\big)$ and a homeomorphism

$$\gamma^\mu_{\mathcal{T}}\colon \mathcal{F}\mathcal{T}_\delta \longrightarrow \bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu),$$

onto an open neighborhood of $\mathcal{U}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ such that $\gamma^\mu_{\mathcal{T}}|\mathcal{U}_{\mathcal{T}}(\mu)$ is the identity, $\gamma^\mu_{\mathcal{T}}(\mathcal{F}\mathcal{T}_\delta-\mathcal{F}^\emptyset\mathcal{T})$ is contained in $\partial\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$, and $\gamma^\mu_{\mathcal{T}}|\mathcal{F}^\emptyset\mathcal{T}_\delta$ is an orientation-preserving diffeomorphism onto an open



$$\rightsquigarrow \mathcal{D}_{\tilde{\mathcal{T}},k}\gamma^{\mu}_{\mathcal{T}}(\upsilon)=\big\{\mathcal{D}_{\mathcal{T},h_1}+\varepsilon_{\mathcal{T},h_1}(\upsilon)\big\}\upsilon_{h_1}+\big\{\mathcal{D}_{\mathcal{T},h_3}+\varepsilon_{\mathcal{T},h_3}(\upsilon)\big\}\upsilon_{h_2}\otimes\upsilon_{h_3}$$

Figure 6: An Example of the Estimate of Proposition 2.7

subset of $\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$. Furthermore, for all $k\in\tilde{I}$, with appropriate identifications,

$$\Big|\mathcal{D}_{\tilde{\mathcal{T}},k}\gamma^{\mu}_{\mathcal{T}}(\upsilon)-\alpha_{\mathcal{T},k}\big(\rho_{\mathcal{T}}(\upsilon)\big)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big|\quad\forall\upsilon\in\mathcal{FT}_{\delta},\qquad \textit{where}$$

$$\rho_{\mathcal{T}}(\upsilon)=\big((\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)\in\tilde{\mathcal{F}}\mathcal{T}\equiv\bigoplus_{h\in\chi(\mathcal{T})}L_h\mathcal{T}\otimes L^*_{\tilde{\iota}_h}\mathcal{T};\quad \tilde{\upsilon}_h=\bigotimes_{i\in\hat{I},h\in\bar{D}_i\mathcal{T}}\upsilon_i;\quad \tilde{\iota}_h\in I-\hat{I},\ h\in\bar{D}_{\tilde{\iota}_h}\mathcal{T};$$

$$\alpha_{\mathcal{T},k}\big((\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)=\sum_{h\in I_k\cap\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},h}\tilde{\upsilon}_h,$$

and $I_k\subset I$ is the rooted tree containing k.

Figure 6 illustrates the analytic estimate of Proposition 2.7 in a case when $\tilde{I}=\{k\}$ is a single-element set. Note that, while the stratum $\mathcal{U}_{\mathcal{T}}(\mu)$ of Figure 6 has codimension three in $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$, the section $\mathcal{D}_{\tilde{\mathcal{T}},k}$ depends only on two parameters of the normal bundle, υ_{h_1} and $\upsilon_{h_2}\otimes\upsilon_{h_3}$, at least up to negligible terms. Such bubble types $\mathcal{T}$ will always be hollow in the sense of Definition 2.3 and will not effect our computations.

Proposition 2.7 is a special case of Theorem 2.8 in [Z2]; see also the remark following the theorem. The dimension of $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ is obtained as follows:

$$\begin{aligned}\frac{1}{2}\dim\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)=\dim_{\mathbb{C}}\mathcal{U}_{\mathcal{T}^*}(\mu)&=\sum_{i\in I^*}\big(\tilde{d}_i(n+1)+n-2\big)-(|\tilde{I}|-1)n-\big(\mathrm{codim}_{\mathbb{C}}\mu+|M_0|\big)\\&=n+1-2|\tilde{I}|-|M_0|.\end{aligned}$$

The analytic estimate on $\mathcal{D}_{\tilde{\mathcal{T}},k}$ is crucial for the implementation of the topological tools of Subsection 2.1 in Subsection 3.1. If $\mathcal{T}$ is semiprimitive, the bundle $\mathcal{FT}=\tilde{\mathcal{F}}\mathcal{T}$ and the section $\alpha_{\mathcal{T}}=\alpha_{\mathcal{T}}\circ\rho_{\mathcal{T}}$ extend over $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ via the decomposition (2.5). In terms of the notions of Subsection 2.1, $(\mathcal{FT},\mathcal{FT}-F^{\emptyset}\mathcal{T},\gamma^{\mu}_{\mathcal{T}})$ is a normal-bundle model for $\mathcal{U}_{\mathcal{T}}(\mu)\subset\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$. This normal-bundle model admits a closure if $\mathcal{T}$ is semiprimitive. Note that $\mathcal{FT}$ is not usually the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ if both spaces are viewed as algebraic stacks; see [P2]. Proposition 2.7 implies only that the restrictions to $\mathcal{U}_{\mathcal{T}}(\mu)$ of $\mathcal{FT}$ and of the normal bundle of $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$ in $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ are isomorphic as topological vector bundles.

For any $k,m\in\mathbb{Z}$, we define bundle $E_{k,m}\longrightarrow\bar{\mathcal{V}}_{k,m}(\mu)$ and homomorphism $\alpha_{k,m}\colon E_{k,m}\longrightarrow\mathrm{ev}^*T\mathbb{P}^n$ over $\bar{\mathcal{V}}_{k,m}(\mu)$ by

$$E_{k,m}|\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)=\bigoplus_{i\in\tilde{I}}L_i\tilde{\mathcal{T}},\qquad \alpha_{k,m}\big((\upsilon_i)_{i\in\tilde{I}}\big)=\sum_{i\in\tilde{I}}\mathcal{D}_{\tilde{\mathcal{T}},i}\upsilon_i,$$

whenever $\tilde{\mathcal{T}}=(S^2,[N]-M_0,\tilde{I};\tilde{j},\tilde{d})$ is a basic bubble type such that $\sum\tilde{d}_i=d$, $|\tilde{I}|=k$, and $|M_0|=m$. The following lemma will be used in Section 3.

Lemma 2.8 *Suppose $n\geq 2$, $d\geq 1$, $N\geq 1$, and $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that $\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-1$. If $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})$ is a bubble type such that $\mathcal{U}_{\mathcal{T}}(\mu)\subset\bar{\mathcal{V}}_{k,m}(\mu)$, the restriction of $\alpha_{k,m}$ to the subbundle*

$$E\mathcal{T}^{\perp}\equiv\bigoplus_{i\in\chi(\mathcal{T})-\hat{I}}L_i\mathcal{T}\subset E_{k,m}$$

is nondegenerate over $\mathcal{U}_{\mathcal{T}}(\mu)$.

Proof: The linear map $\alpha_{k,m}$ has full rank on $E\mathcal{T}^{\perp}$ over $\mathcal{U}_{\mathcal{T}}(\mu)$ if and only if the section

$$\big\{\widetilde{\alpha_{k,m}|E\mathcal{T}^{\perp}}\big\}\in\Gamma\big(\mathbb{P}E\mathcal{T}^{\perp}|\mathcal{U}_{\mathcal{T}}(\mu);\gamma^*_{E\mathcal{T}^{\perp}}\otimes\mathrm{ev}^*T\mathbb{P}^n\big)$$

has no zeros. Note that

$$\dim_{\mathbb{C}}\mathbb{P}E\mathcal{T}^{\perp}|\mathcal{U}_{\mathcal{T}}(\mu)\leq\dim_{\mathbb{C}}\mathcal{V}_k(\mu)+(k-1)=n-k<n.$$

Thus, it is enough to show that $\big\{\widetilde{\alpha_{k,m}|E\mathcal{T}^{\perp}}\big\}$ is transversal to the zero set in $\mathbb{P}E\mathcal{T}^{\perp}|\mathcal{U}_{\mathcal{T}}(\mu)$ if the constraints μ are in general position. This last fact is immediate from Lemma 2.9.

Lemma 2.9 *If $u\colon S^2\longrightarrow\mathbb{P}^n$ is a holomorphic map of positive degree and $e_\infty\in T_\infty S^2$ is a nonzero vector, the linear maps*

$$\begin{aligned} &H^0_{\bar\partial}(S^2;u^*T\mathbb{P}^n)\longrightarrow T_{u(\infty)}\mathbb{P}^n, && \xi\longrightarrow\xi(\infty),\\ &\big\{\xi\in H^0_{\bar\partial}(S^2;u^*T\mathbb{P}^n)\colon\xi(\infty)=0\big\}\longrightarrow T_{u(\infty)}\mathbb{P}^n, && \xi\longrightarrow\nabla_{e_\infty}\xi, \end{aligned}$$

are onto.

This lemma is well-known; see Corollary 6.3 in [Z2] for example.

3 Computations

3.1 Summary and Motivation

In this section, we prove

Proposition 3.1 *Suppose $n\geq 2$, $d\geq 1$, and $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that*

$$\mathrm{codim}_{\mathbb{C}}\mu\equiv\sum_{l=1}^{l=N}\mathrm{codim}_{\mathbb{C}}\mu_l-N=d(n+1)-1.$$

Then the number of degree-d genus-one curves that have a fixed generic complex structure on the normalization and pass through the constraints μ is given by

$$\begin{aligned} n_{1,d}(\mu)&=\frac{1}{2}\big(RT_{1,d}(\mu_1;\mu_2,\ldots,\mu_N)-CR_1(\mu)\big), \qquad \textit{where}\\ CR_1(\mu)&=\sum_{k=1}^{2k\leq n+1}(-1)^{k-1}(k-1)!\sum_{l=0}^{n+1-2k}\binom{n+1}{l}\big\langle a^l\eta_{n+1-2k-l},\big[\bar{\mathcal{V}}_k(\mu)\big]\big\rangle. \end{aligned}$$

Proposition 3.1 follows from Proposition 3.2 and Corollaries 3.6 and 3.10. We use the topological tools of Subsection 2.1 and the analytic estimate of Proposition 2.7 to obtain the first corollary in Subsection 3.2. The derivation of Corollary 3.10 in Subsection 3.3 is essentially combinatorics.

Proposition 3.2 *Suppose $n \geq 2$, $d \geq 1$, and $\mu = (\mu_1, \ldots, \mu_N)$ is an N-tuple of proper subvarieties of $\mathbb{P}^n$ in general position such that $\mathrm{codim}_{\mathbb{C}}\mu = d(n+1)-1$. Then the number of degree-d genus-one curves that have a fixed generic complex structure on the normalization and pass through the constraints μ is given by*

$$n_{1,d}(\mu) = \frac{1}{2}\big(RT_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - CR_1(\mu)\big), \quad \textit{where} \quad CR_1(\mu) = N(\alpha_{1,0}),$$

i.e. $CR_1(\mu)$ is the number of zeros of the affine map

$$\psi_{\alpha_{1,0},\bar{\nu}} \colon E_{1,0} = L_1 \longrightarrow ev^*T\mathbb{P}^n, \qquad \psi_{\alpha_{1,0},\bar{\nu}}(\upsilon) = \bar{\nu}_\upsilon + \alpha_{1,0}(\upsilon),$$

over $\bar{\mathcal{V}}_1(\mu)$ for a generic section $\bar{\nu} \in \Gamma\big(\bar{\mathcal{V}}_1(\mu); ev^*T\mathbb{P}^n\big)$.

Proposition 3.2 is basically the main result of the analytic part of [I]. The exact statement is not made in [I], but it can be deduced from the arguments in [I] by comparing with the methods of [Z2].

The general meaning of Proposition 3.2 is the following. The number $\mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_N)$ can be viewed as the "euler class" of a bundle $\Gamma^{0,1}$ over a closure $\bar{C}^\infty$ of the space C^∞ of smooth maps from a fixed elliptic curve that pass through the constraints $\mu_1, \ldots, \mu_N$; see [LT]. Then,

$$2n_{1,d}(\mu) = \big|\bar{\partial}^{-1}(0) \cap C^\infty\big| = \mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_N) - \sum \mathcal{C}_{\mathcal{M}_{\mathcal{T}}(\mu)}(\bar{\partial}), \tag{3.1}$$

where $\big\{\mathcal{M}_{\mathcal{T}}(\mu)\big\}$ are complex finite-dimensional, usually non-compact, manifolds that stratify $\bar{\partial}^{-1}(0) \cap (\bar{C}^\infty - C^\infty)$. Equation (3.1) is an infinite-dimensional analogue of (2) of Proposition (2.4). In the finite-dimensional case, computation of a contribution to the euler class from an s-regular stratum $\mathcal{Z}$ of the zero set of section s reduces to counting the zeros of a polynomial map between *finite*-rank vector bundles over $\bar{\mathcal{Z}}$, unless $\mathcal{Z}$ is s-hollow. The goal in the infinite-dimensional case under consideration is a reduction to the same problem and involves an adoption of the obstruction-bundle idea of [T]. It turns out that $\mathcal{C}_{\mathcal{M}_{\mathcal{T}}(\mu)}(\bar{\partial}) = 0$ for all but one stratum $\mathcal{M}_{\mathcal{T}}(\mu)$ of $\bar{\partial}^{-1}(0) \cap (\bar{C}^\infty - C^\infty)$. The number $CR_1(\mu)$ described by Proposition 3.2 is the contribution $\mathcal{C}_{\mathcal{M}_{\mathcal{T}}(\mu)}(\bar{\partial})$ from the only stratum $\mathcal{M}_{\mathcal{T}}(\mu)$ of $\bar{\partial}^{-1}(0) \cap (\bar{C}^\infty - C^\infty)$ that does contribute to the "euler class" $\mathrm{RT}_{1,d}(\mu_1; \mu_2, \ldots, \mu_N)$ of $\Gamma^{0,1}$.

As Subsection 2.1 suggests, the computation of $N(\alpha_{1,0})$ may require going through a possibly large tree of steps. We construct this tree in the next subsection. However, as a motivation, in the rest of this subsection, we go through the initial steps of this computation, without introducing any additional combinatorial notation. In fact, there are no more steps to go through if $n = 2$ and all the constraints are points or if $n = 3$ and all the constraints are points or lines.

Since the domain of the linear map $\alpha_{1,0}$ is a line bundle, $\tilde{\alpha}_{1,0} = \alpha_{1,0}$. Thus, by Lemma 2.5,

$$N\big(\alpha_{1,0}\big) = \big\langle c(\mathrm{ev}^*T\mathbb{P}^n)c(E_{1,0})^{-1}, \big[\bar{\mathcal{V}}_{1,0}(\mu)\big]\big\rangle - \mathcal{C}_{\alpha_{1,0}^{-1}(0)}\big(\alpha_{1,0}^{\perp}\big), \tag{3.2}$$

where $\alpha_{1,0}^{\perp}\colon E_{1,0}\longrightarrow \mathrm{ev}^*T\mathbb{P}^n/\mathbb{C}\bar{\nu}_0$ denotes the composition of the linear map $\alpha_{1,0}\colon E_{1,0}\longrightarrow \mathrm{ev}^*T\mathbb{P}^n$ with the quotient projection map π_0. As in Lemma 2.5, $\bar{\nu}_0\in\Gamma(\bar{\mathcal{V}}_{1,0}(\mu);\mathrm{ev}^*T\mathbb{P}^n)$ is a generic non-vanishing section. Such a section exists, since the dimension of $\bar{\mathcal{V}}_{1,0}(\mu)$ is $n-1$. We denote the quotient bundle $\mathrm{ev}^*T\mathbb{P}^n/\mathbb{C}\bar{\nu}_0$ by $\mathcal{O}_1$. Let $\tilde{\mathcal{T}}=(S^2,[N],\{\hat{0}\};\hat{0},d)$. By definition, $\mathcal{V}_{1,0}(\mu)=\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$. Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})\le\tilde{\mathcal{T}}$ is a bubble type, i.e. $\mathcal{U}_{\mathcal{T}}(\mu)$ is one of the spaces of stable maps that stratify $\bar{\mathcal{V}}_{1,0}(\mu)$. If $d_{\hat{0}}\neq 0$, by Lemma 2.8,

$$\alpha_{1,0}^{-1}(0)\cap\mathcal{U}_{\mathcal{T}}(\mu)=\emptyset.$$

On the other hand, if $d_{\hat{0}}=0$, by definition, $\alpha_{1,0}$ vanishes on $\mathcal{U}_{\mathcal{T}}(\mu)$. Thus,

$$\alpha_{1,0}^{-1}(0)=\bigsqcup_{[\mathcal{T}]}\mathcal{U}_{\mathcal{T}}(\mu),\tag{3.3}$$

where the union is taken over all equivalence classes of bubble types

$$\mathcal{T}\equiv(S^2,[N],I;j,\underline{d})<\tilde{\mathcal{T}}$$

such that $d_{\hat{0}}=0$. By Proposition 2.7 and Lemma 2.8, the decomposition (3.3) satisfies the requirements of (2) of Proposition 2.4, if $\bar{\nu}_0$ is generic. Indeed, by Proposition 2.7,

$$\Big|\alpha_{1,0}\gamma_{\mathcal{T}}^{\mu}(\upsilon)-\alpha_{\mathcal{T},\hat{0}}\rho_{\mathcal{T}}(\upsilon)\Big|\le C(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}(\upsilon)\big|\quad\forall\upsilon\in\mathcal{F}\mathcal{T}_\delta,\qquad\text{where}\tag{3.4}$$
$$\rho_{\mathcal{T}}(\upsilon)=\big((\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)\in\tilde{\mathcal{F}}\mathcal{T}\equiv L_{\hat{0}}^*\mathcal{T}\otimes\bigoplus_{h\in\chi(\mathcal{T})}L_h\mathcal{T};\quad\tilde{\upsilon}_h=\bigotimes_{i\in\hat{I},h\le i}\upsilon_i;\quad\alpha_{\mathcal{T},\hat{0}}\big((\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)=\sum_{h\in\chi(\mathcal{T})}\mathcal{D}_{\mathcal{T},h}\tilde{\upsilon}_h.$$

By Lemma 2.8 and the decomposition (2.4), the linear map

$$\alpha_{\mathcal{T},\hat{0}}\colon\tilde{\mathcal{F}}\mathcal{T}\longrightarrow\mathrm{Hom}(E_{1,0},\mathrm{ev}^*T\mathbb{P}^n)$$

is injective on every fiber of $\tilde{\mathcal{F}}\mathcal{T}$. If the section $\bar{\nu}_0$ is generic, the same is true of the linear map

$$\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\colon\tilde{\mathcal{F}}\mathcal{T}\longrightarrow\mathrm{Hom}(E_{1,0},\mathcal{O}_1),\qquad\big\{\{\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\}(\tilde{\upsilon})\big\}(v)=\pi_0\big(\{\alpha_{\mathcal{T},\hat{0}}(\tilde{\upsilon})\}(v)\big),\tag{3.5}$$

as can been seen from a dimension count. Thus, (3.4) implies that there exists $\tilde{C}\in C^\infty(\mathcal{U}_{\mathcal{T}}(\mu);\mathbb{R})$ such that

$$\Big|\alpha_{1,0}^{\perp}\gamma_{\mathcal{T}}^{\mu}(\upsilon)-\{\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\}\rho_{\mathcal{T}}(\upsilon)\Big|\le\tilde{C}(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\{\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\}\rho_{\mathcal{T}}(\upsilon)\big|\quad\forall\upsilon\in\mathcal{F}\mathcal{T}_\delta.\tag{3.6}$$

By definition, the ranks of $\mathcal{F}\mathcal{T}$ and $\tilde{\mathcal{F}}\mathcal{T}$ are $|\hat{I}|$ and $|\chi(\mathcal{T})|$, respectively, while $\chi(\mathcal{T})\subset\hat{I}$. Thus, by Definition 2.1, $\mathcal{U}_{\mathcal{T}}(\mu)$ is $\alpha_{1,0}^{\perp}$-hollow if $\chi(\mathcal{T})\neq\hat{I}$. In such a case, by Proposition 2.4, $\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_{1,0}^{\perp})=0$. On the other hand, if $\chi(\mathcal{T})=\hat{I}$, i.e. $\mathcal{T}$ is a semiprimitive bubble type, $\rho_{\mathcal{T}}$ is the identity map, and thus $\pi_0\circ\alpha_{\mathcal{T},\hat{0}}$ is the resolvent of $\alpha_{1,0}^{\perp}$ near $\mathcal{U}_{\mathcal{T}}(\mu)$. By Proposition 2.7,

$$\mathcal{C}_{\mathcal{U}_{\mathcal{T}}(\mu)}(\alpha_{1,0}^{\perp})=N\big(\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\big),\quad\text{where}\quad\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\in\Gamma\big(\bar{\mathcal{U}}_{\mathcal{T}}(\mu);\mathrm{Hom}(\mathcal{F}\mathcal{T},E_{1,0}^*\otimes\mathcal{O}_1)\big),\tag{3.7}$$

provided $\pi_0\circ\alpha_{\mathcal{T},\hat{0}}$ is a regular linear map. By a slight abuse of notation, we now denote by $\pi_0\circ\alpha_{\mathcal{T},\hat{0}}$ the extension of the linear map over $\mathcal{U}_{\mathcal{T}}(\mu)$ defined in (3.5) to $\bar{\mathcal{U}}_{\mathcal{T}}(\mu)$. The existence of an extension

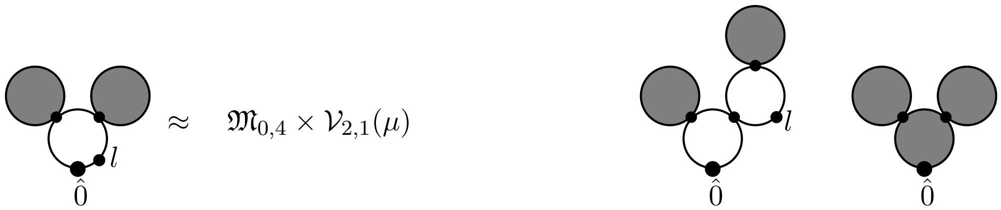


Figure 7: A Boundary Stratum that Contributes to $\mathcal{C}_{\alpha_{1,0}^{-1}(0)}(\alpha_{1,0}^{\perp})$ and Two That Do Not

follows from the decompositions (2.4) and (2.5). With respect to the latter decomposition,

$$\mathcal{F}\mathcal{T} \approx \pi_1^*\gamma_{\hat{0}}^* \otimes \pi_2^* \bigoplus_{h\in\chi(\mathcal{T})} L_h\bar{\mathcal{T}} \longrightarrow \overline{\mathfrak{M}}_{\chi(\mathcal{T})\sqcup M_{\hat{0}}\mathcal{T}} \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu), \qquad E_{1,0}^*\otimes\mathcal{O}_1 \approx \pi_1^*\gamma_{\hat{0}}^* \otimes \pi_2^*\mathcal{O}_1;$$

$$\big\{\{\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\}\big(u_{\hat{0}}\otimes(\upsilon_h)_{h\in\chi(\mathcal{T})}\big)\big\}(v_{\hat{0}}) = u_{\hat{0}}(v_{\hat{0}})\cdot\pi_0\sum_{h\in\chi(\mathcal{T})}\mathcal{D}_{\bar{\mathcal{T}},h}\upsilon_h,$$

where $\gamma_{\hat{0}} \longrightarrow \overline{\mathfrak{M}}_{\chi(\mathcal{T})\sqcup M_{\hat{0}}\mathcal{T}}$ denotes the universal tangent bundle for the marked point $\hat{0}$. Thus, summing (3.7) over all equivalence classes of semiprimitive bubble types $\mathcal{T} < \tilde{\mathcal{T}}$, we obtain

$$\mathcal{C}_{\alpha_{1,0}^{-1}(0)}(\alpha_{1,0}^{\perp}) = \sum_{[\mathcal{T}]} N\big(\pi_0\circ\alpha_{\mathcal{T},\hat{0}}\big) = \sum_{(k,m)>(1,0)} N\big(\alpha_{1;k,m}\big), \qquad \text{where}$$
$$\alpha_{1;k,m} \in \Gamma\big(\overline{\mathfrak{M}}_{0,k+m+1}\times\bar{\mathcal{V}}_{k,m}(\mu); \mathrm{Hom}(\gamma_{\hat{0}}^*\otimes E_{k,m}; \gamma_{\hat{0}}^*\otimes\mathcal{O}_1)\big),$$
$$\big\{\alpha_{1;k,m}(u_{\hat{0}}\otimes\upsilon)\big\}(v_{\hat{0}}) = u_{\hat{0}}(v_{\hat{0}})\cdot\pi_0\alpha_{k,m}(\upsilon).$$

Above $(k,m) > (1,0)$ means that $k \geq 1$, $m \geq 0$, and at least one of the inequalities is strict. In the process of computing the numbers $N(\alpha_{1;k,m})$, we will show that $\pi_0\circ\alpha_{\mathcal{T},\hat{0}}$ is indeed a regular linear map, as needed.

In Figure 7, we give examples of one type of boundary strata $\mathcal{U}_{\mathcal{T}}(\mu)$ that contributes to $\mathcal{C}_{\alpha_{1,0}^{-1}(0)}(\alpha_{1,0}^{\perp})$ and of two that do not. As before, each disk denotes a sphere, and we represent the entire space $\mathcal{U}_{\mathcal{T}}(\mu)$ by drawing the domain of an element of $\mathcal{U}_{\mathcal{T}}(\mu)$. We shade the components of the domain on which every map in $\mathcal{U}_{\mathcal{T}}(\mu)$ is nonconstant and leave blank the components on which every map in $\mathcal{U}_{\mathcal{T}}(\mu)$ is constant. In this figure, we also illustrate the splitting and the summation of over all equivalence classes of semiprimitive bubble types used in the previous paragraph. In short, the strata $\mathcal{U}_{\mathcal{T}}(\mu)$ that contribute to $\mathcal{C}_{\alpha_{1,0}^{-1}(0)}(\alpha_{1,0}^{\perp})$ consist of the stable maps that are constant on the principle component, i.e. the one containing the special marked point $\hat{0}$, have only one level of bubbles, i.e. all the non-principle components are attached directly to the principle component, and the maps are nonconstant on each of the bubbles.

We next apply the topological method of Subsection 2.1 to counting the zeros of an affine map with the linear term $\alpha_{1;k,m}$. By Lemma 2.5,

$$N\big(\alpha_{1;k,m}\big) = \big\langle c\big(\gamma_{\hat{0}}^*\otimes E_{k,m}\big)c\big(\gamma_{\hat{0}}^*\otimes\mathcal{O}_1\big)^{-1}, \big[\overline{\mathfrak{M}}_{0,k+m+1}\times\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle - \mathcal{C}_{\tilde{\alpha}_{1;k,m}^{-1}(0)}\big(\tilde{\alpha}_{1;k,m}^{\perp}\big), \tag{3.8}$$

where

$$\tilde{\alpha}^{\perp}_{1;k,m}\colon \gamma_{\hat{0}}^*\otimes\gamma_{E_{k,m}} \longrightarrow \big(\gamma_{\hat{0}}^*\otimes\pi^*_{\mathbb{P}E_{k,m}}\mathcal{O}_1\big)/\mathbb{C}\bar{\nu}_1$$

denotes the composition of the linear map

$$\tilde{\alpha}_{1;k,m}\colon \gamma_{\hat{0}}^*\otimes\gamma_{E_{k,m}} \longrightarrow \gamma_{\hat{0}}^*\otimes\pi^*_{\mathbb{P}E_{k,m}}\mathcal{O}_1$$

with the quotient projection map π_1. As before,

$$\bar{\nu}_1 \in \Gamma\big(\overline{\mathfrak{M}}_{0,k+m+1}\times\mathbb{P}E_{k,m};\gamma_{\hat{0}}^*\otimes\pi^*_{\mathbb{P}E_{k,m}}\mathcal{O}_1\big)$$

is a generic non-vanishing section. We put

$$\mathcal{O}_2=\gamma_{\hat{0}}\otimes\Big(\big(\gamma_{\hat{0}}^*\otimes\pi^*_{\mathbb{P}E_{k,m}}\mathcal{O}_1\big)/\mathbb{C}\bar{\nu}_1\Big)\approx\big(\pi^*_{\mathbb{P}E_{k,m}}\mathrm{ev}^*T\mathbb{P}^n/\mathbb{C}\bar{\nu}_0\big)/\bar{\nu}_1(\gamma_{\hat{0}}).$$

Let $\tilde{\mathcal{T}}=(S^2,[N]-[M_0],\tilde{I};\tilde{j},\tilde{d})$ be a bubble type such that $|M_0|=m$, $|\tilde{I}|=k$, $\tilde{d}_i>0$, and $\sum\tilde{d}_i=d$, i.e. $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$ is one of the components of the space $\bar{\mathcal{V}}_{k,m}(\mu)$. Suppose $\mathcal{T}=(S^2,[N],I;j,\underline{d})\le\tilde{\mathcal{T}}$ is a bubble type, i.e. $\mathcal{U}_{\mathcal{T}}(\mu)$ is one of the spaces of stable maps that stratify $\bar{\mathcal{U}}_{\tilde{\mathcal{T}}}(\mu)$. By Lemma 2.5,

$$\tilde{\alpha}^{-1}_{1;k,m}(0)\cap\overline{\mathfrak{M}}_{0,k+m+1}\times\mathbb{P}E_{k,m}\big|_{\mathcal{U}_{\mathcal{T}}(\mu)}=\big\{\big(b;[(v_i)_{i\in\tilde{I}}]\big)\colon v_i=0 \text{ if } d_i\neq 0\big\}. \tag{3.9}$$

Of course, the set on the right-hand side of (3.9) is empty if $d_i\neq 0$ for all $i\in\tilde{I}$. From (3.9), we conclude that

$$\tilde{\alpha}^{-1}_{1;k,m}(0)=\bigsqcup_{[\mathcal{T}]<[\tilde{\mathcal{T}}]}\big\{\big(b;[(v_i)_{i\in\tilde{I}}]\big)\colon v_i=0 \text{ if } d_i\neq 0\big\}, \tag{3.10}$$

where the union is taken over all equivalence classes of bubble types $\tilde{\mathcal{T}}$ and $\mathcal{T}$ as above. One might think that the decomposition (3.10) is the analogue of (3.3) in this case, i.e. each space on the right-hand side of (3.10) is either $\tilde{\alpha}^{\perp}_{1;k,m}$-hollow or $\tilde{\alpha}^{\perp}_{1;k,m}$-regular. In general, this is not the case, and we need to decompose each space on the right-hand side of (3.10) into the subspaces based on which of the component elements v_i are not zero.

If $\tilde{\mathcal{T}}$ and $\mathcal{T}$ are bubble types as above and J is a subset of $\tilde{I}$, we set

$$\mathcal{Z}^J_{\mathcal{T}}\equiv\overline{\mathfrak{M}}_{0,k+m+1}\times\Big(\mathbb{P}E\mathcal{T}^J-\bigcup_{J'\subsetneq J}\mathbb{P}E\mathcal{T}^{J'}\Big),\qquad\text{where}\qquad E\mathcal{T}^J=\bigoplus_{i\in J}L_i\mathcal{T}\longrightarrow\mathcal{U}_{\mathcal{T}}(\mu).$$

Let $\tilde{I}_0(\mathcal{T})=\{i\in\tilde{I}\colon d_i=0\}$. This is the subset of the principle components on which every k-tuple of stable maps in $\mathcal{U}_{\mathcal{T}}(\mu)$ is constant. By (3.10),

$$\tilde{\alpha}^{-1}_{1;k,m}(0)=\bigsqcup_{[\mathcal{T}]<[\tilde{\mathcal{T}}]}\;\bigsqcup_{J\subset\tilde{I}_0(\mathcal{T})}\mathcal{Z}^J_{\mathcal{T}}. \tag{3.11}$$

We will show that the set $\mathcal{Z}^J_{\mathcal{T}}$ is $\tilde{\alpha}^{\perp}_{1;k,m}$-regular if $\mathcal{T}$ is a semiprimitive bubble type and $J=\tilde{I}_0(\mathcal{T})$. Otherwise, $\mathcal{Z}^J_{\mathcal{T}}$ is $\tilde{\alpha}^{\perp}_{1;k,m}$-hollow and thus does not contribute to $\mathcal{C}_{\tilde{\alpha}^{-1}_{1;k,m}(0)}(\tilde{\alpha}^{\perp}_{1;k,m})$. Figure 8 shows one of a typical main stratum $\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$ of $\bar{\mathcal{V}}_{2,m}(\mu)$, in a case when $\tilde{I}=\{i_1,i_2\}$ is a two-element set, and two strata $\mathcal{U}_{\mathcal{T}}(\mu)$ such that $\mathcal{T}<\tilde{\mathcal{T}}$ is a semiprimitive bubble type.

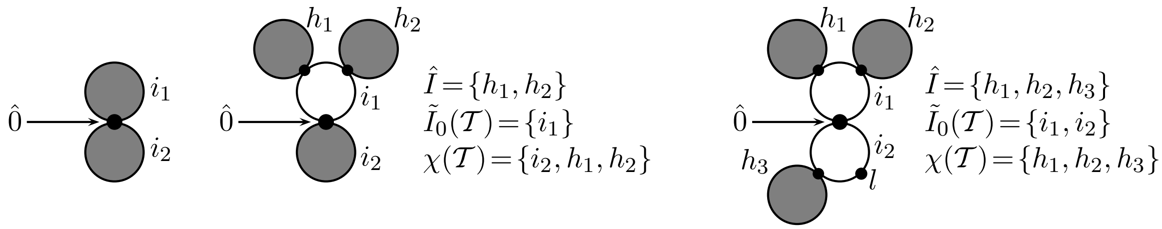


Figure 8: A Stratum $\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$, s.t. $|\tilde{I}|=2$, and Two Strata $\mathcal{U}_{\mathcal{T}}(\mu)$, s.t. $\mathcal{T}$ is Semiprimitive

The map $\gamma_{\mathcal{T}}^{\mu}$ of Proposition 2.7 induces an orientation-preserving homeomorphism $\gamma_{\mathcal{T}}^{J}$ between open neighborhoods of $\mathcal{Z}_{\mathcal{T}}^{J}$ in

$$\mathcal{N}\mathcal{Z}_{\mathcal{T}}^{J} \equiv \mathcal{F}\mathcal{T} \oplus \gamma_{E\mathcal{T}^J}^{*}\otimes\big(E\mathcal{T}^{\tilde{I}_0(\mathcal{T})-J}\oplus E\mathcal{T}^{\tilde{I}-\tilde{I}_0(\mathcal{T})}\big) \longrightarrow \mathcal{Z}_{\mathcal{T}}^{J}$$

and in $\overline{\mathfrak{M}}_{0;k+m+1}\times\mathbb{P}E_{k,m}$. The estimate of Proposition 2.7 implies that for some $\delta, C\in C^{\infty}(\mathcal{Z}_{\mathcal{T}}^{J};\mathbb{R}^{+})$,

$$\big|\tilde{\alpha}_{k,m}\gamma_{\mathcal{T}}^{J}(b;\upsilon,u)-\alpha_{\mathcal{T}}^{J}\rho_{\mathcal{T}}^{J}(b;\upsilon,u)\big| \le C(b)|\upsilon|^{\frac{1}{p}}\big|\rho_{\mathcal{T}}^{J}(b;\upsilon,u)\big| \qquad \forall(\upsilon,u)\in\mathcal{N}\mathcal{Z}_{\mathcal{T},\delta}^{J}, \tag{3.12}$$
$$\text{where}\qquad \rho_{\mathcal{T}}^{J}\colon\mathcal{N}\mathcal{Z}_{\mathcal{T}}^{J}\longrightarrow\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J}\equiv\bigoplus_{h\in\chi(\mathcal{T})}\tilde{\mathcal{N}}_h\mathcal{Z}_{\mathcal{T}}^{J},$$

$$\tilde{\mathcal{N}}_h\mathcal{Z}_{\mathcal{T}}^{J}=\begin{cases}L_{\tilde{\imath}_h}^{*}\mathcal{T}\otimes L_h\mathcal{T}, & \text{if } h\in\hat{I},\tilde{\imath}_h\in J;\\ \gamma_{E\mathcal{T}^J}^{*}\otimes L_h\mathcal{T}, & \text{otherwise;}\end{cases}\qquad \rho_{\mathcal{T};h}^{J}(\upsilon,u)=\begin{cases}\rho_{\mathcal{T};h}(\upsilon), & \text{if } h\in\hat{I},\tilde{\imath}_h\in J;\\ u_{\tilde{\imath}_h}\otimes\rho_{\mathcal{T};h}(\upsilon), & \text{if } h\in\hat{I},\tilde{\imath}_h\notin J;\\ u_h, & \text{if } h\in\tilde{I}-\tilde{I}_0(\mathcal{T});\end{cases}$$

$$\alpha_{\mathcal{T}}^{J}\in\Gamma\Big(\mathcal{Z}_{\mathcal{T}}^{J};\mathrm{Hom}\big(\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J},\mathrm{Hom}(\gamma_{E\mathcal{T}^J},\mathrm{ev}^{*}T\mathbb{P}^n)\big)\Big)$$
$$\big\{\alpha_{\mathcal{T}}^{J}(\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big)\big\}\big((\upsilon_i)_{i\in J}\big)=\sum_{i\in J}\sum_{h\in\chi(\mathcal{T})\cap D_i\mathcal{T}}\mathcal{D}_{\mathcal{T},h}(\tilde{\upsilon}_h\upsilon_i)+\sum_{i\in\tilde{I}-J}\sum_{h\in\chi(\mathcal{T})\cap\bar{D}_i\mathcal{T}}\mathcal{D}_{\mathcal{T},h}(\tilde{\upsilon}_h\upsilon)\in\mathrm{ev}^{*}T\mathbb{P}^n.$$

Above $\rho_{\mathcal{T};h}$ denotes the hth component of $\rho_{\mathcal{T}}$, i.e. $\tilde{\upsilon}_h$ in the notation of Proposition 2.7, and $\tilde{\imath}_h\in\tilde{I}$ is defined by $\tilde{\imath}_h\le h$ whenever $h\in I$. By Lemma 2.8 and the decomposition (2.4), the linear map

$$\alpha_{\mathcal{T}}^{J}\colon\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J}\longrightarrow\mathrm{Hom}\big(\gamma_{E\mathcal{T}^J},\mathrm{ev}^{*}T\mathbb{P}^n\big)$$

is injective on every fiber of $\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J}$. If the sections $\bar{\nu}_0$ and $\bar{\nu}_1$ are generic, the same is true of the linear map

$$\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^{J}\colon\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J}\longrightarrow\mathrm{Hom}\big(\gamma_{\hat{0}}^{*}\otimes\gamma_{E\mathcal{T}^J},\gamma_{\hat{0}}^{*}\otimes\mathcal{O}_2\big)\approx\mathrm{Hom}\big(\gamma_{E\mathcal{T}^J},\mathcal{O}_2\big)$$
$$\big\{\{\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^{J}\}(\tilde{\upsilon})\big\}(\upsilon)=\pi_1\pi_0\big(\{\alpha_{\mathcal{T}}^{J}(\tilde{\upsilon})\}(\upsilon)\big), \tag{3.13}$$

as can been seen from a dimension count. Thus, by (3.12),

$$\Big|\tilde{\alpha}_{1;k,m}^{\perp}\gamma_{\mathcal{T}}^{J}(\upsilon)-\{\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^{J}\}\rho_{\mathcal{T}}^{J}(\upsilon)\Big|\le\tilde{C}(b_\upsilon)|\upsilon|^{\frac{1}{p}}\big|\{\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^{J}\}\rho_{\mathcal{T}}^{J}(\upsilon)\big| \quad \forall\upsilon\in\mathcal{N}\mathcal{Z}_{\mathcal{T},\delta}^{J}. \tag{3.14}$$

By definition, the ranks of $\mathcal{N}\mathcal{Z}_{\mathcal{T}}^{J}$ and $\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^{J}$ are $|\hat{I}|+|\tilde{I}-J|$ and $|\chi(\mathcal{T})|$, respectively, while

$$\chi(\mathcal{T})-\hat{I}=\tilde{I}-\tilde{I}_0(\mathcal{T}).$$

Thus, if $\hat{I}-\chi(\mathcal{T})\neq\emptyset$, i.e. $\mathcal{T}$ is not semiprimitive, or $J\neq\tilde{I}_0(\mathcal{T})$, the rank of $\mathcal{N}\mathcal{Z}_{\mathcal{T}}^J$ is less than the rank of $\tilde{\mathcal{N}}\mathcal{Z}_{\mathcal{T}}^J$ and thus $\mathcal{Z}_{\mathcal{T}}^J$ is $\tilde{\alpha}_{1;k,m}^{\perp}$-hollow. On the other hand, if $\hat{I}-\chi(\mathcal{T})=\emptyset$ and $J=\tilde{I}_0(\mathcal{T})$, $\rho_{\mathcal{T}}^J$ is the identity map, and thus $\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J$ is the resolvent of $\tilde{\alpha}_{1;k,m}^{\perp}$ near $\mathcal{Z}_{\mathcal{T}}^J$. By Proposition 2.7,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}^J}(\tilde{\alpha}_{1;k,m}^{\perp})=N\big(\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\big),\quad\text{where}\quad \pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\in\Gamma\big(\bar{\mathcal{Z}}_{\mathcal{T}}^J;\mathrm{Hom}(\mathcal{N}\mathcal{Z}_{\mathcal{Z}}^J,\gamma_{E\mathcal{T}^J}^*\otimes\mathcal{O}_2)\big).\tag{3.15}$$

As before, we now denote by $\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J$ the natural extension of the map defined in (3.13) over $\mathcal{Z}_{\mathcal{T}}^J$.

While we can proceed by computing the numbers $N\big(\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\big)$, where $\mathcal{T}$ is a semiprimitive bubble type and $J=\tilde{I}_0(\mathcal{T})$, we simplify the computation a little by replacing the linear map $\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J$ by another linear map $\alpha_{2;\mathcal{T}}$, such that

$$N\big(\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\big)=N\big(\alpha_{2;\mathcal{T}}\big)\tag{3.16}$$

and $\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J$ is a regular linear map if and only if $\alpha_{2;\mathcal{T}}$ is. With respect to the decomposition (2.5),

$$\mathcal{N}\mathcal{Z}_{\mathcal{T}}^J\approx\bigoplus_{i\in J}\gamma_{\hat{0};i}^*\otimes\bigoplus_{\iota_h=i}L_h\bar{\mathcal{T}}\;\oplus\gamma_{E\mathcal{T}^J}^*\otimes\bigoplus_{h\in\tilde{I}-J}L_h\bar{\mathcal{T}},\quad E\mathcal{T}^J=\bigoplus_{i\in J}\gamma_{\hat{0};i}\longrightarrow\overline{\mathfrak{M}}_{2;\mathcal{T}}\equiv\prod_{i\in J}\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}},$$
$$\big\{\{\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\}(\tilde{\upsilon}_h)_{h\in\chi(\mathcal{T})}\big\}\big((\upsilon_i)_{i\in J}\big)=\pi_1\pi_0\Big(\sum_{i\in J}\sum_{\iota_h=1}\mathcal{D}_{\mathcal{T},h}(\tilde{\upsilon}_h\upsilon_i)+\sum_{i\in\tilde{I}-J}\mathcal{D}_{\mathcal{T},i}(\tilde{\upsilon}_i\upsilon)\Big),$$

where $\gamma_{\hat{0};i}\longrightarrow\overline{\mathfrak{M}}_{H_i\mathcal{T}\sqcup M_i\mathcal{T}}$ is the tautological line bundle. We define the linear map $\alpha_{2;\mathcal{T}}$ by

$$\alpha_{2;\mathcal{T}}\in\Gamma\big(\overline{\mathfrak{M}}_{0,k+m+1}\times\mathbb{P}E\mathcal{T}^J\times\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu);\mathrm{Hom}\big(\gamma_{E\mathcal{T}^J}^*\otimes E\bar{\mathcal{T}},\gamma_{E\mathcal{T}^J}^*\otimes\mathcal{O}_2)\big),$$
$$\big\{\alpha_{2;\mathcal{T}}\big(u\otimes(\upsilon_h)_{h\in\bar{I}}\big)\big\}(\tilde{\upsilon})=u(\tilde{\upsilon})\cdot\pi_1\pi_0\sum_{h\in\bar{I}}\mathcal{D}_{\bar{\mathcal{T}},h}\upsilon_h.\tag{3.17}$$

Let $\rho\colon\gamma_{E\mathcal{T}^J}^*\otimes E\bar{\mathcal{T}}\longrightarrow\mathcal{N}\mathcal{Z}_{\mathcal{T}}^J$ be the vector-bundle map defined by

$$\rho(u\otimes\upsilon_h)=\begin{cases}(u\circ\pi_{\iota_h})\otimes\upsilon_h, & \text{if } h\in\hat{I};\\ u\otimes\upsilon_h, & \text{if } h\in\tilde{I}-J,\end{cases}$$

where $\pi_i\colon E\mathcal{T}^J\longrightarrow L_i\mathcal{T}$ is the projection map. The map ρ is an isomorphism over the dense open subset $\mathcal{Z}_{\mathcal{T}}$ of $\bar{\mathcal{Z}}_{\mathcal{T}}$ and

$$\alpha_{2;\mathcal{T}}=\pi_1\circ\pi_0\circ\alpha_{\mathcal{T}}^J\circ\rho.$$

Thus, (3.16) holds by definition of $N(\alpha)$; see Subsection 2.1. Summing (3.15) over all equivalence classes of bubble types $\mathcal{T}<\tilde{\mathcal{T}}$ of the appropriate form and using (3.16), we conclude that

$$\mathcal{C}_{\tilde{\alpha}_{1;k,m}^{-1}(0)}(\tilde{\alpha}_{1;k,m}^{\perp})=\sum_{[\mathcal{T}]}N(\alpha_{2;\mathcal{T}})=\sum_{\sigma}N\big(\alpha_{\sigma}\big),\qquad\text{where}$$
$$\alpha_\sigma\in\Gamma\big(\overline{\mathfrak{M}}_{k+m+1}\times\mathbb{P}F_\sigma\times\bar{\mathcal{V}}_\sigma;\mathrm{Hom}\big(\gamma_{F_\sigma}^*\otimes E_\sigma,\gamma_{F_\sigma}^*\otimes\mathcal{O}_2)\big),$$
$$\big\{\alpha_\sigma(u\otimes\upsilon)\big\}(\tilde{\upsilon})=u(\tilde{\upsilon})\cdot\pi_1\pi_0\alpha'_\sigma(\upsilon).$$

This sum is taken over all tuples $\sigma=(2;k_2,m_2;\phi)$, where $(k_2,m_2)>(k,m)$ and ϕ specifies a splitting of the set $[k_2]$ into k-disjoint subsets and an assignment of m_2-m of the elements of the set $[m_2]$ to these subsets. For such a tuple σ, we put

$$\bar{\mathcal{V}}_\sigma=\bar{\mathcal{V}}_{k_2,m_2}(\mu);\quad E_\sigma=E_{k_2,m_2};\quad \alpha'_\sigma=\alpha_{k_2,m_2};\quad F_\sigma=\bigoplus_{i\in\mathrm{Im}\phi}\gamma_{\sigma;i}\longrightarrow\overline{\mathfrak{M}}_\sigma\equiv\prod_{i\in\mathrm{Im}\phi}\overline{\mathfrak{M}}_{i\sqcup\phi^{-1}(i)}.$$

For the purposes of the last line, we view ϕ as a map from $[k_2]-[k]$ and a subset of $[m_2]$ to $[k]$, and $\gamma_{\sigma;i} \longrightarrow \overline{\mathfrak{M}}_{i\sqcup\phi^{-1}(i)}$ denotes the tautological line bundle.

3.2 A Tree of Chern Classes

In this subsection, we prove Corollary 3.6 by setting up a possibly large, but finite, tree. If each node of the tree is assigned the chern class that appears in the statement of Lemma 3.3, the sum of these chern classes, counted with a sign dependent on the distance to the root, is the number of Corollary 3.6. The reader is referred to the previous subsection for a more explicit description of the first two levels of the tree and for the proof of Lemma 3.3 in the corresponding cases. The proof of Lemma 3.3 in general is nearly the same as the one given for the second-level nodes in the previous subsection.

Each node in the tree is a tuple $\sigma=(r;k,m;\phi)$, where $r\geq 0$ is the distance to the root $\sigma_0=(0;1,0;\cdot)$, $k\geq 1$, and $m\geq 0$. The tree satisfies the following properties. If $r>0$ and $\sigma^*=(r-1;k^*,m^*;\phi^*)$ is the node from which σ is directly descendent, we require that $(k^*,m^*)<(k,m)$. Furthermore, ϕ specifies a splitting of the set $[k]$ into k^*-disjoint subsets and an assignment of $m-m^*$ of the elements of the set $[m]$ to these subsets. This description inductively constructs an infinite tree. However, we will need to consider only the nodes $\sigma=(r;k,m;\phi)$ with $2k+m\leq n+1$. We will write $\sigma\vdash\sigma^*$ to indicate that σ is directly descendent from σ^*.

For each node in the above tree, we now define a linear map between vector bundles over an ms-manifold. If $\sigma=(r;k,m;\phi)$, let $\{\sigma_s=(s;k_s,m_s;\phi_s)\colon 0\leq s\leq r\}$ be the sequence of nodes such that $\sigma_r=\sigma$ and $\sigma_s\vdash\sigma_{s-1}$ for all $s>0$. Put

$$\bar{\mathcal{V}}_\sigma=\bar{\mathcal{V}}_{k,m}(\mu),\quad E_\sigma=E_{k,m}\longrightarrow\bar{\mathcal{V}}_\sigma,\quad \alpha'_\sigma=\alpha_{k,m},\quad \mathcal{X}_\sigma=\mathcal{Y}_\sigma\times\bar{\mathcal{V}}_\sigma,\quad \mathcal{X}_{\sigma,s}=\mathcal{Y}_{\sigma,s}\times\bar{\mathcal{V}}_\sigma,$$
$$\text{where}\quad \mathcal{Y}_\sigma=\mathcal{Y}_{\sigma,r},\quad \mathcal{Y}_{\sigma,0}=\{pt\},\quad \mathcal{Y}_{\sigma,s}=\mathbb{P}F_{\sigma_s}\times\mathcal{Y}_{\sigma,s-1}\ \text{ if }\ s>0,$$
$$\overline{\mathfrak{M}}_\sigma=\prod_{i\in\mathrm{Im}\,\phi}\overline{\mathfrak{M}}_{i\sqcup\phi^{-1}(i)},\qquad F_\sigma=\bigoplus_{i\in\mathrm{Im}\,\phi}\gamma_{\sigma;i}\longrightarrow\overline{\mathfrak{M}}_\sigma.$$

For the purposes of the last line above, we view ϕ as a map from $[k]-[k^*]$ and a subset of $[m]$ to $[k^*]$ in the notation of the previous paragraph. Then, $\gamma_{\sigma;i}\longrightarrow\overline{\mathfrak{M}}_{i\sqcup\phi^{-1}(i)}$ is the tautological line bundle; see Subsection 2.2. Denote by $\gamma_{F_{\sigma,0}}$ the (trivial) line bundle over $\mathcal{Y}_{\sigma,0}$. Let

$$\mathcal{O}_\sigma=\mathcal{O}_{\sigma,r},\quad \mathcal{O}_{\sigma,0}=\mathrm{ev}^*T\mathbb{P}^n,\quad \mathcal{O}_{\sigma,s}=\mathcal{O}_{\sigma,s-1}\big/\mathrm{Im}\,\bar{\nu}_{\sigma,s-1}\ \text{ if } s>0,$$

where $\bar{\nu}_{\sigma,s}\in\Gamma\big(\mathcal{X}_{\sigma,s};\mathrm{Hom}(\gamma_{F_{\sigma_s}},\mathcal{O}_{\sigma,s})\big)$ is a generic section. Since $k_{s-1}\leq k_s$, $m_{s-1}\leq m_s$, and one of the inequalities is strict,

$$\frac{1}{2}\dim\mathcal{X}_{\sigma,s}\leq\frac{1}{2}\dim\mathcal{X}_\sigma=\big(n+1-2k-m\big)+\sum_{s=1}^{s=r}\big(\big|\mathrm{Im}\,\phi_s\big|-1\big)=n-k-r<\mathrm{rk}\,\mathcal{O}_{\sigma,0}-r.$$

Thus, we see inductively that each bundle $\mathcal{O}_{\sigma,s}$ is well-defined and a generic section $\bar{\nu}_{\sigma,s}$ of $\mathrm{Hom}(\gamma_{F_{\sigma,s}},\mathcal{O}_{\sigma,s})$ does not vanish. Let $\pi_\sigma\colon\mathrm{ev}^*T\mathbb{P}^n\longrightarrow\mathcal{O}_\sigma$ be the projection map. We define

$$\alpha_\sigma\in\Gamma\big(\mathcal{X}_\sigma;\mathrm{Hom}(\gamma^*_{F_\sigma}\otimes E_\sigma;\gamma^*_{F_\sigma}\otimes\mathcal{O}_\sigma)\big),\quad\text{by}\quad \big\{\alpha_\sigma(u\otimes v)\big\}(\tilde{v})=u(\tilde{v})\cdot\pi_\sigma\alpha'_\sigma(v)\in\mathcal{O}_\sigma.$$

Note that $\tilde{\alpha}_{\sigma_0}=\alpha_{1,0}$.

Lemma 3.3 *For every node σ^*,*

$$N\big(\alpha_{\sigma^*}\big) = \big\langle c\big(\gamma_{F_{\sigma^*}}^* \otimes \mathcal{O}_{\sigma^*}\big) c\big(\gamma_{F_{\sigma^*}}^* \otimes E_{\sigma^*}\big)^{-1}, \big[\mathcal{X}_{\sigma^*}\big]\big\rangle - \sum_{\sigma \vdash \sigma^*} N\big(\alpha_\sigma\big).$$

Remark: For a dense open subset of tuples $\{\bar{\nu}_{\sigma,s}\}$, the corresponding linear map α_σ constructed above is regular and $N(\alpha_\sigma)$ is independent of the choice of $\{\bar{\nu}_{\sigma,s}\}$. What we need is that for every bubble type $\mathcal{T}$ such that $\mathcal{U}_{\mathcal{T}}(\mu) \subset \bar{\mathcal{V}}_{k_r,m_r}(\mu)$ the intersection of the image of the linear map

$$\alpha_{\mathcal{T}} \in \Gamma\Big(\mathcal{Y}_\sigma \times \mathcal{U}_{\mathcal{T}}(\mu); \mathrm{Hom}\big(\bigoplus_{i \in \chi(\mathcal{T})} L_i\mathcal{T}, \mathrm{ev}^* T\mathbb{P}^n\big)\Big), \quad \alpha_{\mathcal{T}}(\upsilon) = \sum_{i \in \chi(\mathcal{T})} \mathcal{D}_{\mathcal{T},i} \upsilon_i,$$

with the subbundle

$$\mathrm{Im}\, \bar{\nu}_{\sigma,0} \oplus \ldots \oplus \mathrm{Im}\, \bar{\nu}_{\sigma,r-1} \subset \mathcal{O}_{\sigma,0} = \mathrm{ev}^* T\mathbb{P}^n$$

is $\{0\}$. The fact that this condition is satisfied for a dense open subset of tuples $\{\bar{\nu}_{\sigma,s}\}$ follows by a dimension count as above, along with an argument similar to the proof of Lemma 3.10 in [Z2].

Proof of Lemma 3.3: (1) By Lemma 2.5,

$$N\big(\alpha_{\sigma^*}\big) = \big\langle c\big(\gamma_{F_{\sigma^*}}^* \otimes \mathcal{O}_{\sigma^*}\big) c\big(\gamma_{F_{\sigma^*}}^* \otimes E_{\sigma^*}\big)^{-1}, \big[\mathcal{X}_{\sigma^*}\big]\big\rangle - \mathcal{C}_{\tilde{\alpha}_{\sigma^*}^{-1}(0)}\big(\tilde{\alpha}_{\sigma^*}^{\perp}\big). \tag{3.18}$$

Let $\sigma^* = (r^*; k^*, m^*; \phi^*)$. By Lemma 2.8, $\tilde{\alpha}_{\sigma^*}^{-1}(0)$ is the union of the sets

$$\mathcal{Z}_{\mathcal{T}}^J \equiv \mathcal{Y}_{\sigma^*} \times \Big(\mathbb{P}E\mathcal{T}^J - \bigcup_{J' \subsetneq J} \mathbb{P}E\mathcal{T}^{J'}\Big), \quad \text{where} \quad E\mathcal{T}^J = \bigoplus_{i \in J} L_i\mathcal{T} \longrightarrow \mathcal{U}_{\mathcal{T}}(\mu),$$

taken over non-basic bubble types $\mathcal{T} = (S^2, [N] - M_0, I; j, \underline{d})$, with $|I - \hat{I}| = k^*$, $|M_0| = m^*$, and $\sum d_i = d$, and nonempty subsets J of $I - \hat{I} - \chi(\mathcal{T})$.
(2) The map $\gamma_{\mathcal{T}}^{\mu}$ of Proposition 2.7 induces an orientation-preserving homeomorphism $\gamma_{\mathcal{T}}^J$ between open neighborhoods of $\mathcal{Z}_{\mathcal{T}}^J$ in

$$\mathcal{N}\mathcal{Z}_{\mathcal{T}}^J \equiv \mathcal{F}\mathcal{T} \oplus \gamma_{E\mathcal{T}^J}^* \otimes \big(E\mathcal{T}^{I - \hat{I} - \chi(\mathcal{T}) - J} \oplus E\mathcal{T}^{\chi(\mathcal{T}) - \hat{I}}\big) \longrightarrow \mathcal{Z}_{\mathcal{T}}^J$$

and in $\mathcal{Y}_{\sigma^*} \times \mathbb{P}E_{\sigma^*}$. Furthermore, the estimate (3.12) holds. Proceeding as in the previous subsection, we conclude that $\mathcal{Z}_{\mathcal{T}}^J$ is $\tilde{\alpha}_{\sigma}^{\perp}$-hollow unless $\mathcal{T}$ is semiprimitive and $J = I - \hat{I} - \chi(\mathcal{T})$. Thus,

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}^J}\big(\tilde{\alpha}_{\sigma^*}^{\perp}\big) = 0 \qquad \text{if } \mathcal{T} \text{ is not semiprimitive or } J \neq I - \hat{I} - \chi(\mathcal{T}). \tag{3.19}$$

If $\mathcal{T}$ is semiprimitive and $J = I - \hat{I} - \chi(\mathcal{T})$, we find that

$$\mathcal{C}_{\mathcal{Z}_{\mathcal{T}}^J}\big(\tilde{\alpha}_{\sigma^*}^{\perp}\big) = N\big(\alpha_{\sigma^*,\mathcal{T}}\big) \qquad \text{if } \mathcal{T} \text{ is semiprimitive and } J = I - \hat{I} - \chi(\mathcal{T}), \tag{3.20}$$

$$\text{where} \quad \alpha_{\sigma^*,\mathcal{T}} \in \Gamma\big(\mathcal{Y}_{\sigma^*} \times \mathbb{P}E\mathcal{T}^J \times \bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu); \mathrm{Hom}(\gamma_{E\mathcal{T}^J}^* \otimes E\bar{\mathcal{T}}, \gamma_{E\mathcal{T}^J}^* \otimes \mathcal{O}_{\sigma^*,r^*+1})\big),$$

$$E\mathcal{T}^J \equiv \bigoplus_{i \in J} \gamma_{\mathcal{T};i} \longrightarrow \overline{\mathfrak{M}}_{\sigma^*,\mathcal{T}} \equiv \prod_{i \in J} \overline{\mathfrak{M}}_{H_i\mathcal{T} \sqcup M_i\mathcal{T}}, \quad \mathcal{O}_{\sigma^*,r+1} = \mathcal{O}_{\sigma^*} / \mathrm{Im}\, \bar{\nu}_{\sigma^*}$$

$$\big\{\alpha_{\sigma^*,\mathcal{T}}(u \otimes \upsilon)\big\}(\tilde{\upsilon}) = u(\tilde{\upsilon}) \cdot \pi_{\sigma^*} \alpha_{k,m}(\upsilon), \quad k = |\chi(\mathcal{T})| = |\bar{I}|, \quad m = m^* + \sum_{i \in I - \chi(\mathcal{T})} |M_i\mathcal{T}|;$$

see (3.17).
(3) From equations (3.18)-(3.20) we conclude that that

$$\begin{aligned} N\big(\tilde{\alpha}_{\sigma^*}\big) &= \big\langle c\big(\gamma^*_{F_{\sigma^*}}\otimes\mathcal{O}_{\sigma^*}\big)c\big(\gamma^*_{F_{\sigma^*}}\otimes E_{\sigma^*}\big)^{-1}, \big[\mathcal{X}_{\sigma^*}\big]\big\rangle - \sum_{(k,m)>(k^*,m^*)} \sum_{\substack{|\chi(\mathcal{T})|=k,\\ \sum\limits_{i\in I-\chi(\mathcal{T})}|M_i\mathcal{T}|=m-m^*}} N\big(\alpha_{\sigma^*,\mathcal{T}}\big) \\ &= \big\langle c\big(\gamma^*_{F_{\sigma^*}}\otimes\mathcal{O}_{\sigma^*}\big)c\big(\gamma^*_{F_{\sigma^*}}\otimes E_{\sigma^*}\big)^{-1}, \big[\mathcal{X}_{\sigma^*}\big]\big\rangle - \sum_{\sigma\vdash\sigma^*} N\big(\alpha_\sigma\big). \end{aligned}$$

The inner sum on the first line above is taken over all equivalence classes of semiprimitive bubble types $\mathcal{T}=(S^2,N-M_0,I;j,\underline{d})$ such that $|I-\hat{I}|=k^*$, $|M_0|=m^*$, and $\sum d_i=d$.

Lemma 3.4 *For every node $\sigma=(r;k,m;\phi)$ and positive integer $s\le r-1$,*

$$\big\langle c\big(\mathcal{O}_{\sigma,s+1}\big)c\big(E_\sigma\big)^{-1}, \big[\mathcal{X}_{\sigma,s}\big]\big\rangle = \big\langle c\big(\mathcal{O}_{\sigma,s}\big)c\big(E_\sigma\big)^{-1}, \big[\mathcal{X}_{\sigma,s-1}\big]\big\rangle,$$

where $\{\sigma_s\}$ is the sequence corresponding to σ defined in the paragraph preceding Lemma 3.3.

Proof: Since $\mathcal{O}_{\sigma,s+1}\approx\mathcal{O}_{\sigma,s}/\gamma_{F_{\sigma_s}}$,

$$\big\{c\big(\mathcal{O}_{\sigma,s+1}\big)c\big(E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_{\sigma,s}} = \sum_{l=0}^{\dim\mathcal{X}_{\sigma,s}} \sum_{l_1+l_2=l} \lambda^{l_1}_{F_{\sigma_s}} c_{l_2}\big(\mathcal{O}_{\sigma,s}\big)\big\{c\big(E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_{\sigma,s}-2l}. \tag{3.21}$$

By construction, $\lambda_{F_{\sigma_s}}\in H^*(\mathbb{P}F_{\sigma_s})$, while $c(\mathcal{O}_{\sigma,s}),c(E_\sigma)\in H^*(\mathcal{X}_{\sigma,s-1})$. Thus, (3.21) gives

$$\begin{aligned} \big\{c\big(\mathcal{O}_{\sigma,s+1}\big)c\big(E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_{\sigma,s}} &= \lambda^{n_\sigma}_{F_{\sigma_s}} \sum_{l=0}^{\dim\mathcal{X}_{\sigma,s}} c_{l-n_\sigma}\big(\mathcal{O}_{\sigma,s}\big)\big\{c\big(E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_{\sigma,s}-2l} \\ &= \lambda^{n_\sigma}_{F_{\sigma_s}}\big\{c\big(\mathcal{O}_{\sigma,s}\big)c\big(E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_{\sigma,s-1}}, \end{aligned} \tag{3.22}$$

where $n_\sigma=\dim\mathbb{P}F_{\sigma_s}$. By (2.1),

$$\big\langle \lambda^{n_\sigma}_{F_{\sigma_s}}, \big[\mathbb{P}F_{\sigma_s}\big]\big\rangle = \big\langle c\big(F_{\sigma_s}\big)^{-1}, \big[\bar{\mathcal{M}}_{\sigma_s}\big]\big\rangle = \prod_{i\in\mathrm{Im}\ \phi_s} \big\langle c\big(\gamma_{\sigma_s;i}\big)^{-1}, \big[\bar{\mathcal{M}}_{0,i+\phi_s^{-1}(i)}\big]\big\rangle = 1. \tag{3.23}$$

The last identity is a consequence of (1) of Lemma 3.11. The claim follows from (3.21)-(3.23).

Corollary 3.5 *For every node $\sigma=(r;k,m;\phi)$,*

$$\big\langle c\big(\gamma^*_{F_\sigma}\otimes\mathcal{O}_\sigma\big)c\big(\gamma^*_{F_\sigma}\otimes E_\sigma\big)^{-1}, \big[\mathcal{X}_\sigma\big]\big\rangle = \big\langle c(ev^*T\mathbb{P}^n)c(E_{k,m})^{-1}, \big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle.$$

Proof: Since $\mathrm{rk}\,\mathcal{O}_\sigma=\mathrm{rk}\,E_\sigma+\frac{1}{2}\dim\mathcal{X}_\sigma$, we can identify E_σ with a subbundle of $\mathcal{O}_\sigma$. Then,

$$\begin{aligned} & c\big(\gamma^*_{F_\sigma}\otimes\mathcal{O}_\sigma\big)c\big(\gamma^*_{F_\sigma}\otimes E_\sigma\big)^{-1} = c\big(\gamma^*_{F_\sigma}\otimes\mathcal{O}_\sigma/\gamma^*_{F_\sigma}\otimes E_\sigma\big) = c\big(\gamma^*_{F_\sigma}\otimes(\mathcal{O}_\sigma/E_\sigma)\big) \Longrightarrow \\ & \big\{c\big(\gamma^*_{F_\sigma}\otimes\mathcal{O}_\sigma\big)c\big(\gamma^*_{F_\sigma}\otimes E_\sigma\big)^{-1}\big\}_{\dim\mathcal{X}_\sigma} = \sum_{l=0}^{\dim\mathcal{X}_\sigma} \lambda^l_{F_\sigma}\big\{c(\mathcal{O}_\sigma)c(E_\sigma)^{-1}\big\}_{\dim\mathcal{X}_\sigma-2l}. \end{aligned} \tag{3.24}$$

Similarly to the proof of Lemma 3.4, (3.24) gives

$$\begin{aligned}\big\langle c\big(\gamma_{F_\sigma}^*\otimes\mathcal{O}_\sigma\big)c\big(\gamma_{F_\sigma}^*\otimes E_\sigma\big)^{-1},\big[\mathcal{X}_\sigma\big]\big\rangle &= \big\langle c\big(\mathcal{O}_\sigma\big)c\big(E_\sigma\big)^{-1},\big[\mathcal{X}_{\sigma,r-1}\big]\big\rangle \\ &= \big\langle c\big(\mathcal{O}_{\sigma,r}\big)c\big(E_\sigma\big)^{-1},\big[\mathcal{X}_{\sigma,r-1}\big]\big\rangle.\end{aligned} \tag{3.25}$$

Applying Lemma 3.4 to the last expression in (3.25) and using $\mathcal{O}_{\sigma,1}\approx(\mathrm{ev}^*T\mathbb{P}^n)/\mathbb{C}$, we obtain

$$\big\langle c\big(\gamma_{F_\sigma}^*\otimes\mathcal{O}_\sigma\big)c\big(\gamma_{F_\sigma}^*\otimes E_\sigma\big)^{-1},\big[\mathcal{X}_\sigma\big]\big\rangle = \big\langle c\big(\mathcal{O}_{\sigma,1}\big)c\big(E_\sigma\big)^{-1},\big[\mathcal{X}_{\sigma,0}\big]\big\rangle = \big\langle c\big(\mathrm{ev}^*T\mathbb{P}^n\big)c(E_{k,m})^{-1},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle.$$

We now combine Lemma 3.3 and Corollary 3.5 to obtain a topological formula for the number $N(\alpha_{1,0})$. For any integers k and k^*, let $\theta_{k^*}^k$ denote the number of ways of splitting a set of k^*-elements into k nonempty subsets. For every pair $(k^*,m^*)\geq(1,0)$ of integers, we define $\Theta(k^*,m^*)$ inductively by

$$\Theta(1,0)=1, \quad \Theta(k^*,m^*) = -\sum_{(1,0)\leq(k,m)<(k^*,m^*)}\binom{m^*}{m}k^{m^*-m}\theta_{k^*}^k\Theta(k,m) \ \text{ if } \ (k^*,m^*)>(1,0). \tag{3.26}$$

Corollary 3.6 *With notation as above,*

$$N(\alpha_{1,0}) = \sum_{(1,0)\leq(k,m)}\Theta(k,m)\sum_{l=0}^{n+1-(2k+m)}\binom{n+1}{l}\big\langle a^l\tilde{\eta}_{n+1-(2k+m)-l},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle.$$

Proof: Note that the coefficient in front of $\Theta(k,m)$ in (3.26) is the number of ways of splitting the set $[k^*]$ into k nonempty subsets and assigning m^*-m elements of the set $[m^*]$ to these subsets. Thus, by Lemma 3.3 and Corollary 3.5,

$$N\big(\alpha_{1,0}\big) = N\big(\tilde{\alpha}_{\sigma_0}\big) = \sum_{(1,0)\leq(k,m)}\Theta(k,m)\big\langle c(\mathrm{ev}^*T\mathbb{P}^n)c(E_{k,m})^{-1},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle. \tag{3.27}$$

Since $E_{k,m}=\bigoplus L_i$,

$$c(E_{k,m})^{-1} = \prod_{i=1}^{i=k}\big(1+c_1(L_i)\big)^{-1} = \prod_{i=1}^{i=k}\sum_{l=0}^{\infty}c_1^l(L_i^*) = \sum_{l=0}^{\infty}\tilde{\eta}_l. \tag{3.28}$$

The last equality above is immediate from the definition of $\tilde{\eta}_l$; see Subsection 2.2. The claim follows from (3.27) and (3.28), along with $c(\mathrm{ev}^*T\mathbb{P}^n)=(1+a)^{n+1}$.

3.3 Combinatorics

In this subsection, we show that the topological expression for $N(\alpha_{1,0})$ given in Corollary 3.6 is the same as the topological expression for $CR_1(\mu)$ given in Proposition 3.1. This fact is immediate from Corollary 3.10. We start by proving an explicit formula for the numbers $\Theta(k,m)$.

Lemma 3.7 *If* $(k,m)\geq(1,0)$, $\Theta(k,m) = (-1)^{k+m-1}k^m(k-1)!$.

(1) We first start verify this formula in the case $k=1$. By (3.26),

$$\Theta(1,0)=1, \qquad \Theta(1,m^*)=-\sum_{m=0}^{m^*-1}\binom{m^*}{m}\Theta(1,m) \quad \text{if} \quad m^*>(1,0). \tag{3.29}$$

We need to show that $\Theta(1,m)=(-1)^m$. If $m=0$, this is the case. Suppose $m^*\geq 1$ and $\Theta(1,m)=(-1)^m$ for all $m<m^*$. Then, by (3.29),

$$\Theta(1,m^*)=-\sum_{m=0}^{m^*-1}\binom{m^*}{m}\Theta(1,m)=-\sum_{m=0}^{m^*}\binom{m^*}{m}(-1)^m+(-1)^{m^*}=-(1-1)^{m^*}+(-1)^{m^*}=(-1)^{m^*},$$

as needed.
(2) We now verify the formula in the general case. It is easy to see from the definition of $\theta^k_{k^*}$ in the previous subsection that

$$\theta^k_k=1 \ \text{ if } k\geq 1 \quad \text{and} \quad \theta^k_{k^*}=k\theta^k_{k^*-1}+\theta^{k-1}_{k^*-1} \ \text{ if } k\geq 2. \tag{3.30}$$

Suppose $k^*\geq 2$ and the claimed formula holds for all (k,m) with $(1,0)\leq(k,m)<(k^*,m^*)$. Then by (3.26),

$$\begin{aligned}\Theta(k^*,m^*)&=-\sum_{(1,0)\leq(k,m)<(k^*,m^*)}\binom{m^*}{m}k^{m^*-m}\theta^k_{k^*}\Theta(k,m)\\&=k^{m^*}\sum_{(1,0)\leq(k,m)<(k^*,m^*)}(-1)^{k+m}\binom{m^*}{m}\theta^k_{k^*}(k-1)!\end{aligned} \tag{3.31}$$

Using (3.30), we obtain

$$\begin{aligned}\sum_{(1,0)\leq(k,m)<(k^*,m^*)}(-1)^{k+m}\binom{m^*}{m}\theta^k_{k^*}(k-1)! &= \sum_{(1,0)\leq(k,m)<(k^*,m^*)}(-1)^{k+m}\binom{m^*}{m}\big(k\theta^k_{k^*-1}+\theta^{k-1}_{k^*-1}\big)(k-1)!\\&=\sum_{m=0}^{m^*-1}(-1)^m\binom{m^*}{m}\sum_{k=1}^{k^*}(-1)^k\big(\theta^k_{k^*-1}k!+\theta^{k-1}_{k^*-1}(k-1)!\big)\\&\quad+(-1)^{m^*}\sum_{k=1}^{k^*-1}(-1)^k\big(\theta^k_{k^*-1}k!+\theta^{k-1}_{k^*-1}(k-1)!\big).\end{aligned} \tag{3.32}$$

Note that

$$\begin{aligned}\sum_{k=1}^{k^*}(-1)^k\big(\theta^k_{k^*-1}k!+\theta^{k-1}_{k^*-1}(k-1)!\big)&=\sum_{k=1}^{k^*}(-1)^k\theta^k_{k^*-1}k!-\sum_{k=0}^{k^*-1}(-1)^k\theta^k_{k^*-1}k!=0;\\ \sum_{k=1}^{k^*-1}(-1)^k\big(\theta^k_{k^*-1}k!+\theta^{k-1}_{k^*-1}(k-1)!\big)&=\sum_{k=1}^{k^*-1}(-1)^k\theta^k_{k^*-1}k!-\sum_{k=0}^{k^*-2}(-1)^k\theta^k_{k^*-1}k!=(-1)^{k^*-1}(k^*-1)!,\end{aligned} \tag{3.33}$$

since $c^{k^*}_{k^*-1}=0$, $c^{k^*-1}_{k^*-1}=1$, and $c^0_{k^*-1}=0$ if $k^*>1$. Combining equations (3.31)-(3.33), we verify the claimed identity for $(k,m)=(k^*,m^*)$.

We next need to relate the intersection numbers $a^l\tilde{\eta}_{l'}$ and $a^l\tilde{\eta}_{l'}$. We break the computation into several steps.

Lemma 3.8 *Suppose $\mathcal{T}=\big(S^2,M,I;j,\underline{d}\big)$ is a basic bubble type, $i\in I$, and $M_i\subset M_i\mathcal{T}$. Then, under the splitting (2.2), with $\bar{\mathcal{T}}=\mathcal{T}/M_i$,*

$$c_1(L_{i'}^*\mathcal{T})\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=\begin{cases}\gamma_{\mathcal{T};i}^*\times 1, & \text{if } i'=i;\\ 1\times c_1(L_{i'}^*\bar{\mathcal{T}}), & \text{if } i'\neq i;\end{cases}\qquad c_1(\mathcal{L}_{i'}^*\mathcal{T})\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=1\times c_1(\mathcal{L}_{i'}^*\bar{\mathcal{T}}).$$

Proof: The first identity and the case $i'\neq i$ of the second identity are immediate from the definitions. In the remaining case, by (2.3), we have

$$c_1(\mathcal{L}_i^*\mathcal{T})\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=c_1(L_i^*\mathcal{T})\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)-\sum_{\emptyset\neq M_i'\subset M_i\mathcal{T}}\mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\bar{\mathcal{U}}_{\mathcal{T}(M_i')}(\mu)\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu).\tag{3.34}$$

By definition of the spaces,

$$\mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\bar{\mathcal{U}}_{\mathcal{T}(M_i')}(\mu)\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=\begin{cases}0, & \text{if } M_i'\not\subset M_i \text{ and } M_i\not\subset M_i';\\ 1\times\mathrm{PD}_{\bar{\mathcal{U}}_{\bar{\mathcal{T}}}(\mu)}\bar{\mathcal{U}}_{\bar{\mathcal{T}}(M_i'-M_i)}(\mu), & \text{if } M_i\subsetneq M_i';\\ \mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}_0}}\bar{\mathcal{U}}_{\mathcal{T}_0(M_i-M_i')}\times 1, & \text{if } M_i'\subsetneq M_i.\end{cases}\tag{3.35}$$

where $\mathcal{T}_0=(S^2,\hat{1}+M_i,\{i\};i,0)$, i.e. $\bar{\mathcal{U}}_{\mathcal{T}_0}=\bar{\mathcal{M}}_{0,\hat{1}+M_i}$. Plugging (3.35), (2) of Lemma 3.11, and the case $i'=i$ of the first statement of this lemma into (3.34), we obtain the remaining claim.

Corollary 3.9 *For all $k\geq 1$, $m\geq 0$, and $l\geq 0$,*

$$\big\langle a^l\tilde{\eta}_{n+1-(2k+m)-l},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle=\sum_{m^*\geq m}\binom{m^*}{m}k^{m^*-m}\big\langle a^l\eta_{n+1-(2k+m^*)-l},\big[\bar{\mathcal{V}}_{k,m^*}(\mu)\big]\big\rangle.$$

Proof: Let $\mathcal{T}=(S^2,[N]-M_0,I;j,\underline{d})$ be a basic bubble type such that $|I|=k$, $|M_0|=m$, and $\sum d_i=d$. By Lemma 3.8 and (1) of Lemma 3.11,

$$\big\langle a^l\tilde{\eta}_{n+1-(2k+m)-l},\big[\bar{\mathcal{U}}_{\mathcal{T}}(\mu)\big]\big\rangle=\sum_{M_0\subset M_0^*\subset[N]}\big\langle a^l\eta_{n+1-(2k+|M_0^*|)-l},\big[\bar{\mathcal{U}}_{\mathcal{T}/M_0^*}(\mu)\big]\big\rangle,\tag{3.36}$$

where $\mathcal{T}/M_0^*=(S^2,[N]-M_0^*,I;j,\underline{d})$. The claim is obtained by summing (3.36) over all equivalence classes of bubble types $\mathcal{T}$ of the above form.

Corollary 3.10 *For all $k\geq 1$ and $l\geq 0$,*

$$\sum_{m\geq 0}\Theta(k,m)\big\langle a^l\tilde{\eta}_{n+1-(2k+m)-l},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle=(-1)^{k-1}(k-1)!\big\langle a^l\eta_{n+1-2k-l},\big[\bar{\mathcal{V}}_k(\mu)\big]\big\rangle.$$

Proof: By Lemma 3.7 and Corollary 3.9,

$$\begin{aligned}&\sum_{m\geq 0}\Theta(k,m)\big\langle a^l\tilde{\eta}_{n+1-(2k+m)-l},\big[\bar{\mathcal{V}}_{k,m}(\mu)\big]\big\rangle\\
&\qquad=(-1)^{k-1}(k-1)!\sum_{m\geq 0}\sum_{m^*\geq m}(-1)^m\binom{m^*}{m}k^{m^*}\big\langle a^l\eta_{n+1-(2k+m^*)-l},\big[\bar{\mathcal{V}}_{k,m^*}(\mu)\big]\big\rangle\\
&\qquad=(-1)^{k-1}(k-1)!\sum_{m^*\geq 0}k^{m^*}\bigg(\sum_{m\leq m^*}(-1)^m\binom{m^*}{m}\bigg)\big\langle a^l\eta_{n+1-(2k+m^*)-l},\big[\bar{\mathcal{V}}_{k,m^*}(\mu)\big]\big\rangle\\
&\qquad=(-1)^{k-1}(k-1)!\big\langle a^l\eta_{n+1-2k-l},\big[\bar{\mathcal{V}}_{k,0}(\mu)\big]\big\rangle,\end{aligned}$$

since

$$\sum_{m\le m^*}(-1)^m\binom{m^*}{m}k^{m^*}=(1-1)^{m^*}=0 \qquad \text{if } \ m^*\neq 0.$$

Lemma 3.11 *(1) If J is a finite set of cardinality at least two, $\big\langle c_1^{|J|-2}(\gamma_J^*),\big[\overline{\mathfrak{M}}_J\big]\big\rangle=1$, where $\gamma_J\longrightarrow\overline{\mathfrak{M}}_J$ is the tautological line bundle.*
(2) If $\mathcal{T}=(S^2,M,I;j,\underline{d})$ is a basic bubble type, $i\in I$, and M_i is nonempty subset of $M_i\mathcal{T}$, under the splitting (2.2),

$$PD_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=-1\times c_1(L_i^*\bar{\mathcal{T}})+c_1(\gamma^*_{\mathcal{T};i})\times 1-\sum_{\emptyset\neq M_i'\subsetneq M_i}PD_{\bar{\mathcal{U}}_{\mathcal{T}_0}}\bar{\mathcal{U}}_{\mathcal{T}_0(M_i-M_i')}\times 1,$$

where $\mathcal{T}_0=(S^2,\hat{1}+M_i,\{i\};i,0)$ and $\bar{\mathcal{T}}=\mathcal{T}/M_i$.

Proof: (1) Both statements are straightforward consequences of well-known facts in algebraic geometry; see [P2]. In our notation, $\overline{\mathfrak{M}}_J$ is the Deligne-Mumford moduli space of rational curves with points marked by the set $\{\hat{0}\}+J$ and $c_1(\gamma_J^*)=\psi_{\hat{0}}$. Thus, if $j_1,j_2\in J$ and $j_1\neq j_2$,

$$c_1(\gamma_J^*)=\psi_{\hat{0}}=\sum_{\emptyset\neq J'\subset J-\{j_1,j_2\}}\mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}_0}}\bar{\mathcal{U}}_{\mathcal{T}_0(J')}, \tag{3.37}$$

where $\mathcal{T}_0=(S^2,J,\{i\};i,0)$. Since $c_1(\gamma_J^*)\big|\bar{\mathcal{U}}_{\mathcal{T}_0(J')}=c_1(\gamma^*_{J'+\hat{1}})$ under the decomposition (2.2), the first claim of the lemma follows from (3.37).
(2) Equation (3.37) implies that for any $\hat{1}\in J$,

$$c_1(\gamma_J^*)+\psi_{\hat{1}}=\sum_{\emptyset\neq J'\subsetneq J-\{\hat{1}\}}\mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}_0}}\bar{\mathcal{U}}_{\mathcal{T}_0(J')}. \tag{3.38}$$

If $\mathcal{T}$, i, and M_i are as in (2) of the lemma, under the splitting (2.2),

$$\mathrm{PD}_{\bar{\mathcal{U}}_{\mathcal{T}}(\mu)}\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)\big|\bar{\mathcal{U}}_{\mathcal{T}(M_i)}(\mu)=-\psi_{\hat{1}}\times 1-1\times\psi_{\hat{0}}. \tag{3.39}$$

The second claim of the lemma follows from (3.38), applied with $J=\{\hat{1}\}+M_i$, and (3.39), since $1\times\psi_{\hat{0}}=1\times c_1(L_i\bar{\mathcal{T}})$.

4 Comparison of $n_d^{(1)}(\mu)$ and $n_{1,d}(\mu)$

4.1 Summary

In this section, we prove

Proposition 4.1 *Suppose $n\ge 2$, $d\ge 1$, and $\mu=(\mu_1,\ldots,\mu_N)$ is an N-tuple of proper linear subspaces of $\mathbb{P}^n$ in general position such that $\mathrm{codim}_{\mathbb{C}}\mu=d(n+1)-1$. Then*

$$n_d^{(1)}(\mu)=n_{1,d}(\mu).$$

Denote by $\overline{\mathfrak{M}}_{1,1}$ the Deligne-Mumford moduli space of stable genus-one curves with one marked point and by $\mathfrak{M}_{1,1}$ the main stratum of $\overline{\mathfrak{M}}_{1,1}$, i.e. the complement of the point ∞ in $\overline{\mathfrak{M}}_{1,1}$. The elements of $\mathfrak{M}_{1,1}$ parameterize (equivalence classes of) smooth genus-one curves with one marked point. The point $\infty \in \overline{\mathfrak{M}}_{1,1}$ corresponds to a sphere with one marked point and with two other points identified.

Denote by $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{1,N}(\mathbb{P}^n, d)$ the moduli space of stable degree-d maps from N-pointed genus-one curves to $\mathbb{P}^n$. Let

$$\overline{\mathfrak{M}}(\mu) = \big\{ b \in \overline{\mathfrak{M}} \colon \mathrm{ev}_l(b) \in \mu_l \ \forall l \in [N] \big\}.$$

We denote by $\pi \colon \overline{\mathfrak{M}} \longrightarrow \overline{\mathfrak{M}}_{1,1}$ the forgetful functor sending each stable map $b = [S, [N], I; x, (j, y), u]$ to the one-marked curve $[S, y_1]$ and contracting all unstable components of (S, y_1). The resulting complex curve is either a torus or a sphere with two points identified. For any $\sigma \in \overline{\mathfrak{M}}_{1,1}$, let

$$\overline{\mathfrak{M}}_\sigma = \pi^{-1}(\sigma), \qquad \overline{\mathfrak{M}}_\sigma(\mu) = \overline{\mathfrak{M}}_\sigma \cap \overline{\mathfrak{M}}(\mu).$$

If the j-invariant σ is different from infinity, i.e. the stable curve $\mathcal{C}_\sigma$ corresponding to σ is smooth, the cardinality of $\overline{\mathfrak{M}}_\sigma(\mu)$ is $\big|\mathrm{Aut}(\mathcal{C}_\sigma)\big|$ times the number of genus-one degree-d curves with j-invariant σ that pass through the constraints μ, i.e.

$$\big|\overline{\mathfrak{M}}_\sigma(\mu)\big| = 2n_{1,d}(\mu). \tag{4.1}$$

If $\{\sigma_k\} \subset \mathfrak{M}_{1,1}$ converges to $\infty \in \overline{\mathfrak{M}}_{1,1}$ and $b_k \in \overline{\mathfrak{M}}_{\sigma_k}(\mu)$, a subsequence of $\{b_k\}$ converges in $\overline{\mathfrak{M}}$ to some $b \in \overline{\mathfrak{M}}_\infty(\mu)$. It will be shown that Σ_b is a sphere with two points identified; see Lemma 4.2 and Corollary 4.5. Conversely, for every

$$b = (S, [N], \{\hat{0}\};, (\hat{0}, y), u) \in \overline{\mathfrak{M}}_\infty(\mu)$$

such that Σ_b is a sphere with two points identified and for every $\sigma \in \mathfrak{M}_{1,1}$ sufficiently close to ∞, there exists a unique stable map $b(\sigma) \in \overline{\mathfrak{M}}_\sigma(\mu)$ close to b in $\overline{\mathfrak{M}}$; see Lemma 4.3. Since the number of stable maps

$$b = (S, [N], \{\hat{0}\};, (\hat{0}, y), u) \in \overline{\mathfrak{M}}_\infty(\mu)$$

such that Σ_b is a sphere with two points identified is $2n_d^{(1)}(\mu)$, Proposition 4.1 follows from the two lemmas, the corollary, and equation (4.1).

4.2 Dimension Counts

In this subsection, we show that if

$$[b] = \big[S, [N], I; x, (j, y), u\big] \in \overline{\mathfrak{M}}_\infty(\mu)$$

and $u_{\hat{0}} = u_b|S$ is not constant, then $\Sigma_b = S$ is a sphere with two points identified; see Lemma 4.2. This lemma is proved by dimension counting. We then observe that for each such stable map b and every $\sigma \in \mathfrak{M}_{1,1}$ sufficiently close to ∞, there exists a unique stable map $b(\sigma) \in \overline{\mathfrak{M}}_\sigma(\mu)$ close to b in $\overline{\mathfrak{M}}$; see Lemma 4.3.

Lemma 4.2 *If* $[b]=\big[S,[N],I;x,(j,y),u\big]\in\overline{\mathfrak{M}}_\infty(\mu)$ *and* $u_{\hat{0}}=u_b|S$ *is not constant, then* $\Sigma_b=S$ *is a sphere with two points identified.*

Proof: Suppose $\mathcal{T}=(S,[N],I;j,\underline{d})$ is a simple bubble type such that S is a circle of k spheres, $d_{\hat{0}}\neq 0$, and $\sum d_i=d$. Let $\mathcal{U}_{\mathcal{T},\underline{d}_{\hat{0}}}$ denote the subspace of $\mathcal{U}_{\mathcal{T}}$ such that the nonconstant restrictions of u_b to the components of S have degrees $d_{\hat{0},1},\ldots,d_{\hat{0},k'}$ for all $b\in U_{\mathcal{T},\underline{d}_{\hat{0}}}$. We must have $\sum d_{\hat{0},l}=d_{\hat{0}}$. Then, the dimension of $\mathcal{U}_{\mathcal{T},\underline{d}_{\hat{0}}}(\mu)$ is given by

$$\Big(\sum_{l=1}^{k'}\big(d_{\hat{0},l}(n+1)+n-1\big)-nk'+\sum_{i\in\hat{I}}\big(d_i(n+1)+n-2-(n-1)\big)+\big(N-(k-k')\big)\Big)-\big(\text{codim}_{\mathbb{C}}\mu+N\big)$$
$$=1-|k|-|\hat{I}|.$$

Thus, $\mathcal{U}_{\mathcal{T}}(\mu)=\emptyset$ unless $k=1$ and $\hat{I}=\emptyset$, i.e. $\Sigma_b=S$ is a sphere with two points identified.

Lemma 4.3 *For every* $[b]=\big[S,[N],\{\hat{0}\};,(\hat{0},y),u\big]\in\overline{\mathfrak{M}}_\infty(\mu)$ *such that* S *is a sphere with two points identified, there exists neighborhood* U_b *of* ∞ *in* $\overline{\mathfrak{M}}_{1,1}$ *and* W_b *of* b *in* $\overline{\mathfrak{M}}_{1,N}(\mathbb{P}^n,d)$ *such that*

$$\big|\overline{\mathfrak{M}}_\sigma(\mu)\cap W_b\big|=1\qquad\forall\sigma\in U_b-\{\infty\}.$$

Proof: Since $d\geq 1$,
$$H^1(S;u_b^*T\mathbb{P}^n)=(n+1)H^1\big(S;u_b^*\mathcal{O}(1_{\mathbb{P}^n})\big)=0;\tag{4.2}$$
see Corollary 6.5 in [Z2] for example. The lemma follows from (4.2) by standard arguments. A purely analytic proof can be found in [RT].

4.3 A Property of Limits in $\overline{\mathfrak{M}}_{1,N}(\mathbb{P}^n,d)$

Suppose $\{\sigma_k\}\subset\mathfrak{M}_{1,1}$ converges to $\infty\in\overline{\mathfrak{M}}_{1,1}$ and $b_k\in\overline{\mathfrak{M}}_{\sigma_k}$ converges to

$$[b]=\big[S,[N],I;x,(j,y),u\big]\in\overline{\mathfrak{M}}_\infty$$

such that $u_b|S$ is constant. In this subsection, we describe a condition such a limit b must satisfy; see Lemma 4.4. This lemma is the key part of Section 4. Its proof extends the argument of [P1] for the $n=2$ case and makes use of the explicit notation described in Subsection 2.2. We conclude by observing that no element of $\overline{\mathfrak{M}}_\infty(\mu)$ can satisfy this condition if the constraints μ are in general position.

Figure 9 illustrates in some cases the condition described by Lemma 4.4. The second picture, however, is somewhat misleading. The two nodes of the domain at which the arrows point are mapped to the same point, which is a “tacnode,” according to Lemma 4.4. It is a “tacnode” in the sense that the span of the lines tangent to the two branches at the node of the image curve in $\mathbb{P}^n$ is at most one-dimensional. In particular, one or both of the branches might be cuspidal. The proof of Lemma 4.4 shows that the branch corresponding to the upper node is in fact a cusp.

Lemma 4.4 *Suppose*

$$[b]=\big[S,[N],I;x,(j,y),u\big]\in\overline{\bigcup_{\sigma\in\mathfrak{M}_{1,1}}\overline{\mathfrak{M}}_\sigma}\cap\mathcal{M}_{\mathcal{T}},$$

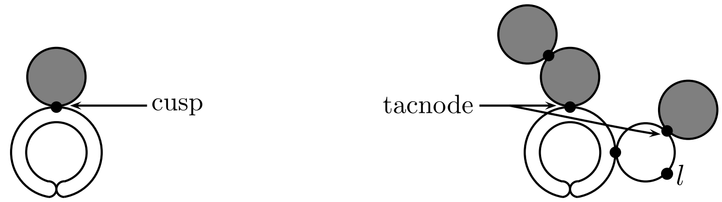


Figure 9: Properties of Images of Some Elements in the Closure of $\bigcup \mathfrak{M}_\sigma$

where $\mathcal{T}=(S,[N],I;j,\underline{d})$ *is a simple bubble type such that* S *is a circle of spheres and* $d_{\hat{0}}=0$. *Then the dimension of the linear span of the set* $\{du_h\big|_\infty e_\infty : h\in\chi(\mathcal{T})\}$ *is less than* $\big|\chi(\mathcal{T})\big|$.

Proof: (1) By the algebraic geometry definition of stable-map convergence, there exist
(i) a one-parameter family of curves $\tilde{\kappa}: \tilde{\mathcal{F}} \longrightarrow \Delta$ such that Δ is a neighborhood of 0 in $\mathbb{C}$, $\tilde{\mathcal{F}}$ is a smooth space, $\tilde{\kappa}^{-1}(0)=\Sigma_b$, and $\Sigma_t\equiv\tilde{\kappa}^{-1}(t)$ is a smooth genus-one curve for all $t\in\Delta^*\equiv\Delta-\{0\}$;
(ii) a holomorphic map $\tilde{u}: \tilde{\mathcal{F}}\longrightarrow\mathbb{P}^n$ such that $\tilde{u}|\kappa^{-1}(0)=u_b$.
This family $\tilde{\kappa}: \tilde{\mathcal{F}}\longrightarrow\Delta$ can be obtained from another family of curves $\kappa_{\hat{0}}: \mathcal{F}_{\hat{0}}\longrightarrow\Delta$ that satisfies (i), except $\kappa_{\hat{0}}^{-1}(0)=S$, by a sequence of blowups at smooth points of the central fiber as we now describe. Choose an ordering $\prec$ of the set I consistent with its partial ordering. If $h\in I$, let

$$I^h=\{i\in I: i\prec h\},\quad i(h)=\max I^h \text{ if } h\in\hat{I},\quad I^{(h)}=I^h\cup\{h\},\quad M(h)=\{l\in[N]: j_l\preceq h\},$$
$$b(h)=\big(S^2, M(h), I^{(h)}; x|\hat{I}^{(h)}, (j,y)|M(h), u|I^{(h)}\big).$$

Suppose $h\in\hat{I}$ and we have constructed a one-parameter family of curves $\kappa_{i(h)}: \mathcal{F}_{i(h)}\longrightarrow\Delta$ that satisfies (i), except $\kappa_{i(h)}^{-1}(0)=\Sigma_{b(i(h))}$. Let $\mathcal{F}_h$ be the blowup of $\mathcal{F}_{i(h)}$ at the smooth point of (ι_h, x_h) of $\Sigma_{b(i(h))}$ and let $\kappa_h: \mathcal{F}_h\longrightarrow\Delta$ be the induced projection map. Choose coordinates (t,w_h) near $(\iota_h,x_h)\in\mathcal{F}_{i(h)}$ such that $d\kappa_{i(h)}\frac{\partial}{\partial w_h}=0$, i.e. w_h is a coordinate in $\kappa_{i(h)}^{-1}(t)$ for $t\in\Delta$ sufficiently small. We define coordinates (t,z_h) on a neighborhood in $\mathcal{F}_h$ of the complement of the node of the new exceptional divisor by

$$(t,z_h)\longrightarrow\big(t, w_h=tz_h, [1,z_h]\big).$$

For a good choice of the family $\kappa_{\hat{0}}: \mathcal{F}_{\hat{0}}\longrightarrow\Delta$, $\tilde{\mathcal{F}}=\mathcal{F}_{h^*}$ and $\tilde{\pi}=\pi_{h^*}$, where h^* is the largest element of I with respect to the ordering $\prec$.
(2) Let $\psi\in H^0(S;w_S)$ be a nonzero differential, i.e. ψ is a holomorphic $(1,0)$-form on the components of S, which has simple poles at the singular points of S with residues that add up to zero at each node. Then, for each $h\in H_{\hat{0}}\mathcal{T}$, there exists $a_h\in\mathbb{C}^*$ such that

$$\psi|_{(0,w_h)}=a_h\big(1+o(1)\big)dw_h.$$

Thus, we can extend ψ to a family of elements $\psi_t\in H^0(\Sigma_t;\omega_{\Sigma_t})$ such that

$$\psi|_{(t,w_h)}=a_h\big(1+o(1)\big)dw_h,\quad \text{with}\;\; a_h\in\mathbb{C}^*. \tag{4.3}$$

If $h\in\hat{I}$, let $|h|=\big|\{i\in I: i<h\}\big|$. Denote by $\tilde{h}$ the element of $H_{\hat{0}}\mathcal{T}$ such that $h\in\bar{D}_{\tilde{h}}\mathcal{T}$. By (4.3), we have

$$\psi|_{(t,z_h)}=t^{|h|}a_{\tilde{h}}\big(1+o(1_t)\big)dz_h,\quad \text{with}\;\; a_{\tilde{h}}\in\mathbb{C}^*. \tag{4.4}$$

(3) Let H_1 and H_2 be any two hyperplanes in $\mathbb{P}^n$ that intersect the image of u_b transversally and miss the image of the nodes of Σ_b. Then for all t sufficiently small and $i=1,2$,

$$u_t^{-1}(H_i) = \{z_{1,h_1}^{(i)}(t), \ldots, z_{d,h_d}^{(i)}\} \subset \Sigma_t, \quad \text{where} \quad h_j \in \hat{I}, \quad z_{j,h_j}^{(i)}(t) = z_{j,h_j}^{(i)}(0) + o(1_t), \tag{4.5}$$

$z_{j,h_j}^{(i)}(0) \in \Sigma_{b,h}$, and $u_t = \tilde{u}|\Sigma_t$. Since $\sum z_{h_j}^{(1)}(t)$ and $\sum z_j^{(2)}(t)$ are linearly equivalent divisors in Σ_t,

$$\sum_{j=1}^{j=d} \int_{z_{j,h_j}^{(1)}(t)}^{z_{j,h_j}^{(2)}(t)} \psi_t = 0 \qquad \forall t \in \Delta^*, \tag{4.6}$$

where each line integral is taken inside of an appropriate coordinate chart (t, z_h). Plugging (4.4) and (4.5) into (4.6) gives

$$\sum_{j=1}^{j=d} t^{|h_j|} a_{\tilde{h}_j} \big(z_{j,h_j}^{(2)}(0) - z_{j,h_j}^{(1)}(0) + o(1_t)\big) = 0 \qquad \forall t \in \Delta^*. \tag{4.7}$$

Let $k = \min\{|h| \colon h \in \chi(\mathcal{T})\}$; then $k = \min\{|h_j| \colon j \in [d]\}$. Thus, dividing equation (4.7) by t^k and then taking the limit as $t \longrightarrow 0$, we conclude that

$$\sum_{|h_j|=k} a_{\tilde{h}_j} z_{j,h_j}^{(1)}(0) = \sum_{|h_j|=k} a_{\tilde{h}_j} z_{j,h_j}^{(2)}(0). \tag{4.8}$$

(4) Equality (4.8) holds for a dense subset of pairs (H_1, H_2). The consequences of this fact can be interpreted as follows. For each $h \in \hat{I}$, let $[u_h, v_h]$ be homogeneous coordinates on $\Sigma_{b,h}$ such that $z_h = v_h/u_h$. Each map u_h corresponds to an $(n+1)$-tuple of homogeneous polynomials

$$p_{h,i} = \sum_{l=0}^{l=d_h} p_{h,i;l} u^l v^{d-l}, \qquad i = 0, \ldots, n, \quad p_{h,i;l} \in \mathbb{C}.$$

Equality (4.8) implies that there exists $K \in \mathbb{C}$ such that

$$\sum_{|h|=k, d_h \neq 0} a_{\tilde{h}} \frac{\sum_{i=0}^{i=n} c_i p_{h,i;d_h-1}}{\sum_{i=0}^{i=n} c_i p_{h,i;d_h}} = K \qquad \forall [c_0, \ldots, c_n] \in \mathbb{P}^n. \tag{4.9}$$

On the other hand, $u_{h_1}(\infty) = u_{h_2}(\infty)$ for all $h_1, h_2 \in \chi(\mathcal{T})$. Thus, for all $h_1, h_2 \in \chi(\mathcal{T})$, there exists $K_{h_1,h_2} \in \mathbb{C}^* - \{0\}$ such that

$$\big(p_{h_1,0;d_{h_1}}, \ldots, p_{h_1,n;d_{h_1}}\big) = K_{h_1,h_2} \big(p_{h_2,0;d_{h_2}}, \ldots, p_{h_2,n;d_{h_2}}\big).$$

It follows that (4.9) is equivalent to

$$\begin{aligned} &\sum_{i=0}^{i=n} \sum_{|h|=k, d_h \neq 0} \tilde{a}_h p_{h,i;d_h-1} c_i = K \sum_{i=0}^{i=n} p_{h_1,i;d_{h_1}} c_i \quad \forall c_i \in \mathbb{C} \Longrightarrow \\ &\qquad \sum_{|h|=k, d_h \neq 0} \tilde{a}_h p_{h,i;d_h-1} = K p_{h_1,i;d_{h_1}}, \qquad i = 0, \ldots, n. \end{aligned} \tag{4.10}$$

where h_1 is a fixed element of the set $\{h\in\hat{I}\colon |h|=k, d_h\neq 0\}$ and $\tilde{a}_h\in\mathbb{C}^*$. It is straightforward to deduce from (4.10) that

$$\sum_{|h|=k,d_h\neq 0}\tilde{a}_h du_h\big|_\infty e_\infty = 0.$$

The lemma is now proved, since $\{h\in\hat{I}\colon |h|=k, d_h\neq 0\}\subset\chi(\mathcal{T})$.

Corollary 4.5 *Suppose*

$$[b]=\big[S,[N],I;x,(j,y),u\big]\in\overline{\bigcup_{\sigma\in\mathfrak{M}_{1,1}}\overline{\mathfrak{M}}_\sigma}\cap\overline{\mathfrak{M}}_\infty(\mu).$$

Then $u_b|S$ is not constant.

Proof: Suppose $u_b|S$ is constant. Let

$$\tilde{I}=\big\{i\in I\colon \chi_{\mathcal{T}}i\neq 0\big\}\subset\hat{I},\quad M_0=\bigcup_{i\in I-\tilde{I}}M_i\mathcal{T},\quad \tilde{x}=x|\hat{\tilde{I}},\quad (\tilde{j},\tilde{y})=(j,y)\big|\big([N]-M_0\big),\quad \tilde{d}=d|\tilde{I},\quad \tilde{u}=u|\tilde{I};$$
$$\tilde{\mathcal{T}}=\big(S^2,[N]-M_0,\tilde{I};\tilde{j},\tilde{d}\big),\quad \tilde{b}=\big(S^2,[N]-M_0,\tilde{I};\tilde{x},(\tilde{j},\tilde{y}),\tilde{u}\big).$$

Then, $\tilde{\mathcal{T}}$ is a bubble type such that $\sum\tilde{d}_i=d$ and $\tilde{d}_i>0$ for all $i\in\tilde{I}-\hat{\tilde{I}}$. The latter property implies that $\chi(\tilde{\mathcal{T}})=\tilde{I}-\hat{\tilde{I}}$. Furthermore, $\tilde{b}\in\mathcal{U}_{\tilde{\mathcal{T}}}(\mu)$. By Lemma 4.4, the linear map

$$\alpha_{|\chi(\tilde{\mathcal{T}})|,|M_0|}\colon\bigoplus_{i\in\chi(\tilde{\mathcal{T}})}L_i\tilde{\mathcal{T}}\longrightarrow \mathrm{ev}^*T\mathbb{P}^n,\quad \alpha_{|\chi(\tilde{\mathcal{T}})|,|M_0|}(\upsilon)=\sum_{i\in\chi(\tilde{\mathcal{T}})}\mathcal{D}_{\tilde{\mathcal{T}},i}\upsilon_i,$$

does not have full rank at $\tilde{b}$. However, this is impossible by Lemma 2.8.

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