A Desingularization of
(the Main Component of)

the Moduli Space of Genus-One Stable Maps into $\mathbb{P}^n$

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slides to appear at
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Overview

- $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) = \{$genus-1 stable maps into $\mathbb{P}^n$ with k marked pts$\}$

  $\bigcup$

- $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) =$ *main* irred. component of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$

Goal: construct desingularization

$$\pi : \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$$

Approach:

Blow up other components of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$ *and* similar subvarieties

Good Properties of $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$

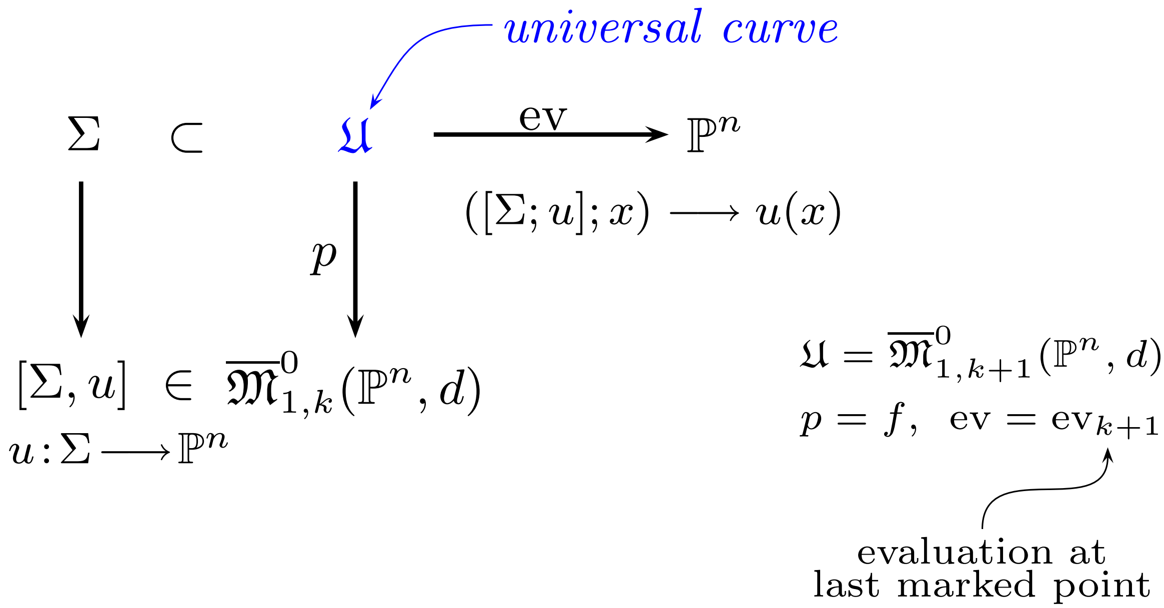
1. $m<n \implies \mathbb{P}^m \hookrightarrow \mathbb{P}^n$:

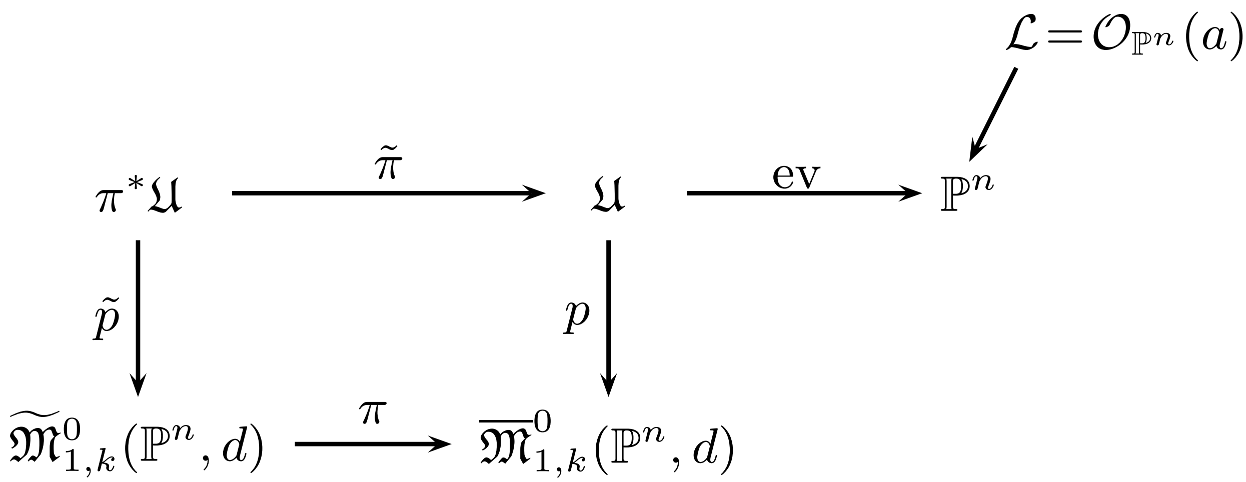
$$\begin{CD}
\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^m,d) @>{\tilde{i}}>> \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\\
@V{\pi}VV @V{\pi}VV\\
\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^m,d) @>{i}>> \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)
\end{CD}$$

2.

$$\begin{CD}
\widetilde{\mathfrak{M}}^0_{1,k+1}(\mathbb{P}^n,d) @>{\tilde{f}}>> \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)\\
@V{\pi}VV @V{\pi}VV\\
\overline{\mathfrak{M}}^0_{1,k+1}(\mathbb{P}^n,d) @>{f}>> \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)
\end{CD}$$

3.


universal curve
$\Sigma \subset \mathfrak{U} \xrightarrow{\mathrm{ev}} \mathbb{P}^n$
$([\Sigma; u]; x) \longrightarrow u(x)$
p
$[\Sigma, u] \in \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$
$u\colon \Sigma \longrightarrow \mathbb{P}^n$
$\mathfrak{U} = \overline{\mathfrak{M}}^0_{1,k+1}(\mathbb{P}^n, d)$
$p = f, \quad \mathrm{ev} = \mathrm{ev}_{k+1}$
evaluation at
last marked point


$\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(a)$
$\pi^*\mathfrak{U}$
$\tilde{\pi}$
$\mathfrak{U}$
ev
$\mathbb{P}^n$
$\tilde{p}$
p
$\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$
π
$\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)$

$$
\begin{array}{ccc}
\tilde{\mathcal{V}}(a) = \tilde{p}_* \tilde{\pi}^* \mathrm{ev}^* \mathcal{L} & & \mathcal{V}(a) = p_* \mathrm{ev}^* \mathcal{L} \\
\big\downarrow & & \big\downarrow \\
\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) & \xrightarrow{\ \pi\ } & \overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d)
\end{array}
$$

Thm: $\tilde{\mathcal{V}}(a)$ is locally free

$$
\begin{array}{ccc}
\tilde{\mathcal{V}}(a) \sim \widetilde{\mathfrak{M}}^0_{1,k}(\mathcal{L}, d) & \xrightarrow{\ \tilde{\pi}\ } & \mathcal{V}(a) \sim \overline{\mathfrak{M}}_{1,k}(\mathcal{L}, d) \\
\big\downarrow & & \big\downarrow \\
\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) & \xrightarrow{\ \pi\ } & \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d)
\end{array}
$$

Applications

Enumerative Geometry

counts of genus-1 curves in $\mathbb{P}^n$

$$= \int_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)} e(V) \qquad = \underbrace{\int_{\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)} \pi^* e(V)}_{\text{compute by localization}}$$

v.b.

Gromov-Witten Theory

$Y_a \subset \mathbb{P}^n$ hypersurface of degree a

Thm (J. Li, Z.-):

$$\mathrm{GW}_{1,k}(Y_a) = \int_{\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)} e(\mathcal{V}(a)) + \mathrm{GW}_{0,*}(Y_a)$$

$$= \underbrace{\int_{\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)} e(\tilde{\mathcal{V}}(a))}_{\text{compute by localization}} \quad + \mathrm{GW}_{0,*}(Y_a)$$

Approach to Proof:

- Describe structure of $\overline{\mathfrak{M}}_1$

- Describe changes after each step of blowup

- Do *idealized* blowup at each step:
  - blow up along variety $\Pr \overline{\mathfrak{M}}_1^l$
  - attach $\mathbb{P}\mathcal{N}_l^{\text{ide}}$
    $\mathcal{N}_l^{\text{ide}} =$ *idealized* normal bundle for $\Pr \iota_l$

  helps describing $\Pr \overline{\mathfrak{M}}_1^0$

Notation: Genus-1 Curves

- $\overline{\mathcal{M}}_{1,l} = \{\text{genus-1 curves with } l \text{ marked pts}\}$

- $L_k, \mathbb{E} \longrightarrow \overline{\mathcal{M}}_{1,l}$ natural line bundles; $k=1,\ldots,l$
  - $\mathbb{E}$ = Hodge l.b.: $\mathbb{E}|_{[\Sigma,y_1,\ldots,y_l]} = H^0(\Sigma; T^*\Sigma)$
  - L_k = universal tangent l.b. for y_k:
    $$L_k|_{[\Sigma,y_1,\ldots,y_l]} = T_{y_k}\Sigma$$

- $s_k : L_k \longrightarrow \mathbb{E}^*$ natural homomorphism:
  $$\big\{s_k(v)\big\}(\psi) = \psi|_{y_k}(v) \in \mathbb{C}$$

Notation: Genus-0 Maps

- $\overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d) = \{\text{genus-0 maps w. 1 marked pt}\}$
- $\text{ev}_1 \colon \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d) \longrightarrow \mathbb{P}^n$ evaluation at marked pt

  $$[\Sigma, y_1, u] \longrightarrow u(y_1)$$

- $L_1 \longrightarrow \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d)$ universal tangent l.b. at y_1

- $\mathcal{D}_1 \colon L_1 \longrightarrow \text{ev}_1^* T\mathbb{P}^n$ differential map:

  $$\mathcal{D}_1|_{[\Sigma, y_1, u]}(v) = du|_{y_1} v \in T_{u(y_1)}\mathbb{P}^n$$

- $$\overline{\mathfrak{M}}_{0,(l)} \subset \bigcup_{\substack{d_1+\ldots+d_l=d \\ d_1,\ldots,d_l>0}} \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d_1) \times \ldots \times \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d_l)$$

  ev_1 agree on $\overline{\mathfrak{M}}_{0,(l)}$

- $$\pi_k \colon \overline{\mathfrak{M}}_{0,(l)} \longrightarrow \bigcup_{d_k=1}^{d_k=d} \overline{\mathfrak{M}}_{0,1}(\mathbb{P}^n, d_k) \quad \text{projection}$$

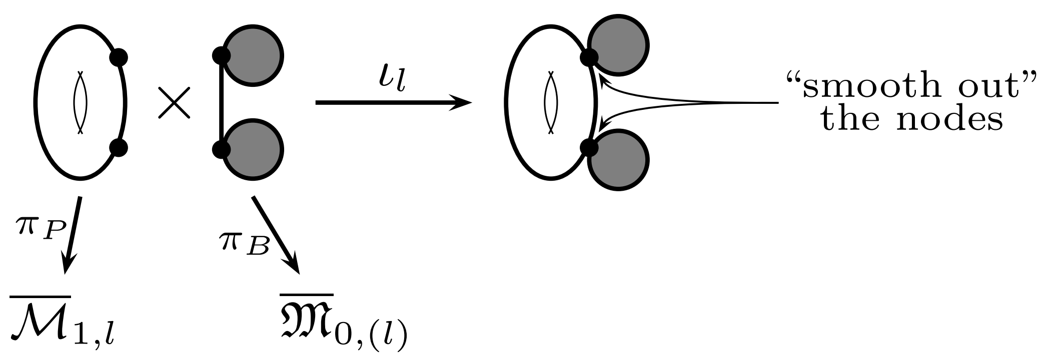
Structure of $\overline{\mathfrak{M}}_1$

- $\overline{\mathfrak{M}}_1 = \overline{\mathfrak{M}}_1^0 \cup \bigcup_{l\ge 1} \overline{\mathfrak{M}}_1^l$; $\overline{\mathfrak{M}}_1^0 - \bigcup_{l\ge 1} \overline{\mathfrak{M}}_1^l$ is smooth

- $\iota_l : (\overline{\mathcal{M}}_{1,l} \times \overline{\mathfrak{M}}_{0,(l)})/S_l \longrightarrow \overline{\mathfrak{M}}_1^l \subset \overline{\mathfrak{M}}_1$
  node-identifying immersion
  embedding outside of $\bigcup_{l'<1} \overline{\mathfrak{M}}_1^{l'}$

- *idealized* normal bundle for ι_l:

$$\mathcal{N}_l^{\text{ide}} = \bigoplus_{k=1}^{k=l} \pi_P^* L_k \otimes \pi_B^* \pi_k^* L_1 \longrightarrow \overline{\mathcal{M}}_{1,l} \times \overline{\mathfrak{M}}_{0,(l)}$$



Describe structure of $\overline{\mathfrak{M}}_1^0 \cap \overline{\mathfrak{M}}_1^l$:

$\overline{\mathcal{Z}}_l = \iota_l^{-1}(\overline{\mathfrak{M}}_1^0)$, $\mathcal{Z}_l = \overline{\mathcal{Z}}_l \cap (\mathcal{M}_{1,l} \times \mathfrak{M}_{0,(l)})$,
normal cone $\mathcal{N}\overline{\mathcal{Z}}_l \subset \mathcal{N}_l^{\mathrm{ide}}$ for immersion $\iota_l : \overline{\mathcal{Z}}_l \longrightarrow \overline{\mathfrak{M}}_1^0$

Define bundle homomorphism

$$\mathcal{D}_{(l)} : \mathcal{N}_l^{\mathrm{ide}} = \bigoplus_{k=1}^{k=l} \pi_P^* L_k \otimes \pi_B^* \pi_k^* L_1 \longrightarrow \pi_P^* \mathbb{E}^* \otimes \mathrm{ev}_1^* T\mathbb{P}^n$$

$$\mathcal{D}_{(l)}|_{\pi_P^* L_k \oplus \pi_B^* \pi_l^* L_1} = \pi_P^* s_k \otimes \pi_B^* \pi_k^* \mathcal{D}_1$$

Proposition

- $\mathcal{Z}_l = \{u \in \mathcal{M}_{1,l} \times \mathfrak{M}_{0,(l)} : \ker \mathcal{D}_{(l)}|_u \neq 0\}$
- $\mathcal{N}\overline{\mathcal{Z}}_l|_{\mathcal{Z}_l} = \ker \mathcal{D}_{(l)}$
- $\overline{\mathcal{Z}}_l$ = closure of $\mathcal{Z}_l$ in $\overline{\mathcal{M}}_{1,l} \times \overline{\mathfrak{M}}_{0,(l)}$
- $\mathcal{N}\overline{\mathcal{Z}}_l$ = closure of $\mathcal{N}\overline{\mathcal{Z}}_l|_{\mathcal{Z}_l}$ in $\mathcal{N}_l^{\mathrm{ide}}$

The lth step of the blowup construction:

- blow up along proper transform of $\overline{\mathfrak{M}}_1^l$
  a smooth subvariety

- attach *idealized exceptional divisor* $\mathbb{P}\tilde{\mathcal{N}}_l^{\mathrm{ide}}$
  along exceptional divisor $\mathcal{E}_l$
  $\tilde{\mathcal{N}}_l^{\mathrm{ide}}$ = idealized normal bundle for $\mathrm{Pr}\,\iota_l$

- describe changes in $\iota_{l'}$, $\mathcal{Z}_{l'}$, $\mathcal{N}\mathcal{Z}_{l'}$ $\quad \forall\, l'$

End Result:

- Proper Transform of $\widetilde{\mathcal{Z}}_l$ is the zero set of transverse section of a vector bundle over a blowup of $\mathbb{P}\tilde{\mathcal{N}}_l^{\mathrm{ide}}$

- $\widetilde{\mathfrak{M}}^0_{1,k} = \mathfrak{M}_1^{\mathrm{eff}} \cup \bigcup_{l=1}^{l=d} \widetilde{\mathcal{Z}}_l$
    - $\mathfrak{M}^{\mathrm{eff}}_{1,k} = \overline{\mathfrak{M}}_1^0 - \bigcup_{l=1}^{l=d} \overline{\mathfrak{M}}_1^l$ is smooth
    - $\widetilde{\mathcal{Z}}_l$ is smooth
    - Normal sheaf of $\mathcal{Z}_l$ in $\widetilde{\mathfrak{M}}^0_1$ is l.b.

$\Longrightarrow$ $\widetilde{\mathfrak{M}}^0_1$ is smooth