The Smoothability of Normal Crossings Symplectic Varieties

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Abstract

Our previous paper introduces topological notions of normal crossings symplectic divisor and variety and establishes that they are equivalent, in a suitable sense, to the desired geometric notions. Friedman's d-semistability condition is well-known to be an obstruction to the smoothability of a normal crossings variety in a one-parameter family with a smooth total space in the algebraic geometry category. We show that the direct analogue of this condition is the only obstruction to such smoothability in the symplectic topology category. Every smooth fiber of the families of smoothings we describe provides a multifold analogue of the now classical (two-fold) symplectic sum construction; we thus establish an old suggestion of Gromov in a strong form.

Contents

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1 Introduction

Flat one-parameter families of smoothings are an important tool in algebraic geometry and raise considerable interest in related areas of symplectic topology. The Gross-Siebert program [11] for a direct proof of mirror symmetry has highlighted in particular the significance of log smooth degenerations to log smooth algebraic varieties. A central part of this program is the study of Gromov-Witten invariants (which are fundamentally symplectic topology invariants) under such degenerations. Such a study is undertaken from an algebro-geometric perspective in [1, 3, 12]. The almost complex analogue of the log smooth category provided by the exploded manifold category of [19] underlines a similar study of Gromov-Witten invariants in [20]. Log smooth varieties include varieties with normal crossings (or NC) singularities, i.e. singularities of the form $z_1 \ldots z_N = 0$ in complex coordinates. Purely symplectic topology notions of an NC symplectic variety and of a one-parameter family of smoothings of such a variety are introduced in [4] and in this paper, respectively. It is straightforward to show that the direct analogue of the well-known triple point condition of algebraic geometry is an obstruction for an NC symplectic variety to admit a one-parameter family of smoothings. The main construction of this paper produces such a family for every NC variety satisfying this direct analogue and thus establishes the necessity *and* sufficiency of this condition. A non-central fiber of such a family is a representative of the deformation equivalence class of the multifold symplectic sum construction on the central fiber envisioned in [10, p343].

For a symplectic submanifold V in a symplectic manifold (X, ω), the normal bundle

$$\mathcal{N}_X V \equiv \frac{TX|_V}{TV} \approx TV^\omega \equiv \big\{v \in T_x X\colon\ x \in V,\, \omega(v,w) = 0\ \forall\, w \in T_x V\big\} \longrightarrow V \tag{1.1}$$

of V in X inherits a fiberwise symplectic form $\omega|_{\mathcal{N}_X V}$ from ω. The space of complex structures on the fibers of (1.1) compatible with (resp. tamed by) $\omega|_{\mathcal{N}_X V}$ is non-empty and contractible; we call such complex structures ω-compatible (resp. ω-tame). The now classical symplectic sum construction, indicated in [10, p343] and carried out in [9, 14], smooths out the union of two symplectic manifolds (X_1, ω_1) and (X_2, ω_2) glued along a common compact smooth symplectic divisor $V \equiv X_{12}$ such that

$$c_1(\mathcal{N}_{X_1} X_{12}) + c_1(\mathcal{N}_{X_2} X_{12}) = 0 \in H^2(X_{12}; \mathbb{Z}) \tag{1.2}$$

into a new symplectic manifold $(X_\#, \omega_\#)$. From the complex geometry point of view, this construction replaces the nodal singularity $z_1 z_2 = 0$ in $\mathbb{C}^n$, i.e. the union of the two coordinate hyperplanes, by a smoothing $z_1 z_2 = \lambda$ with $\lambda \in \mathbb{C}^*$.

In this paper, we describe a multifold version of the symplectic sum construction of [9, 14]; it in particular smooths out the union of several symplectic manifolds identified along transversely intersecting smooth divisors with a *single* smoothing parameter λ. From the complex geometry point of view, this construction replaces the singularity $z_1 \ldots z_N = 0$ in $\mathbb{C}^n$, i.e. the union of the N coordinate hyperplanes, by a smoothing $z_1 \ldots z_N = \lambda$ with $\lambda \in \mathbb{C}^*$. An inverse degeneration construction, which includes a multifold version of the symplectic cut procedure of [13], is described in [6]. The precise relation between the smoothing/sum construction of the present paper and the degeneration/cut construction of [6] is the subject of [7].

The topological restriction (1.2) is equivalent to the existence of an isomorphism

$$\mathcal{N}_{X_1} X_{12} \otimes_{\mathbb{C}} \mathcal{N}_{X_2} X_{12} \approx X_{12} \times \mathbb{C} \tag{1.3}$$

of complex line bundles over X_{12}. The topological type of $X_\#$ in [9] depends only on the homotopy class of such an isomorphism. With such a choice fixed, the construction of [9] involves choosing an ω_1-compatible almost complex structure on $\mathcal{N}_{X_1}X_{12}$, an ω_2-compatible almost complex structure on $\mathcal{N}_{X_2}X_{12}$, and a representative for the above homotopy class. Because of these choices, the resulting symplectic manifold $(X_\#,\omega_\#)$ is determined by (X_1,ω_1), (X_2,ω_2), and the choice of the homotopy class only up to symplectic deformation equivalence. Since the symplectic deformations of the tuple

$$\big((X_1,X_2,X_{12}),(\omega_1,\omega_2)\big) \tag{1.4}$$

do not affect the deformation equivalence class of $(X_\#,\omega_\#)$, it would have been sufficient to carry out the symplectic sum construction of [9] only on a path-connected set of representatives for each deformation equivalence class of the tuples (1.4). This change in perspective turns out to be very useful for smoothing out NC symplectic varieties, including unions of several symplectic manifolds glued along transversally intersecting smooth divisors.

The one-parameter family $z_1\ldots z_N=\lambda$ of smoothings of the union of the N coordinate hyperplanes $\mathbb{C}^N_i$ in $\mathbb{C}^N$ involves compatible complex structures on $\mathbb{C}^N_i$ that preserve all coordinate subspaces $\mathbb{C}^N_I\subsetneq\mathbb{C}^N$. For each $i=1,\ldots,N$, the union of the codimension 2 coordinate subspaces $\mathbb{C}^N_{ij}$ with $j\neq i$ is a simple crossings (or SC) Kähler divisor in $\mathbb{C}^N_i$. Analogues of this notion and of the related notion of an SC variety in the symplectic category are introduced in [4] and reviewed in Section 2.1 of the present paper; see Definitions 2.1 and 2.5. In the terminology of Definition 2.5, the tuple (1.4) is a 2-fold SC symplectic configuration and the tuple

$$\big(X_1\cup_{X_{12}}X_2,(\omega_1,\omega_2)\big) \tag{1.5}$$

is the associated NC symplectic variety. As noted at the end of Section 2.1, an SC symplectic variety $X_\emptyset$ comes with a natural complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ over its singular locus X_∂; see (2.14). We call it the normal bundle of X_∂ in $X_\emptyset$; it reduces to the left-hand side of (1.3) in the setting of [9]. By Theorem 2.8, an SC symplectic variety $X_\emptyset$ is smoothable in a one-parameter family if and only if the line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ is trivial. Furthermore, the possible families of smoothings are again classified by the homotopy classes of its trivializations. We give two examples in Section 2.3. In [5], we extend Theorem 2.8 to arbitrary NC symplectic varieties and give more elaborate examples of the associated smoothings.

Theorem 2.8 leads to and has further potential for very different applications in symplectic topology. First and foremost, it includes a new surgery construction for symplectic manifolds and thus opens the possibility of generating new symplectic manifolds. Furthermore, it fits naturally with a decade-long program to develop decomposition formulas for Gromov-Witten invariants under one-parameter families of almost Kähler (or projective) degenerations; approaches to this problem appear in [1, 3, 12, 20]. An immediate consequence of Theorem 2.8, along with [4, Theorem 2.17], is that the decomposition formulas arising from [20] include splitting formulas for Gromov-Witten invariants of the N-fold symplectic sums of Theorem 2.8. Since the decomposition formulas of [20] have connections with tropical geometry, Theorem 2.8 may have applications in this field as well. It should also have applications in the theory of singularities, as an isolated singularity can often be studied by smoothing it and then applying symplectic techniques as in [16, 23]. Theorem 2.8 provides a purely topological condition for the smoothability of a singularity symplectically after a sequence of blowups that turns it into a simple crossings form (even though it may not be smooth-

able algebraically).

Surgery constructions on 4-dimensional symplectic manifolds along *pairwise* positively intersecting immersed surfaces are described in [25, 26]. While these are called N-fold symplectic sum constructions, this terminology agrees with ours (which is consistent with algebraic geometry and [10, p343]) only for $N=3$. In particular, the setting of [26, Theorem 2.7] is a specialization of the $N=3$ case of the setting of our Theorem 2.8. The output of [26, Theorem 2.7] is then symplectically deformation equivalent to the smooth fibers of the one-parameter family provided by Theorem 2.8. The perspectives taken in [26] and the present paper are fundamentally different as well. The viewpoint taken in the former is that of surgery on 4-dimensional manifolds; our viewpoint is that of smoothing a variety in a one-dimensional family with a smooth total space. The configurations in [26] with $N\ge 4$ correspond to varieties, such as

$$\big\{(x,y,z,w)\in\mathbb{C}^4 : xy=0,\, zw=0\big\}, \tag{1.6}$$

that do not even admit such smoothings. The total space of the natural one-parameter smoothing of (1.6), i.e. with 0 replaced by $\lambda\in\mathbb{C}$, is singular at the origin. On the other hand, the total space of this family is smooth in the logarithmic category central to the mirror symmetry program of [11] and in the exploded manifold category of [20]. Unfortunately, symplectic topology analogues of the singularity described by (1.6) are yet to be defined.

Notions of symplectic regularizations for an SC divisor $\{V_i\}_{i\in S}$ in X and a configuration $\mathbf{X}$ are introduced in [4, Sections 2.2,2,3] and recalled in Sections 3.1 and 3.2 of the present paper; see Definitions 3.5 and 3.7. Such regularizations provide essential auxiliary data for the multifold symplectic smoothing/sum construction of Theorem 2.8, just as they did in the $N=2$ case addressed in [9, 14]. By [4, Theorem 2.17], the space $\mathrm{Symp}^+(\mathbf{X})$ of symplectic structures on $\mathbf{X}$ is weakly homotopy equivalent to the space $\mathrm{Aux}(\mathbf{X})$ of pairs consisting of a symplectic structure on $\mathbf{X}$ and a compatible regularization. In Section 3.3, we show that a given trivialization of the complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ can be homotoped to be compatible with a given symplectic regularization for $\mathbf{X}$ in a suitable sense and that any two compatible trivializations are homotopic to each other through compatible trivializations. While the projection map from $\mathrm{Aux}(\mathbf{X})$ to $\mathrm{Symp}^+(\mathbf{X})$ need not be surjective in general (in contrast to the $N=2$ case), this is not an issue for typical applications in symplectic topology.

In Section 4, we show that the triviality of $\mathcal{O}_{X_\partial}(X_\emptyset)$ is sufficient for the existence of a one-parameter family of smoothings of the symplectic variety associated to an SC symplectic configuration, up to symplectic deformation equivalence. By Proposition 5.1, this condition is necessary and in fact every one-parameter family of smoothings determines a homotopy class of trivializations of $\mathcal{O}_{X_\partial}(X_\emptyset)$. By Proposition 5.5, the homotopy class determined by a one-parameter family constructed as in Section 4 is the homotopy class used in its construction. Appendix A collects some basic facts concerning connections on vector bundles. Appendix B provides a more intrinsic perspective on the smoothability criterion of Theorem 2.8.

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2 Main theorem

We begin by introducing the most commonly used notation. If $N\in\mathbb{Z}^{\geq 0}$ and $I\subset\{1,\ldots,N\}$, let

$$[N]=\{1,\ldots,N\},\qquad \mathbb{C}^N_I=\big\{(z_1,\ldots,z_N)\in\mathbb{C}^n\colon\ z_i=0\ \forall\, i\in I\big\}.$$

Denote by $\mathcal{P}(N)$ the collection of subsets of $[N]$ and by $\mathcal{P}^*(N)\subset\mathcal{P}(N)$ the collection of nonempty subsets. If $\mathcal{N}\longrightarrow V$ is a vector bundle, $\mathcal{N}'\subset\mathcal{N}$, and $V'\subset V$, we define

$$\mathcal{N}'|_{V'}=\mathcal{N}|_{V'}\cap\mathcal{N}'\,. \tag{2.1}$$

Let $\mathbb{I}=[0,1]$.

2.1 Preliminaries

We first recall the notions of simple crossings (or SC) symplectic divisor and variety introduced, described in more detail, and illustrated with examples in [4, Section 2.1]. We then define a natural complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ over the singular locus X_∂ of an SC symplectic variety $X_\emptyset$.

Let X be a (smooth) manifold. For any submanifold $V\subset X$, let

$$\mathcal{N}_XV\equiv\frac{TX|_V}{TV}\longrightarrow V$$

denote the normal bundle of V in X. For a collection $\{V_i\}_{i\in S}$ of submanifolds of X and $I\subset S$, let

$$V_I\equiv\bigcap_{i\in I}V_i\subset X\,.$$

Such a collection is called transverse if any subcollection $\{V_i\}_{i\in I}$ of these submanifolds intersects transversely, i.e. the homomorphism

$$T_xX\oplus\bigoplus_{i\in I}T_xV_i\longrightarrow\bigoplus_{i\in I}T_xX,\qquad \big(v,(v_i)_{i\in I}\big)\longrightarrow(v+v_i)_{i\in I}\,, \tag{2.2}$$

is surjective for all $x\in V_I$. Each subspace $V_I\subset X$ is then a submanifold of X and the homomorphisms

$$\begin{aligned}\mathcal{N}_XV_I\longrightarrow\bigoplus_{i\in I}\mathcal{N}_XV_i\big|_{V_I}\quad\forall\ I\subset S,\qquad \mathcal{N}_{V_{I-i}}V_I\longrightarrow\mathcal{N}_XV_i\big|_{V_I}\quad\forall\ i\in I\subset S,\\ \bigoplus_{i\in I-I'}\mathcal{N}_{V_{I-i}}V_I\longrightarrow\mathcal{N}_{V_{I'}}V_I\quad\forall\ I'\subset I\subset S\end{aligned} \tag{2.3}$$

induced by inclusions of the tangent bundles are isomorphisms.

If X is an oriented manifold, a transverse collection $\{V_i\}_{i\in S}$ of oriented submanifolds of X of even codimensions induces an orientation of each submanifold $V_I\subset X$ with $|I|\geq 2$, which we call the intersection orientation of V_I. If V_I is zero-dimensional, it is a discrete collection of points in X and the homomorphism (2.2) is an isomorphism at each point $x\in V_I$; the intersection orientation of V_I at $x\in V_I$ then corresponds to a plus or minus sign, depending on whether this isomorphism is orientation-preserving or orientation-reversing. For convenience, we call the original orientations

of $X=V_\emptyset$ and $V_i=V_{\{i\}}$ the intersection orientations of these submanifolds V_I of X with $|I|<2$.

Suppose (X,ω) is a symplectic manifold and $\{V_i\}_{i\in S}$ is a transverse collection of submanifolds of X such that each V_I is a symplectic submanifold of (X,ω). Each V_I then carries an orientation induced by $\omega|_{V_I}$, which we will call the ω-orientation. If V_I is zero-dimensional, it is automatically a symplectic submanifold of (X,ω); the ω-orientation of V_I at each point $x\in V_I$ corresponds to the plus sign by definition. By the previous paragraph, the ω-orientations of X and V_i with $i\in I$ also induce intersection orientations on all V_I.

Definition 2.1. Let (X,ω) be a symplectic manifold. An SC symplectic divisor in (X,ω) is a finite transverse collection $\{V_i\}_{i\in S}$ of closed submanifolds of X of codimension 2 such that V_I is a symplectic submanifold of (X,ω) for every $I\subset S$ and the intersection and ω-orientations of V_I agree.

The intersection and symplectic orientations of V_I agree if $|I|<2$. Thus, an SC symplectic divisor $\{V_i\}_{i\in S}$ with $|S|=1$ is a smooth symplectic divisor in the usual sense. If (X,ω) is a 4-dimensional symplectic manifold, a finite transverse collection $\{V_i\}_{i\in S}$ of closed symplectic submanifolds of X of codimension 2 is an SC symplectic divisor if all points of the pairwise intersections $V_{i_1}\cap V_{i_2}$ with $i_1\neq i_2$ are positive; these are the cases considered in [25, 26].

Definition 2.2. Let X be a manifold and $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2. A symplectic structure on $\{V_i\}_{i\in S}$ in X is a symplectic form ω on X such that V_I is a symplectic submanifold of (X,ω) for all $I\subset S$.

For X and $\{V_i\}_{i\in S}$ as in Definition 2.2, we denote by $\mathrm{Symp}(X,\{V_i\}_{i\in S})$ the space of all symplectic structures on $\{V_i\}_{i\in S}$ in X and by

$$\mathrm{Symp}^+\big(X,\{V_i\}_{i\in S}\big)\subset\mathrm{Symp}\big(X,\{V_i\}_{i\in S}\big)$$

the subspace of the symplectic forms ω such that $\{V_i\}_{i\in S}$ is an SC symplectic divisor in (X,ω).

Definition 2.3. Let $N\in\mathbb{Z}^+$. An N-fold transverse configuration is a tuple $\{X_I\}_{I\in\mathcal{P}^*(N)}$ of manifolds such that $\{X_{ij}\}_{j\in[N]-i}$ is a transverse collection of submanifolds of X_i for each $i\in[N]$ and

$$X_{\{ij_1,\ldots,ij_k\}}\equiv\bigcap_{m=1}^k X_{ij_m}=X_{ij_1\ldots j_k}\qquad\forall\, j_1,\ldots,j_k\in[N]-i.$$

Definition 2.4. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct. A symplectic structure on $\mathbf{X}$ is a tuple

$$(\omega_i)_{i\in[N]}\in\prod_{i=1}^N\mathrm{Symp}\big(X_i,\{X_{ij}\}_{j\in[N]-i}\big)$$

such that $\omega_{i_1}|_{X_{i_1i_2}}=\omega_{i_2}|_{X_{i_1i_2}}$ for all $i_1,i_2\in[N]$.

For an N-fold transverse configuration as in Definition 2.3, let

$$X_\emptyset=\bigg(\bigsqcup_{i=1}^N X_i\bigg)\Big/\!\sim,\quad X_i\ni x\sim x\in X_j\ \ \forall\, x\in X_{ij}\subset X_i,X_j,\ i\neq j\,,\tag{2.4}$$

$$X_\partial\equiv\bigcup_{I\in\mathcal{P}(N),|I|=2}X_I\subset X_\emptyset\,.\tag{2.5}$$

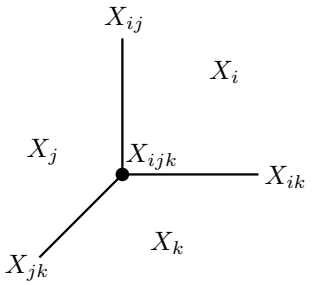

Figure 1: A 3-fold simple crossings configuration and variety.

The SC variety $X_\emptyset$ associated to a 3-fold SC configuration is shown in Figure 1. For $k\in\mathbb{Z}^{\ge 0}$, we call a tuple $(\omega_i)_{i\in[N]}$ a k-form on $X_\emptyset$ if ω_i is a k-form on X_i for each $i\in[N]$ and

$$\omega_i\big|_{X_{ij}} = \omega_j\big|_{X_{ij}} \qquad \forall\ i,j\in[N].$$

For $\mathbf{X}$ as in Definition 2.4, let $\mathrm{Symp}(\mathbf{X})$ denote the space of all symplectic structures on $\mathbf{X}$ and

$$\mathrm{Symp}^+\big(\mathbf{X}\big) = \mathrm{Symp}\big(\mathbf{X}\big) \cap \prod_{i=1}^N \mathrm{Symp}^+\big(X_i, \{X_{ij}\}_{j\in[N]-i}\big)\,. \tag{2.6}$$

Thus, if $(\omega_i)_{i\in[N]}$ is an element of $\mathrm{Symp}^+(\mathbf{X})$, then $\{X_{ij}\}_{j\in[N]-i}$ is an SC symplectic divisor in (X_i,ω_i) for each $i\in[N]$.

Definition 2.5. Let $N\in\mathbb{Z}^+$. An N-fold simple crossings (or SC) symplectic configuration is a tuple

$$\mathbf{X} = \big((X_I)_{I\in\mathcal{P}^*(N)}, (\omega_i)_{i\in[N]}\big) \tag{2.7}$$

such that $\{X_I\}_{I\in\mathcal{P}^*(N)}$ is an N-fold transverse configuration, X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, and $(\omega_i)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X})$. The SC symplectic variety associated to such a tuple $\mathbf{X}$ is the pair $(X_\emptyset,(\omega_i)_{i\in[N]})$.

Suppose (X,ω) is a compact symplectic manifold and $V\subset X$ is a smooth symplectic divisor, i.e. $|S|=1$ in the notation of Definition 2.1. Fix an identification Ψ of a tubular neighborhood $D^\epsilon_X V$ of V in $\mathcal{N}_X V$ with a tubular neighborhood of V in X (i.e. a regularization of V in X in the sense of Definition 3.1) and an ω-tame complex structure $\mathfrak{i}$ on $\mathcal{N}_X V$. Let

$$\begin{aligned}\mathcal{O}_X(V) &= \big(\Psi^{-1\,*}\pi^*_{\mathcal{N}_X V}\mathcal{N}_X V|_{\Psi(D^\epsilon_X V)}\sqcup (X-V)\times\mathbb{C}\big)/\!\sim \longrightarrow X, \\ \Psi^{-1\,*}\pi^*_{\mathcal{N}_X V}\mathcal{N}_X V|_{\Psi(D^\epsilon_X V)} &\ni \big(\Psi(v),v,cv\big)\sim\big(\Psi(v),c\big)\in(X-V)\times\mathbb{C}.\end{aligned} \tag{2.8}$$

This is a complex line bundle over X with $c_1(\mathcal{O}_X(V))=\mathrm{PD}_X([V]_X)$, where $[V]_X$ is the homology class in X represented by V. The space of pairs $(\Psi,\mathfrak{i})$ involved in explicitly constructing this line bundle is contractible.

Suppose $\mathbf{X}$ is an SC symplectic configuration as in (2.7). If $i,j,k\in[N]$ are distinct, the inclusion $(X_{jk},X_{ijk})\longrightarrow(X_j,X_{ij})$ induces an isomorphism

$$\mathcal{N}_{X_{jk}}X_{ijk}\equiv\frac{TX_{jk}|_{X_{ijk}}}{TX_{ijk}}\xrightarrow{\approx}\frac{TX_j|_{X_{ijk}}}{TX_{ij}|_{X_{ijk}}}\equiv\mathcal{N}_{X_j}X_{ij}\big|_{X_{ijk}} \tag{2.9}$$

of rank 2 real vector bundles over X_{ijk}; this is a special case of the second isomorphism in (2.3) for $X=X_j$. Thus, the rank 2 real vector bundles $\mathcal{N}_{X_j}X_{ij}|_{X_{ijk}}$ and $\mathcal{N}_{X_k}X_{ik}|_{X_{ijk}}$ are canonically identified with $\mathcal{N}_{X_{jk}}X_{ijk}$. Let

$$\Psi_{ij;j}\colon \mathcal{N}'_{ij;j} \longrightarrow X_j, \qquad i,j\in[N],\, i\neq j,$$

be a collection of identifications of tubular neighborhoods of X_{ij} in $\mathcal{N}_{X_j}X_{ij}$ and in X_j so that

$$\Psi_{ij;j}\big|_{\mathcal{N}'_{ij;j}\cap\mathcal{N}_{X_{jk}}X_{ijk}} = \Psi_{ik;k}\big|_{\mathcal{N}'_{ik;k}\cap\mathcal{N}_{X_{jk}}X_{ijk}} \tag{2.10}$$

for all $i,j,k\in[N]$ with $k,j\neq i$.

Since the intersection and ω_j-orientations of

$$X_{ijk} = X_{ij}\cap X_{jk} \subset X_j$$

agree, the isomorphism (2.9) is orientation-preserving with respect to the orientations induced by $(\omega_j|_{X_{jk}})|_{\mathcal{N}_{X_{jk}}X_{ijk}}$ and $\omega_j|_{\mathcal{N}_{X_j}X_{ij}}$. Thus, we can choose a collection of ω_j-tame complex structures $\mathfrak{i}_{ij;j}$ on the vector bundles $\mathcal{N}_{ij;j}$ so that

$$\mathfrak{i}_{ij;j}\big|_{\mathcal{N}_{X_{jk}}X_{ijk}} = \mathfrak{i}_{ik;k}\big|_{\mathcal{N}_{X_{jk}}X_{ijk}} \tag{2.11}$$

for all $i,j,k\in[N]$ with $k,j\neq i$.

For $i,j\in[N]$ distinct, let $\mathcal{O}_{X_j}(X_{ij})$ be the complex line bundle over X_j constructed as in (2.8) using the identification $\Psi_{ij;j}$ and the complex structure $\mathfrak{i}_{ij;j}$. By (2.10) and (2.11), there are canonical identifications

$$\mathcal{O}_{X_j}(X_{ij})\big|_{X_{jk}} = \mathcal{O}_{X_{jk}}(X_{ijk}) = \mathcal{O}_{X_k}(X_{ik})\big|_{X_{jk}} \tag{2.12}$$

for all $i,j,k\in[N]$ with $j,k\neq i$. For each $i\in[N]$,

$$\begin{aligned}\mathcal{O}_{X_i^c}(X_i) \equiv \Big(\bigsqcup_{j\in[N]-\{i\}}\mathcal{O}_{X_j}(X_{ij})\Big)\Big/\!\sim \longrightarrow X_i^c \equiv \bigcup_{j\in[N]-\{i\}}X_j \subset X_\emptyset\,,\\ \mathcal{O}_{X_j}(X_{ij})\big|_{X_{jk}}\ni u\sim u\in\mathcal{O}_{X_k}(X_{ik})\big|_{X_{jk}} \quad \forall\, i,j,k\in[N],\, j,k\neq i,\end{aligned} \tag{2.13}$$

is thus a well-defined complex line bundle. Let $\mathcal{O}_{X_\partial}(X_i)=\mathcal{O}_{X_i^c}(X_i)|_{X_\partial}$.

We call the complex line bundle

$$\mathcal{O}_{X_\partial}(X_\emptyset) \equiv \bigotimes_{i=1}^N \mathcal{O}_{X_\partial}(X_i) \tag{2.14}$$

the normal bundle of the singular locus X_∂ in $X_\emptyset$. The space of the collections of pairs $(\Psi_{ij;j},\mathfrak{i}_{ij;j})$ involved in explicitly constructing this line bundle is contractible. By (2.12),

$$\mathcal{O}_{X_\partial}(X_\emptyset)\big|_{X_{ij}} = \mathcal{N}_{X_i}X_{ij}\otimes\mathcal{N}_{X_j}X_{ij}\otimes\bigotimes_{k\in[N]-\{i,j\}}\mathcal{O}_{X_{ij}}(X_{ijk}) \quad \forall\, i,j\in[N],\, i\neq j.$$

In the $N=2$ case, this line bundle is the left-hand side of (1.3).

2.2 Statement

We now describe the setup for our smoothing/sum construction in the symplectic topology category. Theorem 2.8 provides a necessary and sufficient topological condition for when it can be carried out.

Definition 2.6. If $(\mathcal{Z},\omega_{\mathcal{Z}})$ is a symplectic manifold and $\Delta\subset\mathbb{C}$ is a disk centered at the origin, a smooth surjective map $\pi\colon\mathcal{Z}\longrightarrow\Delta$ is a nearly regular symplectic fibration if

- $\mathcal{Z}_0\equiv\pi^{-1}(0)=X_1\cup\ldots\cup X_N$ for some SC symplectic divisor $\{X_i\}_{i\in[N]}$ in $(\mathcal{Z},\omega_{\mathcal{Z}})$,
- π is a submersion outside of the submanifolds X_I with $|I|\ge 2$,
- for every $\lambda\in\Delta-\{0\}$, the restriction ω_λ of $\omega_{\mathcal{Z}}$ to $\mathcal{Z}_\lambda\equiv\pi^{-1}(\lambda)$ is nondegenerate.

Let $(\mathcal{Z},\omega_{\mathcal{Z}})$, π, and X_I be as above. For each $I\subset[N]$, the derivatives of π along the normal bundles $\mathcal{N}_{\mathcal{Z}}X_i$ of X_i in $\mathcal{Z}$ induce a homomorphism

$$D_I\pi\colon\bigotimes_{i\in I}\mathcal{N}_{\mathcal{Z}}X_i\big|_{X_I}\longrightarrow\mathbb{C} \tag{2.15}$$

that vanishes along X_J with $I\subsetneq J\subset[N]$. If $I=\{i\}$, $D_I\pi$ is the homomorphism induced by the differential

$$\mathrm{d}\pi\colon T\mathcal{Z}|_{X_I}\longrightarrow T_0\mathbb{C}=\mathbb{C}$$

of π; its restrictions to TX_I and $T\mathcal{Z}|_{X_J}$ with $I\subsetneq J\subset[N]$ vanish. If $I\subset[N]$ is non-empty and $J=I\sqcup\{j\}$, then D_J is the homomorphism induced by the differential of

$$\mathrm{d}\{D_I\pi\}\colon TX_I|_{X_J}\longrightarrow\mathrm{Hom}_{\mathbb{R}}\Big(\bigotimes_{i\in I}\mathcal{N}_{\mathcal{Z}}X_i,\mathbb{C}\Big)$$

via the natural identification $\mathcal{N}_{\mathcal{Z}}X_j|_{X_J}=\mathcal{N}_{X_I}X_J$.

Definition 2.7. Let $(X_\emptyset,(\omega_i)_{i\in[N]})$ be the SC variety associated to an SC symplectic configuration (2.7). A one-parameter family of smoothings of $(X_\emptyset,(\omega_i)_{i\in[N]})$ is a nearly regular symplectic fibration as in Definition 2.6 with

$$X_I=\bigcap_{i\in I}X_i\subset X_\emptyset=\mathcal{Z}_0\subset\mathcal{Z}\qquad\forall\ I\in\mathcal{P}^*(N)$$

such that the homomorphism

$$\mathcal{N}_{\mathcal{Z}}X_i\big|_x\longrightarrow\mathbb{C},\qquad v_i\longrightarrow D_I\pi\big((v_j)_{j\in I}\big), \tag{2.16}$$

is an orientation-preserving isomorphism for all $i\in I\subset[N]$, $x\in X_I$ with $x\notin X_J$ if $I\subsetneq J\subset[N]$, and $v_j\in\mathcal{N}_{\mathcal{Z}}X_j|_x-\{0\}$ for $j\in I-\{i\}$.

We call (the deformation equivalence class of) an SC symplectic variety $(X_\emptyset,(\omega_i)_{i\in[N]})$ smoothable if some SC symplectic variety $(X_\emptyset,(\omega_i')_{i\in[N]})$ deformation equivalent to $(X_\emptyset,(\omega_i)_{i\in[N]})$ admits a one-parameter family of smoothings. Theorem 2.8 below provides a necessary and sufficient topological condition for the smoothability of an SC symplectic variety.

Theorem 2.8. *Let* $\mathbf{X}$ *be an* N*-fold SC symplectic configuration as in (2.7). The associated SC symplectic variety* $(X_\emptyset, (\omega_i)_{i\in[N]})$ *is smoothable if and only if the normal bundle* $\mathcal{O}_{X_\partial}(X_\emptyset)$ *of its singular locus is trivializable. Furthermore, the germ of the deformation equivalence class of the smoothing* $(\mathcal{Z}, \omega_{\mathcal{Z}}, \pi)$ *provided by the proof of this statement is determined by a homotopy class of trivializations of* $\mathcal{O}_{X_\partial}(X_\emptyset)$*. If in addition* X_∂ *is compact, the deformation equivalence class of a smooth fiber* $(\mathcal{Z}_\lambda, \omega_\lambda)$ *is also determined by a homotopy class of these trivializations.*

Remark 2.9. In a future paper, we expect to show that the deformation equivalence class of *any* one-parameter family of smoothings of $(X_\emptyset, (\omega_i)_{i\in[N]})$ corresponds to a homotopy class of trivializations of (2.14). This is equivalent to every such smoothing being equivalent to a smoothing as constructed in Section 4.

By standard Čech cohomology considerations and (2.14), the complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ is trivializable if and only if

$$\sum_{i=1}^{N} c_1\big(\mathcal{O}_{X_\partial}(X_i)\big) = 0 \in \check{H}^2(X_\partial;\mathbb{Z})\,. \tag{2.17}$$

By [24, Corollary 6.9.5], the Čech and singular cohomologies of X_∂ (as well as of all other spaces in this paper) are canonically isomorphic.

If $N=1$, (2.17) imposes no condition. In this case, we can take $(\mathcal{Z}, \omega_{\mathcal{Z}})$ to be the product symplectic manifold $(X_1,\omega_1)\times(\Delta,\omega_{\mathbb{C}})$, where $\omega_{\mathbb{C}}$ is the standard symplectic form on $\mathbb{C}$. The $N=2$ case of Theorem 2.8 is the symplectic sum construction of [9, 14] for the SC symplectic variety (1.5). It glues two symplectic manifolds (X_1,ω_1) and (X_2,ω_2) along normal circle bundles of a common symplectic divisor (X_{12},ω_{12}) if

$$\begin{aligned} c_1\big(\mathcal{O}_{X_\partial}(X_1)\big) + c_1\big(\mathcal{O}_{X_\partial}(X_2)\big) &\equiv c_1\big(\mathcal{N}_{X_1}X_{12}\big) + c_1\big(\mathcal{N}_{X_2}X_{12}\big) \\ &= 0 \in H^2(X_{12};\mathbb{Z}) \equiv H^2(X_\partial;\mathbb{Z}), \end{aligned}$$

i.e. (1.2) is satisfied.

In general, the condition (2.17) implies that

$$c_1\big(\mathcal{N}_{X_i}X_{ij}\big) + c_1\big(\mathcal{N}_{X_j}X_{ij}\big) + \sum_{k\in[N]-\{i,j\}} \big[X_{ijk}\big]_{X_{ij}} = 0 \qquad \forall\, i,j\in[N],\ i\neq j. \tag{2.18}$$

The latter implies the former if at most one of the restriction homomorphisms

$$H^1(X_{ij};\mathbb{Z}) \longrightarrow \bigoplus_{k\in[N]-\{i,j\}} H^1(X_{ijk};\mathbb{Z}), \qquad i,j\in[N],\ i\neq j, \tag{2.19}$$

is not surjective, but not in general; see Example 2.11. In the most basic case of the $N=3$ situation of Theorem 2.8 with X_{ij} and X_{ik} being symplectic surfaces in a 4-dimensional manifold X_i intersecting transversely and positively at a single point, the conditions (2.17) and (2.18) reduce to the simple condition on the self-intersection numbers of these surfaces stated in [26, Theorem 2.7].

The algebro-geometric analogue of (2.17),

$$\mathcal{O}_{X_\partial}(X_\emptyset) \approx \mathcal{O}_{X_\partial}\,,$$

is called d-semistability in [8, Definition (1.13)]. It is well-known to be an obstruction to the existence of a one-parameter family of smoothings of $X_\emptyset$ in the algebraic geometry category; see [8, Corollary (1.12)]. As shown in [22], it is not the only obstruction in the algebraic category, even in the $N=2$ case. The algebro-geometric analogue of (2.18),

$$\mathcal{N}_{X_i}X_{ij}\otimes\mathcal{N}_{X_j}X_{ij}\otimes\bigotimes_{k\in[N]-\{i,j\}}\mathcal{O}_{X_{ij}}(X_{ijk})\approx\mathcal{O}_{X_{ij}}\qquad\forall\,i,j\in[N],\ i\neq j\,,$$

is known as the triple point condition; see [21, Proposition 2.4.3].

As in the $N=2$ case of Theorem 2.8 addressed in [9], the construction of $\pi\colon(\mathcal{Z},\omega_{\mathcal{Z}})\longrightarrow\Delta$ involves some auxiliary data for $\mathbf{X}$ and a compatible choice of a trivialization of the complex line bundle (2.14) in a given homotopy class. We call the former regularizations and recall their definition in Sections 3.1 and 3.2. Proposition 3.9, proved in Section 3.3, ensures that each homotopy class of trivializations of (2.14) contains a representative compatible with a given regularization for $\mathbf{X}$. The main part of the proof of Theorem 2.8 is carried out in Section 4, where the chosen auxiliary data for $\mathbf{X}$ and a compatible trivialization of (2.14) are used to construct a one-parameter family $\pi\colon\mathcal{Z}\longrightarrow\Delta$ of smoothings of $(X_\emptyset,(\omega_i')_{i\in[N]})$. By Proposition 5.1 proved in Section 5.1, every one-parameter family $\pi\colon\mathcal{Z}\longrightarrow\Delta$ of smoothings of $(X_\emptyset,(\omega_i')_{i\in[N]})$ determines a homotopy class of trivializations of (2.14). By Proposition 5.5 proved in Section 5.2, the homotopy class determined by the family constructed in Section 4 is the homotopy class used to construct it.

2.3 Examples

We now give two examples. The first one describes a 3-fold case of Theorem 2.8. The second example shows that condition (2.18) is in general weaker than condition (2.17).

Example 2.10 ([26, Example 2.8])**.** Let $\widehat{\mathbb{P}}^2$ be the blowup of $\mathbb{P}^2$ at a point p, $E,\bar{L}\subset\widehat{\mathbb{P}}^2$ be the exceptional divisor and the proper transform of a line through p, respectively, and $P=E\cap\bar{L}$. We take $X_1,X_2,X_3=\widehat{\mathbb{P}}^2$ and identify $E\subset X_1$ with $\bar{L}\subset X_2$, $E\subset X_2$ with $\bar{L}\subset X_3$, and $E\subset X_3$ with $\bar{L}\subset X_1$; see Figure 2. By adjusting the size of the blowup as in [15, Section 7.1], we can ensure that the identifications can be made symplectically. Since

$$\big\langle c_1(\mathcal{N}_{\widehat{\mathbb{P}}^2}E),E\big\rangle=-1\qquad\text{and}\qquad\big\langle c_1(\mathcal{N}_{\widehat{\mathbb{P}}^2}\bar{L}),\bar{L}\big\rangle=0,$$

the resulting 3-fold configuration $((X_I)_{I\in\mathcal{P}^*(3)},(\omega_i)_{i\in[3]})$ satisfies (2.18) for all $i,j\in[3]$ distinct. Since all three homomorphisms (2.19) are surjective in this case, this configuration thus satisfies (2.17). The singular locus X_∂ of the NC symplectic variety $X_\emptyset$ to be smoothed out consists of 3 copies of $\mathbb{P}^1$ with one point in common. Since X_∂ is simply connected, there is only one homotopy class of trivializations of (2.14). The symplectic deformation equivalence class (of a smooth fiber) of the corresponding 3-fold symplectic sum is $\mathbb{P}^2$. This is illustrated in the second diagram in Figure 2 from the symplectic cut perspective of [6] applied in a toric setting (the big triangle corresponds to $\mathbb{P}^2$).

Example 2.11. Let $X_1=\mathbb{P}^3$ with its standard symplectic form, $X_{12}\approx\mathbb{P}^2$ be a linear subspace, and $X_{13}\subset\mathbb{P}^3$ be a cubic surface transverse to X_{12}. The intersection X_{123} of X_{12} and X_{13} is then a plane cubic, i.e. a genus 1 curve. For $i=2,3$, define

$$\begin{aligned}\mathcal{L}_{1i}&=\mathcal{N}_{X_1}X_{1i}\otimes\mathcal{O}_{X_{1i}}(X_{123})\approx\mathcal{O}_{\mathbb{P}^3}(4)\big|_{X_{1i}}\longrightarrow X_{1i}\,,\\ X_i&=\mathbb{P}\big(\mathcal{L}_{1i}\oplus\mathcal{O}_{X_{1i}}\big),\qquad X_{1i}^\infty=\mathbb{P}\big(\mathcal{L}_{1i}\oplus 0\big)\approx X_{1i}\,;\end{aligned}$$

Figure 2: The NC variety of Example 2.10 and a toric representation of the corresponding symplectic sum.

see the left diagram in Figure 3. There are canonical isomorphisms

$$\mathcal{L}_{12}|_{X_{123}} \approx \big(\mathcal{N}_{X_1}X_{12}\otimes\mathcal{N}_{X_1}X_{13}\big)|_{X_{123}} \approx \big(\mathcal{N}_{X_1}X_{13}\otimes\mathcal{N}_{X_1}X_{12}\big)|_{X_{123}} \approx \mathcal{L}_{13}|_{X_{123}}\,. \tag{2.20}$$

The homotopy classes of all isomorphisms $\mathcal{L}_{12}|_{X_{123}} \approx \mathcal{L}_{13}|_{X_{123}}$ in the category of complex (not holomorphic) line bundles correspond to the homotopy classes of continuous functions $X_{123}\longrightarrow S^1$, i.e. to the elements of

$$H^1(X_{123};\mathbb{Z}) \approx \mathbb{Z}^2\,.$$

Any such isomorphism ρ induces an identification

$$\psi_{23}^{\rho}\colon \mathbb{P}(\mathcal{L}_{12}\oplus\mathbb{C})\big|_{X_{123}} \longrightarrow \mathbb{P}(\mathcal{L}_{13}\oplus\mathbb{C})\big|_{X_{123}}\,,$$

which can be assumed to be holomorphic by pushing the holomorphic structure forward; the zero element of $H^1(X_{123};\mathbb{Z})$ corresponds to the identification induced by (2.20). The SC variety

$$X_{\emptyset}^{\rho} = X_1\cup X_2\cup X_3, \qquad X_{23} \equiv \mathbb{P}(\mathcal{L}_{12}\oplus\mathcal{O}_{X_{12}})\big|_{X_{123}} \sim_{\psi_{23}^{\rho}} \mathbb{P}(\mathcal{L}_{13}\oplus\mathcal{O}_{X_{13}})\big|_{X_{123}}\,,$$

is then Kähler and satisfies the vanishing condition in (2.18) over X_{12} and X_{13}. If ρ corresponds to a nonzero element of $H^1(X_{123};\mathbb{Z})$, then (2.17) is not satisfied over $X_{12}\cup X_{13}$ because the connecting homomorphism δ in the exact sequence

$$H^1(X_{12};\mathbb{Z})\oplus H^1(X_{13};\mathbb{Z}) \longrightarrow H^1(X_{123};\mathbb{Z}) \stackrel{\delta}{\longrightarrow} H^2\big(X_{12}\cup X_{13};\mathbb{Z}\big)$$

is injective (X_{12} and X_{13} are simply connected). Let

$$X_{123}^0 = \mathbb{P}\big(0\oplus\mathcal{O}_{X_{123}}\big)\,,\ X_{123}^{\infty} = \mathbb{P}(\mathcal{L}_{12}\oplus 0)|_{X_{123}} \subset \mathbb{P}(\mathcal{L}_{12}\oplus\mathcal{O}_{X_{12}})|_{X_{123}}\,,$$

and $\pi\colon X_{23}\longrightarrow X_{123}$ be the projection map. Using

$$\mathcal{N}_{X_i}X_{23} = \pi^*\mathcal{N}_{X_{1i}}X_{123} = \pi^*\big(\mathcal{O}_{\mathbb{P}^3}(7-2i)|_{X_{123}}\big) \qquad \text{for}\ \ i=2,3,$$

we find that

$$\mathcal{N}_{X_2}X_{23}\otimes\mathcal{N}_{X_3}X_{23}\otimes\mathcal{O}_{X_{23}}(X_{123}) = \pi^*\big(\mathcal{O}_{\mathbb{P}^3}(4)|_{X_{123}}\big)\otimes\mathcal{O}_{X_{23}}(X_{123}^{\infty}) \neq \mathcal{O}_{X_{23}}\,.$$

In order to achieve (2.18) over all smooth strata of X_∂, we replace X_2 by its blowup $\widehat{X}_2$ along X_{123}^0; see the right diagram in Figure 3. The proper transform of X_{23} is still X_{23}, but its normal bundle in $\widehat{X}_2$ is

$$\mathcal{N}_{X_2}X_{23}\otimes\mathcal{O}_{X_{23}}(-X_{123}^0) = \mathcal{N}_{X_2}X_{23}\otimes\mathcal{O}_{X_{23}}(-X_{123}^{\infty})\otimes\pi^*\big(\mathcal{O}_{\mathbb{P}^3}(-4)|_{X_{123}}\big)\,.$$

Thus, the modified 3-fold SC symplectic configuration satisfies the vanishing condition in (2.18) over all smooth strata of X_∂.

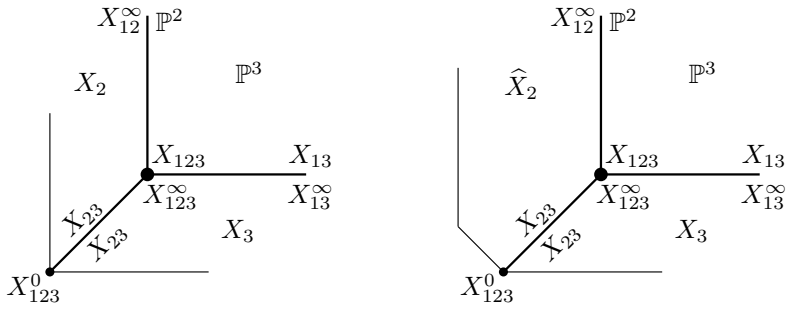

Figure 3: The two NC varieties in Example 2.11.

3 Regularizations

In [4, Sections 2.2,2,3], we introduced notions of symplectic regularizations for an SC divisor $\{V_i\}_{i\in S}$ in X and an SC configuration $\mathbf{X}$; we recall them in Sections 3.1 and 3.2. Such regularizations provide essential auxiliary data for the symplectic smoothing/sum construction of Theorem 2.8, just as they did in the $N=2$ case addressed in [9, 14]. By [4, Theorem 2.17], the space $\mathrm{Symp}^+(\mathbf{X})$ defined in Section 2.1 is weakly homotopy equivalent to the space $\mathrm{Aux}(\mathbf{X})$ of pairs consisting of an element of $\mathrm{Symp}^+(\mathbf{X})$ and a compatible regularization. Proposition 3.9, which is established in Section 3.3, adjusts this weak homotopy equivalence property to incorporate trivializations of (2.14); see also (3.12) and Remark 3.12.

3.1 SC divisors

If B is a manifold, possibly with boundary, and $k\in\mathbb{Z}^{\ge 0}$, we call a family $(\omega_t)_{t\in B}$ of k-forms on X smooth if the k-form $\widetilde{\omega}$ on $B\times X$ given by

$$\widetilde{\omega}_{(t,x)}(v_1,\ldots,v_k)=\begin{cases}\omega_t|_x(v_1,\ldots,v_k), & \text{if } v_1,\ldots v_k\in T_xX;\\ 0, & \text{if } v_1\in T_tB;\end{cases}$$

is smooth. Smoothness for families of other objects is defined similarly.

We call $\pi\colon (L,\rho,\nabla)\longrightarrow V$ a Hermitian line bundle if V is a manifold, $L\longrightarrow V$ is a smooth complex line bundle, ρ is a Hermitian metric on L, and ∇ is a ρ-compatible connection on L. We use the same notation ρ to denote the square of the norm function on L and the Hermitian form on L which is $\mathbb{C}$-antilinear in the second input. Thus,

$$\rho(v)\equiv\rho(v,v),\quad \rho(\mathfrak{i}v,w)=\mathfrak{i}\rho(v,w)=-\rho(v,\mathfrak{i}w)\qquad\forall\ (v,w)\in L\times_V L.$$

Let $\rho^{\mathbb{R}}$ denote the real part of ρ. Each triple (L,ρ,∇) as above induces a connection 1-form $\alpha_{\rho,\nabla}$ on the principal S^1-bundle SL of ρ-unit vectors; see Appendix A. Via the canonical retraction $L\text{–}V\longrightarrow SL$, $\alpha_{\rho,\nabla}$ extends to a 1-form on $L\text{–}V$. A smooth map $h\colon V'\longrightarrow V$ pulls back a Hermitian line bundle (L,ρ,∇) over V to a Hermitian line bundle

$$h^*(L,\rho,\nabla)\equiv(h^*L,h^*\rho,h^*\nabla)\longrightarrow V'.$$

A Riemannian metric on an oriented real vector bundle $L\longrightarrow V$ of rank 2 determines a complex structure on the fibers of L. A Hermitian structure on an oriented real vector bundle $L\longrightarrow V$ of

rank 2 is a pair (ρ,∇) such that (L,ρ,∇) is a Hermitian line bundle with the complex structure $\mathfrak{i}_\rho$ determined by the Riemannian metric $\rho^{\mathbb{R}}$. If Ω is a fiberwise symplectic form on an oriented vector bundle $L\longrightarrow V$ of rank 2, an Ω-compatible Hermitian structure on L is a Hermitian structure (ρ,∇) on L such that $\Omega(\cdot,\mathfrak{i}_\rho\cdot)=\rho^{\mathbb{R}}(\cdot,\cdot)$.

If $(L_i,\rho_i,\nabla^{(i)})_{i\in I}$ is a finite collection of Hermitian line bundles over a symplectic manifold (V,ω),

$$\pi\colon\mathcal{N}\equiv\bigoplus_{i\in I}L_i\longrightarrow V\,,$$

and $\mathrm{pr}_{I;I-i}\colon\mathcal{N}\longrightarrow L_i$ is the component projection map for each $i\in I$, then

$$\widehat{\omega}^{\bullet}_{(\rho_i,\nabla^{(i)})_{i\in I}}\equiv\pi^*\omega+\frac{1}{2}\sum_{i\in I}\mathrm{pr}^*_{I;I-i}\mathrm{d}\big(\rho_i\alpha_{\rho_i,\nabla^{(i)}}\big) \tag{3.1}$$

is a well-defined closed 2-form on the total space of $\mathcal{N}$; it is nondegenerate on a neighborhood of V in $\mathcal{N}$. By (A.9), this definition agrees with [4, (2.10)] whenever $(\rho_i,\nabla^{(i)})$ is an Ω_i-compatible Hermitian structure on L_i

If $\Psi\colon V'\longrightarrow V$ is an embedding, $I'\subset I$, $(L_i,\rho_i,\nabla^{(i)})_{i\in I}$ is a finite collection of Hermitian line bundles over V, and $(L'_i,\rho'_i,\nabla'^{(i)})_{i\in I'}$ is a finite collection of Hermitian line bundles over V', a vector bundle homomorphism

$$\widetilde{\Psi}\colon\bigoplus_{i\in I'}L'_i\longrightarrow\bigoplus_{i\in I}L_i$$

covering Ψ is a product Hermitian inclusion if

$$\widetilde{\Psi}\colon(L'_i,\rho'_i,\nabla'^{(i)})\longrightarrow\Psi^*(L_i,\rho_i,\nabla^{(i)})$$

is an isomorphism of Hermitian line bundles over V' for every $i\in I'$. We call such a morphism a product Hermitian isomorphism covering Ψ if $|I'|=|I|$.

Definition 3.1. Let X be a manifold and $V\subset X$ be a submanifold with normal bundle $\mathcal{N}_XV\longrightarrow V$. A regularization for V in X is a diffeomorphism $\Psi\colon\mathcal{N}'\longrightarrow X$ from a neighborhood of V in $\mathcal{N}_XV$ onto a neighborhood of V in X such that $\Psi(x)=x$ and the isomorphism

$$\mathcal{N}_XV|_x=T^{\mathrm{ver}}_x\mathcal{N}_XV\hookrightarrow T_x\mathcal{N}_XV\xrightarrow{\mathrm{d}_x\Psi}T_xX\longrightarrow\frac{T_xX}{T_xV}\equiv\mathcal{N}_XV|_x$$

is the identity for every $x\in V$.

If (X,ω) is a symplectic manifold and V is a symplectic submanifold in (X,ω), then ω induces a fiberwise symplectic form $\omega|_{\mathcal{N}_XV}$ on the normal bundle $\mathcal{N}_XV$ of V in X via the isomorphism (1.1). We denote the restriction of $\omega|_{\mathcal{N}_XV}$ to a subbundle $L\subset\mathcal{N}_XV$ by $\omega|_L$.

Definition 3.2. Let X be a manifold, $V\subset X$ be a submanifold, and

$$\mathcal{N}_XV=\bigoplus_{i\in I}L_i$$

be a fixed splitting into oriented rank 2 subbundles.

(1) If ω is a symplectic form on X such that V is a symplectic submanifold and $\omega|_{L_i}$ is nondegenerate for every $i\in I$, then an ω-regularization for V in X is a tuple $((\rho_i,\nabla^{(i)})_{i\in I},\Psi)$, where $(\rho_i,\nabla^{(i)})$ is an $\omega|_{L_i}$-compatible Hermitian structure on L_i for each $i\in I$ and Ψ is a regularization for V in X, such that

$$\Psi^*\omega=\widehat{\omega}^{\bullet}_{(\rho_i,\nabla^{(i)})_{i\in I}}\big|_{\mathrm{Dom}(\Psi)}.$$

(2) If B is a manifold, possibly with boundary, and $(\omega_t)_{t\in B}$ is a smooth family of symplectic forms on X which restrict to symplectic forms on V, then an $(\omega_t)_{t\in B}$-family of regularizations for V in X is a smooth family of tuples

$$(\mathcal{R}_t)_{t\in B}\equiv\big((\rho_{t;i},\nabla^{(t;i)})_{i\in I},\Psi_t\big)_{t\in B} \tag{3.2}$$

such that $\mathcal{R}_t$ is an ω_t-regularization for V in X for each $t\in B$ and

$$\big\{(t,v)\in B\times\mathcal{N}_XV:\ v\in\mathrm{Dom}(\Psi_t)\big\}\longrightarrow X,\qquad (t,v)\longrightarrow\Psi_t(v),$$

is a smooth map from a neighborhood of $B\times V$ in $B\times\mathcal{N}_XV$.

Suppose $\{V_i\}_{i\in S}$ is a transverse collection of codimension 2 submanifolds of X. For each $I\subset S$, the last isomorphism in (2.3) with $I'=\emptyset$ provides a natural decomposition

$$\pi_I:\mathcal{N}_XV_I=\bigoplus_{i\in I}\mathcal{N}_{V_{I-i}}V_I\longrightarrow V_I$$

of the normal bundle of V_I in X into oriented rank 2 subbundles. We take this decomposition as given for the purposes of applying Definition 3.2. If in addition $I'\subset I$, let

$$\pi_{I;I'}:\mathcal{N}_{I;I'}\equiv\bigoplus_{i\in I-I'}\mathcal{N}_{V_{I-i}}V_I=\mathcal{N}_{V_{I'}}V_I\longrightarrow V_I\,.$$

There are canonical identifications

$$\mathcal{N}_{I;I-I'}=\mathcal{N}_XV_{I'}|_{V_I},\quad \mathcal{N}_XV_I=\pi^*_{I;I'}\mathcal{N}_{I;I-I'}=\pi^*_{I;I'}\mathcal{N}_XV_{I'}\qquad\forall\ I'\subset I\subset[N]. \tag{3.3}$$

The first equality in the second statement above is used in particular in (3.7).

Definition 3.3. Let X be a manifold and $\{V_i\}_{i\in S}$ be a transverse collection of submanifolds of X. A system of regularizations for $\{V_i\}_{i\in S}$ in X is a tuple $(\Psi_I)_{I\subset S}$, where Ψ_I is a regularization for V_I in X in the sense of Definition 3.1, such that

$$\Psi_I\big(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I)\big)=V_{I'}\cap\mathrm{Im}(\Psi_I) \tag{3.4}$$

for all $I'\subset I\subset S$.

Given a system of regularizations as in Definition 3.3 and $I'\subset I\subset S$, let

$$\mathcal{N}'_{I;I'}=\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I),\qquad \Psi_{I;I'}\equiv\Psi_I\big|_{\mathcal{N}'_{I;I'}}:\mathcal{N}'_{I;I'}\longrightarrow V_{I'}\,.$$

The map $\Psi_{I;I'}$ is a regularization for V_I in $V_{I'}$. As explained in [4, Section 2.2], Ψ_I determines an isomorphism

$$\mathfrak{D}\Psi_{I;I'}:\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow\mathcal{N}_XV_{I'}\big|_{V_{I'}\cap\mathrm{Im}(\Psi_I)} \tag{3.5}$$

of vector bundles covering $\Psi_{I;I'}$ and respecting the natural decompositions of $\mathcal{N}_{I;I-I'}=\mathcal{N}_X V_{I'}|_{V_I}$ and $\mathcal{N}_X V_{I'}$. By the last assumption in Definition 3.1,

$$\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{V_I}}=\mathrm{id}\colon\, \mathcal{N}_{I;I-I'}\longrightarrow \mathcal{N}_X V_{I'}|_{V_I} \tag{3.6}$$

under the canonical identification of $\mathcal{N}_{I;I-I'}$ with $\mathcal{N}_X V_{I'}|_{V_I}$.

Definition 3.4. Let X be a manifold and $\{V_i\}_{i\in S}$ be a transverse collection of submanifolds of X. A regularization for $\{V_i\}_{i\in S}$ in X is a system of regularizations $(\Psi_I)_{I\subset S}$ for $\{V_i\}_{i\in S}$ in X such that

$$\mathrm{Dom}(\Psi_I)=\mathfrak{D}\Psi_{I;I'}^{-1}\big(\mathrm{Dom}(\Psi_{I'})\big),\quad \Psi_I=\Psi_{I'}\circ\mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)} \tag{3.7}$$

for all $I'\subset I\subset S$.

If $(\Psi_I)_{I\subset S}$ is a regularization for $\{V_i\}_{i\in S}$ in X, then (3.6) and (3.7) imply that

$$\begin{aligned}\mathcal{N}'_{I\cup J;I'\cup J}&=\mathcal{N}'_{I;I'}\big|_{V_{I\cup J}},\qquad \Psi_{I\cup J;I'\cup J}=\Psi_{I;I'}\big|_{\mathcal{N}'_{I\cup J;I'\cup J}},\\ \mathfrak{D}\Psi_{I\cup J;I'\cup J}\big|_{\pi^*_{I\cup J;I'\cup J}\mathcal{N}_{I\cup J;(I-I')\cup J}|_{\mathcal{N}'_{I\cup J;I'\cup J}}}&=\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I\cup J;I'\cup J}}}\end{aligned} \tag{3.8}$$

for all $I'\subset I\subset S$ and $J\subset S-I$. Furthermore,

$$\Psi_{I;I''}=\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}\big|_{\mathcal{N}'_{I;I''}},\qquad \mathfrak{D}\Psi_{I;I''}=\mathfrak{D}\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}'_{I;I''}}} \tag{3.9}$$

for all $I''\subset I'\subset I\subset S$.

Definition 3.5. Let X be a manifold and $\{V_i\}_{i\in S}$ be a finite transverse collection of closed submanifolds of X of codimension 2.

(1) If $\omega\in\mathrm{Symp}^+(X,\{V_i\}_{i\in S})$, then an ω-regularization for $\{V_i\}_{i\in S}$ in X is a tuple

$$(\mathcal{R}_I)_{I\subset S}\equiv\big((\rho_{I;i},\nabla^{(I;i)})_{i\in I},\Psi_I\big)_{I\subset S} \tag{3.10}$$

such that $\mathcal{R}_I$ is an ω-regularization for V_I in X for each $I\subset S$, $(\Psi_I)_{I\subset S}$ is a regularization for $\{V_i\}_{i\in S}$ in X, and the induced vector bundle isomorphisms (3.5) are product Hermitian isomorphisms for all $I'\subset I\subset S$.

(2) If B is a manifold, possibly with boundary, and $(\omega_t)_{t\in B}$ is a smooth family of symplectic forms in $\mathrm{Symp}^+(X,\{V_i\}_{i\in S})$, then an $(\omega_t)_{t\in B}$-family of regularizations for $\{V_i\}_{i\in S}$ in X is a smooth family of tuples

$$(\mathcal{R}_{t;I})_{t\in B,I\subset S}\equiv\big((\rho_{t;I;i},\nabla^{(t;I;i)})_{i\in I},\Psi_{t;I}\big)_{t\in B,I\subset S} \tag{3.11}$$

such that $(\mathcal{R}_{t;I})_{I\subset S}$ is an ω_t-regularization for $\{V_i\}_{i\in S}$ in X for each $t\in B$ and $(\mathcal{R}_{t;I})_{t\in B}$ is an $(\omega_t)_{t\in B}$-family of regularizations for V_I in X for each $I\subset S$.

3.2 SC varieties

This section is the analogue of Section 3.1 for SC symplectic configurations, especially those satisfying the topological condition (2.17). Definition 3.7(2) topologizes the set $\widetilde{\mathrm{Aux}}_\hbar(\mathbf{X})$ of triples $((\omega_i)_{i\in[N]},\mathfrak{R},\Phi)$ consisting of a symplectic structure $(\omega_i)_{i\in[N]}$ on a transverse configuration $\mathbf{X}$, an $(\omega_i)_{i\in[N]}$-regularization $\mathfrak{R}$ for $\mathbf{X}$, and a compatible trivialization Φ of (2.14) in a homotopy class $\hbar$. By Proposition 3.9 at the end of this section, the projection

$$\widetilde{\mathrm{Aux}}_\hbar(\mathbf{X})\longrightarrow \mathrm{Symp}_0^+(\mathbf{X})\subset\mathrm{Symp}^+(\mathbf{X}),\quad \big((\omega_i)_{i\in[N]},\mathfrak{R},\Phi\big)\longrightarrow(\omega_i)_{i\in[N]}, \tag{3.12}$$

to the space of SC symplectic configurations satisfying (2.17) induces isomorphisms on π_k for all $k\in\mathbb{Z}^{\ge0}-\{1\}$.

Suppose $\{X_I\}_{I\in\mathcal{P}^*(N)}$ is a transverse configuration in the sense of Definition 2.3. For each $I\in\mathcal{P}^*(N)$ with $|I|\ge2$, let

$$\pi_I\colon\mathcal{N}X_I\equiv\bigoplus_{i\in I}\mathcal{N}_{X_{I-i}}X_I\longrightarrow X_I\,.$$

If in addition $I'\subset I$, let

$$\pi_{I;I'}\colon\mathcal{N}_{I;I'}\equiv\bigoplus_{i\in I-I'}\mathcal{N}_{X_{I-i}}X_I\longrightarrow X_I\,.$$

By the last isomorphism in (2.3) with $X=X_i$ for any $i\in I'$ and $\{V_j\}_{j\in S}=\{X_{ij}\}_{j\in[N]-i}$,

$$\mathcal{N}_{I;I'}=\mathcal{N}_{X_{I'}}X_I\qquad\forall\ I'\subset I\subset[N],\ I'\neq\emptyset.$$

Similarly to (3.3), there are canonical identifications

$$\mathcal{N}_{I;I-I'}=\mathcal{N}X_{I'}|_{X_I},\quad \mathcal{N}X_I=\pi_{I;I'}^*\mathcal{N}_{I;I-I'}=\pi_{I,I'}^*\mathcal{N}X_{I'}\qquad\forall\ I'\subset I\subset[N]; \tag{3.13}$$

the first and last identities above hold if $|I'|\ge2$.

Definition 3.6. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}=\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration. A regularization for $\mathbf{X}$ is a tuple $(\Psi_{I;i})_{i\in I\subset[N]}$, where for each $i\in I$ fixed the tuple $(\Psi_{I;i})_{i\in I\subset[N]}$ is a regularization for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 3.4, such that

$$\Psi_{I;i_1}\big|_{\mathcal{N}_{I;i_1i_2}\cap\mathrm{Dom}(\Psi_{I;i_1})}=\Psi_{I;i_2}\big|_{\mathcal{N}_{I;i_1i_2}\cap\mathrm{Dom}(\Psi_{I;i_2})} \tag{3.14}$$

for all $i_1,i_2\in I\subset[N]$.

Given a regularization as in Definition 3.6 and $I'\subset I\subset[N]$ with $|I|\ge2$ and $I'\neq\emptyset$, let

$$\mathcal{N}'_{I;I'}=\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_{I;i}),\quad \Psi_{I;I'}=\Psi_{I;i}|_{\mathcal{N}'_{I;I'}}\colon\mathcal{N}'_{I;I'}\longrightarrow X_{I'}\qquad\text{if } i\in I'; \tag{3.15}$$

by (3.14), $\Psi_{I;I'}(v)$ does not depend on the choice of $i\in I'$. Let

$$\mathfrak{D}\Psi_{I;i;I'}\colon\pi_{I;I'}^*\mathcal{N}_{I;i\cup(I-I')}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow\mathcal{N}_{I';i}\big|_{\mathrm{Im}(\Psi_{I;I'})} \tag{3.16}$$

be the associated vector bundle isomorphism as in (3.5). If $|I'|\ge2$, we define an isomorphism of split vector bundles

$$\begin{aligned}&\mathfrak{D}\Psi_{I;I'}\colon\pi_{I;I'}^*\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow\mathcal{N}X_{I'}\big|_{\mathrm{Im}(\Psi_{I;I'})},\\ &\mathfrak{D}\Psi_{I;I'}\big|_{\pi_{I;I'}^*\mathcal{N}_{I;i\cup(I-I')}\big|_{\mathcal{N}'_{I;I'}}}=\mathfrak{D}\Psi_{I;i;I'}\quad\forall\ i\in I';\end{aligned} \tag{3.17}$$

by (3.14), the last maps agree on the overlaps.

By (3.15)-(3.17) and (3.8),

$$\begin{aligned}\mathcal{N}'_{I\cup J;I'\cup J}&=\mathcal{N}'_{I;I'}\big|_{X_{I\cup J}}, \qquad \Psi_{I\cup J;I'\cup J}=\Psi_{I;I'}\big|_{\mathcal{N}'_{I\cup J;I'\cup J}},\\ \mathfrak{D}\Psi_{I\cup J;I'\cup J}\big|_{\pi^*_{I\cup J;I'\cup J}\mathcal{N}_{I\cup J;(I-I')\cup J}|_{\mathcal{N}'_{I\cup J;I'\cup J}}}&=\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}'_{I\cup J;I'\cup J}}}\end{aligned} \tag{3.18}$$

for all $I'\subset I\subset[N]$ and $J\subset[N]-I$ with $|I|\ge 2$ in all three cases, $|I'|\ge 1$ in the first two cases, and $|I'|\ge 2$ in the last case. By (3.15), (3.17), and (3.9),

$$\Psi_{I;I''}=\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}\big|_{\mathcal{N}'_{I;I''}}, \quad \mathfrak{D}\Psi_{I;I''}=\mathfrak{D}\Psi_{I';I''}\circ\mathfrak{D}\Psi_{I;I'}\big|_{\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}'_{I;I''}}} \tag{3.19}$$

for all $I''\subset I'\subset I\subset[N]$ with $|I'|\ge 2$ in both cases, $|I''|\ge 2$ in the first case, and $|I''|\ge 2$ in the second case.

Definition 3.7. Let $N\in\mathbb{Z}^+$ and $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration.

(1) If $(\omega_i)_{i\in[N]}$ is a symplectic structure on $\mathbf{X}$ in the sense of Definition 2.4, an $(\omega_i)_{i\in[N]}$-regularization for $\mathbf{X}$ is a tuple

$$\mathfrak{R}\equiv(\mathcal{R}_I)_{I\in\mathcal{P}^*(N)}\equiv\big(\rho_{I;i},\nabla^{(I;i)},\Psi_{I;i}\big)_{i\in I\subset[N]} \tag{3.20}$$

such that $(\Psi_{I;i})_{i\in I\subset[N]}$ is a regularization for $\mathbf{X}$ in the sense of Definition 3.6 and for each $i\in[N]$ the tuple

$$\big((\rho_{I;j},\nabla^{(I;j)})_{j\in I-i},\Psi_{I;i}\big)_{i\in I\subset[N]}$$

is an ω_i-regularization for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 3.5(1).

(2) If B is a smooth manifold, possibly with boundary, and $(\omega_{t;i})_{t\in B,i\in[N]}$ is a smooth family of symplectic structures on $\mathbf{X}$, then an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$ is a family of tuples

$$(\mathfrak{R}_t)_{t\in B}\equiv(\mathcal{R}_{t;I})_{t\in B,I\in\mathcal{P}^*(N)}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]} \tag{3.21}$$

such that $(\mathcal{R}_{t;I})_{I\in\mathcal{P}^*(N)}$ is an $(\omega_{t;i})_{i\in[N]}$-regularization for $\mathbf{X}$ for each $t\in B$ and for each $i\in[N]$ the tuple

$$\big((\rho_{t;I;j},\nabla^{(t;I;j)})_{j\in I-i},\Psi_{t;I;i}\big)_{t\in B,i\in I\subset[N]}$$

is an $(\omega_{t;i})_{t\in B}$-family of regularizations for $\{X_{ij}\}_{j\in[N]-i}$ in X_i in the sense of Definition 3.5(2).

The assumptions in Definition 3.7(1) imply that the corresponding isomorphisms (3.17) are product Hermitian isomorphisms covering the maps (3.15).

The precise definition of the total space of the complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ in (2.14) depends on the choices of identifications $\Psi_{ij;j}$ of neighborhoods of X_{ij} in X_j and in X_j and of the ω_j-tame complex structures $\mathfrak{i}_{ij;j}$ on (the fibers of) $\mathcal{N}_{X_j}X_{ij}$ that satisfy (2.10) and (2.11), respectively. For a smooth family $(X_\emptyset,\omega_{t;i})_{t\in B,i\in[N]}$ of SC symplectic varieties as in Definition 2.5, such choices can be made continuously with respect to $t\in B$. We then obtain a complex line bundle

$$\pi_{B;\partial}\colon\mathcal{O}_{B;X_\partial}(X_\emptyset)\equiv\bigcup_{t\in B}\{t\}\times\mathcal{O}_{t;X_\partial}(X_\emptyset)\longrightarrow B\times X_\partial, \tag{3.22}$$

where $\mathcal{O}_{t;X_\partial}(X_\emptyset) \longrightarrow X_\partial$ is the line bundle corresponding to the symplectic structure $(\omega_{t;i})_{i\in[N]}$ on $X_\emptyset$.

If $\pi\colon L\longrightarrow M$ is a complex line bundle, we call a smooth map $\Phi\colon L\longrightarrow\mathbb{C}$ a trivialization of L if Φ restricts to an isomorphism on each fiber of L. We call a family $(\hbar_t)_{t\in B}$ of homotopy classes of trivializations of $\mathcal{O}_{t;X_\partial}(X_\emptyset)$ continuous if for each $t_0\in B$ there exist a neighborhood U of t_0 in B and a trivialization Φ of $\mathcal{O}_{B;X_\partial}(X_\emptyset)\big|_{\pi_{B;\partial}^{-1}(U)}$ such that the restriction of Φ to $\{t\}\times\mathcal{O}_{t;X_\partial}(X_\emptyset)$ lies in $\hbar_t$ for every $t\in U$.

A regularization $\mathfrak{R}$ for $\mathbf{X}$ as in Definition 3.7(1) specifies the identifications $\Psi_{ij;j}$ and complex structures $\mathfrak{i}_{ij;j}$ needed for the construction of the complex line bundles in (2.12) and (2.13). Given a regularization $\mathfrak{R}$, we thus view the line bundles $\mathcal{O}_{X_{jk}}(X_{ijk})$, $\mathcal{O}_{X_\partial}(X_i)$, and $\mathcal{O}_{X_\partial}(X_\emptyset)$ as explicitly specified and denote them by $\mathcal{O}_{\mathfrak{R};X_{jk}}(X_{ijk})$, $\mathcal{O}_{\mathfrak{R};X_\partial}(X_i)$, and $\mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)$, respectively. By (3.18),

$$\mathcal{O}_{\mathfrak{R};X_{jk}}(X_{ijk})\big|_{X_{jj'k}}=\mathcal{O}_{\mathfrak{R};X_{jj'k}}(X_{ijj'k})\big|_{X_{jj'k}}=\mathcal{O}_{\mathfrak{R};X_{j'k}}(X_{ij'k})\big|_{X_{jj'k}} \tag{3.23}$$

for all $i,j,j',k\in[N]$ with $i,k\neq j,j'$ and $i\neq k$. An $(\omega_{t;i})_{t\in B,i\in[N]}$-family $(\mathfrak{R}_t)_{t\in B}$ of regularizations for $\mathbf{X}$ completely specifies the complex line bundle (3.22).

Let $s_{jk;i}$ denote the standard section of the line bundle $\mathcal{O}_{\mathfrak{R};X_{jk}}(X_{ijk})$:

$$s_{jk;i}(x)=\begin{cases}(x,v,v)\in\Psi_{ijk;jk}^{-1\,*}\pi_{ijk;jk}^*\mathcal{N}_{ijk;jk}\,, & \text{if } x=\Psi_{ijk;jk}(v);\\ (x,1)\in(X_{jk}-X_{ijk})\times\mathbb{C}, & \text{if } x\in X_{jk}-X_{ijk}\,.\end{cases} \tag{3.24}$$

This section is well-defined on the overlap by definition of $\mathcal{O}_{\mathfrak{R};X_{jk}}(X_{ijk})$; see (2.8). By (3.18),

$$s_{jk;i}\big|_{X_{jj'k}}=s_{j'k;i}\big|_{X_{jj'k}}\qquad\forall\, i,j,j',k\in[N],\ i,k\neq j,j',\ i\neq k. \tag{3.25}$$

If $I\subset[N]$, $j,k\in I$ are distinct, and $i\notin I$, let

$$\mathcal{O}_{\mathfrak{R};X_I}(X_i)=\mathcal{O}_{\mathfrak{R};X_{jk}}(X_{ijk})\big|_{X_I}\,,\qquad s_{I;i}=s_{jk;i}\big|_{X_I}\,.$$

By (3.23) and (3.25), $\mathcal{O}_{\mathfrak{R};X_I}(X_i)$ and $s_{I;i}$ are independent of the choice of $j,k\in I$. By (3.24), $s_{I;i}$ does not vanish outside of $X_{I\cup i}\subset X_I$.

For every $I\subset[N]$ with $|I|\ge2$, define a smooth bundle map

$$\begin{aligned}\Pi_{\mathfrak{R};I}\colon\mathcal{N}X_I\equiv\bigoplus_{i\in I}\mathcal{N}_{X_{I-i}}X_I\longrightarrow\mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)\big|_{X_I}\equiv\bigotimes_{i\in I}\mathcal{N}_{X_{I-i}}X_I\ \otimes\bigotimes_{i\notin I}\mathcal{O}_{\mathfrak{R};X_I}(X_i),\\ \Pi_{\mathfrak{R};I}\big((v_{I;i})_{i\in I}\big)=\bigotimes_{i\in I}v_{I;i}\otimes\bigotimes_{i\notin I}s_{I;i}(x)\qquad\forall\,(v_{I;i})_{i\in I}\in\mathcal{N}X_I\big|_x,\ x\in X_I.\end{aligned} \tag{3.26}$$

This map is surjective over the complement $X_I^\star$ of the submanifolds $X_{I'}\subset X_I$ with $I'\supsetneq I$.

Definition 3.8. Let $\mathbf{X}$ be an SC symplectic configuration as in (2.7) and $\mathfrak{R}$ be a regularization for $\mathbf{X}$ as in (3.20). A trivialization Φ of the complex line bundle $\mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)$ over X_∂ is $\mathfrak{R}$-compatible if

$$\Phi\big(\Pi_{\mathfrak{R};I'}\big(\mathfrak{D}\Psi_{I;I'}(v_{I;I'},v_{I;I-I'})\big)\big)=\Phi\big(\Pi_{\mathfrak{R};I}(v_{I;I'},v_{I;I-I'})\big) \tag{3.27}$$

for all $(v_{I;I'},v_{I;I-I'})\in\pi_{I;I'}^*\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}}$ and $I'\subset I\subset[N]$ with $|I'|\ge2$.

Let $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be an N-fold transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct and $(\omega_{t;i})_{t\in B,i\in[N]}$ be a family of symplectic structures on $\mathbf{X}$. Suppose the tuples

$$\big(\mathfrak{R}_t^{(1)}\big)_{t\in B}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}^{(1)}\big)_{t\in B,i\in I\subset[N]},$$
$$\big(\mathfrak{R}_t^{(2)}\big)_{t\in B}\equiv\big(\rho_{t;I;i},\nabla^{(t;I;i)},\Psi_{t;I;i}^{(2)}\big)_{t\in B,i\in I\subset[N]}$$

are $(\omega_{t;i})_{t\in B,i\in[N]}$-families of regularizations for $\mathbf{X}$. We define

$$\big(\mathfrak{R}_t^{(1)}\big)_{t\in B}\cong\big(\mathfrak{R}_t^{(2)}\big)_{t\in B}\tag{3.28}$$

if the two families of regularizations agree on the level of germs, i.e. there exists an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations as in (3.21) such that

$$\mathrm{Dom}(\Psi_{t;I;i})\subset\mathrm{Dom}(\Psi_{t;I;i}^{(1)}),\mathrm{Dom}(\Psi_{t;I;i}^{(2)})\quad\Psi_{t;I;i}=\Psi_{t;I;i}^{(1)}\big|_{\mathrm{Dom}(\Psi_{t;I;i})},\Psi_{t;I;i}^{(2)}\big|_{\mathrm{Dom}(\Psi_{t;I;i})}\tag{3.29}$$

for all $t\in B$ and $i\in I\subset[N]$.

A family (3.21) satisfying (3.29) provides a canonical identification of the line bundles (3.22) determined by $\mathfrak{R}_t^{(1)}$ and $\mathfrak{R}_t^{(2)}$. This identification is independent of the choice of a family (3.21) satisfying (3.29). Thus, the line bundles (3.22) determined by $(\omega_{t;i})_{t\in B,i\in[N]}$-families of regularizations for $\mathbf{X}$ satisfying (3.28) are canonically identified.

Proposition 3.9. *Let $N\in\mathbb{Z}^+$, $\mathbf{X}\equiv\{X_I\}_{I\in\mathcal{P}^*(N)}$ be a transverse configuration such that X_{ij} is a closed submanifold of X_i of codimension 2 for all $i,j\in[N]$ distinct, and $X_i^*\subset X_i$ for each $i\in[N]$ be an open subset, possibly empty, such that $\overline{X_i^*}\cap X_I=\emptyset$ for all $i\in I\subset[N]$ with $|I|=3$. Suppose*

- *B is a compact manifold, possibly with non-empty boundary ∂B, such that the restriction homomorphism $H^1(B;\mathbb{Z})\longrightarrow H^1(\partial B;\mathbb{Z})$ is surjective,*
- *$N(\partial B),N'(\partial B)$ are tubular neighborhoods of $\partial B\subset B$ such that $\overline{N'(\partial B)}\subset N(\partial B)$,*
- *$(\omega_{t;i})_{t\in B,i\in[N]}$ is a smooth family of elements of $\mathrm{Symp}^+(\mathbf{X})$ such that the associated line bundle $\mathcal{O}_{B;X_\partial}(X_\emptyset)$ is trivializable,*
- *$(\hbar_t)_{t\in B}$ is a continuous family of homotopy classes of trivializations of the line bundles $\mathcal{O}_{t;X_\partial}(X_\emptyset)$ determined by $(\omega_{t;i})_{i\in[N]}$ for all $t\in B$,*
- *$(\mathfrak{R}_t)_{t\in N(\partial B)}$ is an $(\omega_{t;i})_{t\in N(\partial B),i\in[N]}$-family of regularizations for $\mathbf{X}$, and*
- *$(\Phi_t)_{t\in N(\partial B)}$ is a smooth family of $\mathfrak{R}_t$-compatible trivializations of the complex line bundles $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)$ in the homotopy class $\hbar_t$.*

Then there exist a smooth family $(\mu_{t,\tau;i})_{t\in B,\tau\in\mathbb{I},i\in[N]}$ of 1-forms on $X_\emptyset$, an $(\omega_{t,1;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$, and a smooth family $(\widetilde{\Phi}_t)_{t\in B}$ of $\widetilde{\mathfrak{R}}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)$ in the homotopy class $\hbar_t$ such that

$$\big(\omega_{t,\tau;i}\equiv\omega_{t;i}+\mathrm{d}\mu_{t,\tau;i}\big)_{i\in[N]}\in\mathrm{Symp}^+(\mathbf{X}),\quad\mu_{t,0;i}=0,\quad\mathrm{supp}\big(\mu_{\cdot,\tau;i}\big)\subset\big(B-N'(\partial B)\big)\times(X_i-X_i^*)$$

for all $t\in B$, $\tau\in\mathbb{I}$, and $i\in[N]$, and

$$\big(\widetilde{\mathfrak{R}}_t\big)_{t\in N'(\partial B)}\cong\big(\mathfrak{R}_t\big)_{t\in N'(\partial B)},\quad\big(\widetilde{\Phi}_t\big)_{t\in N'(\partial B)}=\big(\Phi_t\big)_{t\in N'(\partial B)}.\tag{3.30}$$

3.3 Existence of compatible isomorphisms

We prove Proposition 3.9 below by making trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)$ $\mathfrak{R}_t$-compatible over neighborhoods of the strata of $X_\emptyset$. This argument in a sense adapts the setup of the proof of [4, Theorem 2.17] to deal with bundle trivializations. The key inductive step in this case is carried out by Lemma 3.11.

Let $\mathbf{X}$ be an SC symplectic configuration as in (2.7), $\mathfrak{R}$ be a regularization for $\mathbf{X}$ as in (3.20), and $W \subset X_\partial$. We call a trivialization Φ of $\mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)|_W$ $\mathfrak{R}$-compatible if (3.27) is satisfied whenever $v_{I;I'} \in \Psi_{I;I'}^{-1}(W)|_{X_I \cap W}$.

Proof of Proposition 3.9. With all references to the line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ dropped, Proposition 3.9 is a special case of [4, Theorem 2.17]. Thus, we can assume that $(\mathfrak{R}_t)_{t\in B}$ is an $(\omega_{t;i})_{t\in B, i\in[N]}$-family of regularizations for $\mathbf{X}$ as in (3.21) and that $\mathcal{O}_{B;X_\partial}(X_\emptyset)$ is the line bundle as in (3.22) constructed using this family of regularizations. By Lemma 3.10, we can assume that this line bundle admits a trivialization

$$\Phi_B \equiv (\Phi_t)_{t\in B} \colon \mathcal{O}_{B;X_\partial}(X_\emptyset) \longrightarrow \mathbb{C}$$

so that Φ_t lies in $\hbar_t$ for every $t \in B$. In particular, this trivialization is $\mathfrak{R}_t$-compatible for every $t \in N(\partial B)$. The above applications of [4, Theorem 2.17] and Lemma 3.10 require shrinking $N(\partial B)$ slightly; so $N(\partial B)$ in the remainder of this proof corresponds to $N'(\partial B)$ in the statement of the proposition.

Below we deform Φ_B to make its restriction to $\mathcal{O}_{t;X_\partial}(X_\emptyset)$ compatible with a shrinking of $\mathfrak{R}_t$ for all $t \in B$. Fix a total order $>$ on subsets $I \subset [N]$ with $|I| \ge 2$ so that $I > I^*$ whenever $I \supsetneq I^*$. We will proceed inductively on the strata X_{I^*} of X_∂ using the total order $>$.

Suppose $I^* \subset [N]$ with $|I^*| \ge 2$, $W^>$ is a neighborhood of

$$X_{I^*}^> \equiv \bigcup_{I > I^*} X_I \subset X_\partial$$

in X_∂, and $(\Phi_t^>)_{t\in B}$ is a smooth family of $\mathfrak{R}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_{W^>}$ such that

$$\big(\Phi_t^>\big)_{t\in N(\partial B)} = \big(\Phi_t|_{W^>}\big)_{t\in N(\partial B)}, \quad \big|\Phi_t(x) - \Phi_t^>(x)\big|_t < \big|\Phi_t(x)\big|_t \ \ \forall\, x \in W^>,\, t \in B. \tag{3.31}$$

Let W' be a neighborhood of $X_{I^*}^> \subset X_\partial$ such that $\overline{W'} \subset W$. We apply Lemma 3.11 below with $W = W^>$ and $\Phi_t' = \Phi_t^>$. There thus exist a neighborhood W_{I^*} of $X_{I^*} \subset X_\partial$, an $(\omega_{t;i})_{t\in B, i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$, and a smooth family $(\widetilde{\Phi}_t')_{t\in B}$ of $\widetilde{\mathfrak{R}}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_{W'\cup W_{I^*}}$ satisfying the first condition in (3.30) and the two conditions in (3.31) with $\Phi_t^>$ replaced by $\Phi_t^\ge \equiv \widetilde{\Phi}_t'$ and $W^>$ by $W^\ge \equiv W' \cup W_{I^*}$.

By the downward induction on $\mathcal{P}^*(N)$ with respect to $>$, we thus obtain an $(\omega_{t;i})_{t\in B, i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$ and a smooth family $(\widetilde{\Phi}_t)_{t\in B}$ of $\widetilde{\mathfrak{R}}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)$ satisfying (3.30) such that

$$\big|\Phi_t(x) - \widetilde{\Phi}_t(x)\big|_t < \big|\Phi_t(x)\big|_t \ \ \forall\, x \in X_\partial,\, t \in B.$$

This implies that

$$\Phi_{t;\tau}\equiv(1-\tau)\Phi_t+\tau\widetilde{\Phi}_t\colon\, \mathcal{O}_{\widetilde{\mathfrak{R}}_t;X_\partial}(X_\emptyset)\longrightarrow X_\emptyset\times\mathbb{C},\qquad \tau\in\mathbb{I},$$

is a homotopy from Φ_t to $\widetilde{\Phi}_t$ through trivializations of $\mathcal{O}_{\widetilde{\mathfrak{R}}_t;X_\partial}(X_\emptyset)$ for every $t\in B$. Thus, the trivialization $\widetilde{\Phi}_t$ lies in the homotopy class $\hbar_t$ for every $t\in B$. □

Lemma 3.10. *Let $N'(\partial B)\subset N(\partial B)\subset B$ be as in Proposition 3.9, X be a CW complex, and $\mathcal{L}\longrightarrow B\times X$ be a trivializable complex line bundle. Suppose*

- *$(\hbar_t)_{t\in B}$ is a continuous family of homotopy classes of trivializations of the complex line bundles $\mathcal{L}_t\equiv\mathcal{L}|_{\{t\}\times X}$,*
- *$\Phi_{N(\partial B)}$ is a trivialization of $\mathcal{L}|_{N(\partial B)\times X}$ such that $\Phi_{N(\partial B)}|_{\mathcal{L}_t}$ lies in $\hbar_t$ for every $t\in N(\partial B)$.*

Then there exists a trivialization Φ_B of $\mathcal{L}$ so that $\Phi_B|_{\mathcal{L}_t}$ lies in $\hbar_t$ for every $t\in B$ and

$$\Phi_B|_{N'(\partial B)}=\Phi_{N(\partial B)}|_{N'(\partial B)}\,. \tag{3.32}$$

Proof. Let $\eta_{S^1}\in H^1(S^1;\mathbb{Z})$ be a generator. For any topological space Y, denote by $[Y,S^1]$ the set of homotopy classes of continuous maps $Y\longrightarrow S^1$. If Y is a CW complex, then the map

$$\big[Y,S^1\big]\longrightarrow H^1(Y;\mathbb{Z}),\qquad \big[f\colon Y\longrightarrow S^1\big]\longrightarrow f^*\eta_{S^1}, \tag{3.33}$$

is a bijection.

We can assume that X is connected. Let $\pi_B\colon B\times X\longrightarrow B$ be the projection and

$$\Phi'_B\colon\mathcal{L}\longrightarrow\mathbb{C}$$

be a trivialization of $\mathcal{L}$. For each $t\in B$, denote by $\hbar'_t$ the homotopy class of maps $f\colon X\longrightarrow S^1$ such that the trivialization

$$\Phi'_f\colon\mathcal{L}_t\longrightarrow\mathbb{C},\qquad \Phi'_f(v)=f(\pi_{\mathcal{L}}(v))\Phi'_B(v),$$

lies in $\hbar_t$.

Since $(\hbar_t)_{t\in B}$ is a continuous family of homotopy classes, for each $t_0\in B$ there exist a contractible neighborhood U of t_0 in B and a continuous function $F_U\colon U\times X\longrightarrow S^1$ such that $F_U|_{\{t\}\times X}$ lies in $\hbar'_t$ for every $t\in U$. The class

$$\eta_U\equiv F_U^*\eta_{S^1}\in H^1\big(U\times X;\mathbb{Z}\big)$$

is then independent of the choice of U. These classes agree on the overlaps and thus determine an element $\eta_B\in H^1(B\times X;\mathbb{Z})$. Since the map (3.33) is surjective, there exists a continuous map $F\colon B\times X\longrightarrow S^1$ such that $\eta_B=F^*\eta_{S^1}$. Define

$$\Phi_F\colon\mathcal{L}\longrightarrow\mathbb{C},\qquad \Phi'_F(v)=F(\pi_{\mathcal{L}}(v))\Phi'_B(v)\,.$$

For each $t\in B$, the trivialization $\Phi_F|_{\mathcal{L}_t}$ lies in $\hbar_t$.

Let $F_{N(\partial B)}\colon N(\partial B)\times X\longrightarrow\mathbb{C}^*$ be the continuous function so that

$$\Phi_{N(\partial B)}(v)=F_{N(\partial B)}(\pi_{\mathcal{L}}(v))\Phi'_{F;2}(v)\qquad\forall\,v\in\mathcal{L}|_{N(\partial B)\times X}\,. \tag{3.34}$$

For each $t\in\partial B$, the restrictions of $\Phi_{N(\partial B)}$ and Φ'_F to $\mathcal{L}_t$ are homotopic. Thus, the restriction of $F_{N(\partial B)}$ to $\{t\}\times X$ with $t\in\partial B$ is null-homotopic. This implies that

$$F^*_{N(\partial B)}\eta_{S^1}\big|_{\partial B\times X}\in H^1\big(\partial B;\mathbb{Z}\big)\otimes H^0(X;\mathbb{Z})\subset H^1\big(\partial B\times X;\mathbb{Z}\big).$$

Since the restriction homomorphism $H^1(B;\mathbb{Z})\longrightarrow H^1(\partial B;\mathbb{Z})$ is surjective and the map (3.33) is bijective, there thus exists a continuous map $f\colon B\longrightarrow S^1$ such that

$$F^*_{N(\partial B)}\eta_{S^1}\big|_{\partial B\times X}=\big(f^*\eta_{S^1}\big|_{\partial B}\big)\otimes 1,\quad \big[f\circ\pi_B|_{\partial B\times X}\big]=\big[F_{N(\partial B)}|_{\partial B\times X}\big]\in\big[\partial B\times X,S^1\big].$$

Since $N(\partial B)$ is a tubular neighborhood of ∂B, the last equality above implies that there exists a continuous function

$$G\colon B\times X\longrightarrow S^1\qquad\text{s.t.}\quad G(t,x)=\begin{cases}F_{N(\partial B)}(t,x), & \text{if } t\in N'(\partial B);\\ f(t), & \text{if } t\notin N(\partial B).\end{cases} \tag{3.35}$$

Define

$$\Phi_B\colon\mathcal{L}\longrightarrow\mathbb{C},\qquad \Phi_B(v)=G(\pi_{\mathcal{L}}(v))\Phi'_{F;2}(v)\,.$$

By the first case in (3.35) and (3.34), this trivialization of $\mathcal{L}$ satisfies (3.32). Since $N(\partial B)$ is a tubular neighborhood of ∂B, this implies that the restrictions of Φ_B and $\Phi_{N(\partial B)}$ to $\mathcal{L}_t$ are homotopic for every $t\in N(\partial B)$. By the second case in (3.35), the restrictions of Φ_B and Φ'_F to $\mathcal{L}_t$ are homotopic for every $t\notin N(\partial B)$ as trivializations of $\mathcal{L}_t$. Thus, $\Phi_B|_{\mathcal{L}_t}$ lies in $\hbar_t$ for every $t\in B$. □

Lemma 3.11. *Let $\mathbf{X}$, B, and $(\omega_{t;i})_{t\in B,i\in[N]}$ be as in Proposition 3.9 and $N(\partial B)$ be a neighborhood of $\partial B\subset B$. Suppose*

- *$I^*\in\mathcal{P}(N)$ and $W,W'\subset X_\partial$ are open subsets such that*

$$|I^*|\ge 2,\quad \overline{W'}\subset W,\quad X_I\subset W'\ \ \forall\, I\in\mathcal{P}(N),\, I\supsetneq I^*\,, \tag{3.36}$$

- *$(\mathfrak{R}_t)_{t\in B}$ is an $(\omega_{t;i})_{t\in B,i\in[N]}$-family of regularizations for $\mathbf{X}$,*
- *$(\Phi_t)_{t\in B}$ is a smooth family of trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)$ over X_∂ which are $\mathfrak{R}_t$-compatible for $t\in N(\partial B)$,*
- *$(\Phi'_t)_{t\in B}$ is a smooth family of $\mathfrak{R}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_W$ such that*

$$\big(\Phi_t|_W\big)_{t\in N(\partial B)}=\big(\Phi'_t\big)_{t\in N(\partial B)},\quad \big|\Phi_t(x)-\Phi'_t(x)\big|_t<\big|\Phi_t(x)\big|_t\ \ \forall\, x\in W,\, t\in B. \tag{3.37}$$

Then there exist a neighborhood W_{I^*} of $X_{I^*}\subset X_\partial$, an $(\omega_{t;i})_{t\in B,i\in[N]}$-family $(\widetilde{\mathfrak{R}}_t)_{t\in B}$ of regularizations for $\mathbf{X}$, and a smooth family $(\widetilde{\Phi}'_t)_{t\in B}$ of $\widetilde{\mathfrak{R}}_t$-compatible trivializations of $\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_{W'\cup W_{I^*}}$ such that

$$\big(\widetilde{\mathfrak{R}}_t\big)_{t\in B}\cong\big(\mathfrak{R}_t\big)_{t\in B},\quad \big(\widetilde{\Phi}'_t|_{W'}\big)_{t\in B}=\big(\Phi'_t|_{W'}\big)_{t\in B}\,, \tag{3.38}$$

$$\big(\Phi_t|_{W'\cup W_{I^*}}\big)_{t\in N(\partial B)}=\big(\widetilde{\Phi}'_t\big)_{t\in N(\partial B)},\quad \big|\Phi_t(x)-\widetilde{\Phi}'_t(x)\big|<\big|\Phi_t(x)\big|\ \ \forall\, x\in W'\cup W_{I^*},\, t\in B, \tag{3.39}$$

Proof. Let $(\mathfrak{R}_t)_{t\in B}$ be as in (3.21). For each $x\in X_\partial$ and $x\in W$, let

$$\begin{aligned}\Phi_t(x)&=\Phi_t|_{\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x}\colon \mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x\longrightarrow\mathbb{C}\qquad\text{and}\\ \Phi'_t(x)&=\Phi'_t|_{\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x}\colon \mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x\longrightarrow\mathbb{C},\end{aligned}$$

respectively.

Choose open subsets $W''\subset W'''\subset X_\partial$ such that

$$\overline{W'}\subset W'',\quad \overline{W''}\subset W''',\quad \overline{W'''}\subset W. \tag{3.40}$$

By (3.36) and [4, Lemma 5.8], we can shrink the domains of the maps $\Psi_{t;I;i}$ so that

$$\Psi^{-1}_{t;I^*;\emptyset}\big(W'\big)\subset \mathcal{N}X_{I^*}|_{X_{I^*}\cap W''},\quad \Psi_{t;I;\emptyset}\big(\mathrm{Dom}\big(\Psi_{t;I;\emptyset}|_{X_I\cap W'}\big)\big)\subset W''\ \ \forall\, I\in\mathcal{P}^*(N),\ |I|\ge 2. \tag{3.41}$$

Let $\rho\colon X_{I^*}\longrightarrow[0,1]$ a smooth function such that

$$\rho(x)=\begin{cases}1, & \text{if } x\in X_{I^*}\cap W''';\\ 0, & \text{if } x\notin X_{I^*}\cap W.\end{cases} \tag{3.42}$$

Define

$$\begin{aligned}\widetilde{\Phi}_{t;I^*}\colon\ &\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_{X_{I^*}}\longrightarrow\mathbb{C},\\ \widetilde{\Phi}_{t;I^*}(x)\equiv\widetilde{\Phi}_{t;I^*}|_{\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x}&\equiv\rho(x)\Phi'_t(x)+\big(1-\rho(x)\big)\Phi_t(x)\quad\forall\, x\in X_{I^*},\,t\in B.\end{aligned} \tag{3.43}$$

By (3.43) and (3.42),

$$\big(\widetilde{\Phi}_{t;I^*}|_{X_{I^*}\cap W'''}\big)_{t\in B}=\big(\Phi'_t|_{X_{I^*}\cap W'''}\big)_{t\in B}. \tag{3.44}$$

By (3.43) and (3.37),

$$\big(\Phi_t|_{X_{I^*}}\big)_{t\in N(\partial B)}=\big(\widetilde{\Phi}_{t;I^*}\big)_{t\in N(\partial B)},\quad \big|\Phi_t(x)-\widetilde{\Phi}_{t;I^*}(x)\big|<\big|\Phi_t(x)\big|\ \ \forall\, x\in X_{I^*},\,t\in B. \tag{3.45}$$

In particular, $\widetilde{\Phi}_{t;I^*}(x)$ is a complex linear isomorphism for all $x\in X_{I^*}$ and $t\in B$.

Since B is compact, there exists a neighborhood W'_{I^*} of $X_{I^*}\subset X_\emptyset$ such that

$$B\times W'_{I^*}\subset\bigcup_{t\in B}\{t\}\times\mathrm{Im}\big(\Psi_{t;I^*;\emptyset}\big). \tag{3.46}$$

Since $\overline{W''}\subset W'''$ and $X_I\subset W'$ for all $I\in\mathcal{P}(N)$ with $I\supsetneq I^*$, we can shrink W'_{I^*} so that

$$W''\cap W'_{I^*}\subset\Psi_{t;I^*;\emptyset}\big(\mathrm{Dom}(\Psi_{t;I^*;\emptyset})|_{X_{I^*}\cap W'''}\big)\ \ \forall\, t\in B,\quad X_I\cap W'_{I^*}\subset W''\ \ I\in\mathcal{P}(N),\,I\supsetneq I^*. \tag{3.47}$$

Let $W^\star_{I^*}\subset W'_{I^*}$ be the complement of the subspaces $X_I\subset X_\emptyset$ with $I\subset[N]$ such that $I\not\subset I^*$.

Fix $t\in B$ and $x\in W^\star_{I^*}$. Let $I_x\subset I^*$ be the largest subset such that $x\in X_{I_x}$. By (3.46),

$$x=\Psi_{t;I^*;I_x}(v_{I^*;I_x})$$

for a unique $v_{I^*;I_x}\in\mathcal{N}_{I^*;I_x}\subset\mathcal{N}X_{I^*}$. Since $W'_{I^*}\subset X_\partial$, $|I_x|\ge 2$. Let $x_0=\pi_{I^*}(v_{I^*;I_x})$. By the maximality of I_x, the bundle map $\Pi_{\mathfrak{R}_t;I_x}$ is surjective over x for every $t\in B$. Given $v\in\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x$, choose

$$v_{I^*;I^*-I_x}\in\mathcal{N}_{I^*;I^*-I_x}\big|_{x_0}\quad\text{s.t.}\quad v=\Pi_{\mathfrak{R}_t;I_x}\big(\mathfrak{D}\Psi_{t;I^*;I_x}(v_{I^*;I_x},v_{I^*;I^*-I_x})\big).$$

Since $\mathfrak{D}\Psi_{t;I^*;I_x}$ is an isomorphism of split vector bundles,

$$\Pi_{\mathfrak{R}_t;I^*}(v_{I^*;I_x},v_{I^*;I^*-I_x})\in\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)\big|_{x_0}$$

is determined by v. Thus, the map

$$\widetilde{\Phi}_t(x)\colon\mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)\big|_x\longrightarrow\mathbb{C},\quad\big\{\widetilde{\Phi}_t(x)\big\}(v)\equiv\widetilde{\Phi}_{t;I^*}\big(\Pi_{\mathfrak{R}_t;I^*}(v_{I^*;I_x},v_{I^*;I^*-I_x})\big),\tag{3.48}$$

is well-defined. Since $x_0\in X^\star_{I^*}$ and all components of $v_{I^*;I_x}$ are nonzero, this map is an isomorphism. Since $\widetilde{\Phi}_t|_{X^\star_{I^*}}=\widetilde{\Phi}_{t;I^*}|_{X^\star_{I^*}}$, (3.44) and (3.45) imply that

$$\big(\widetilde{\Phi}_t|_{X^\star_{I^*}\cap W'''}\big)_{t\in B}=\big(\Phi'_t|_{X^\star_{I^*}\cap W'''}\big)_{t\in B},\quad\big(\widetilde{\Phi}_t|_{X^\star_{I^*}}\big)_{t\in N(\partial B)}=\big(\Phi_t|_{X^\star_{I^*}}\big)_{t\in N(\partial B)}.\tag{3.49}$$

With x as above, suppose $I_x\subset I'\subset I^*$ and $x=\Psi_{t;I';I_x}(v_{I';I_x})$ for some

$$v_{I';I_x}\in\mathcal{N}_{I';I_x}\big|_{x'},\quad x'=\Psi_{t;I^*;I'}(v_{I^*;I'})\in W^\star_{I^*},\quad v_{I^*;I'}\in\mathcal{N}_{I^*;I'}\subset\mathcal{N}X_{I^*}\,.$$

By (3.7) with $I=I^*$ and the injectivity of $\Psi_{t;I^*;I'}$, $v_{I';I_x}=\mathfrak{D}\Psi_{t;I^*;I'}(v_{I^*;I_x})$. By (3.48) and the second statement in (3.19) with $I''\subset I'\subset I$ replaced by $I_x\subset I'\subset I^*$,

$$\begin{aligned}\big\{\widetilde{\Phi}_t(x)\big\}\big(\Pi_{\mathfrak{R}_t;I_x}\big(\mathfrak{D}\Psi_{t;I';I_x}(\mathfrak{D}\Psi_{t;I^*;I'}(v_{I^*;I_x},v_{I^*;I^*-I_x}))\big)\big)&=\big\{\widetilde{\Phi}_{t;I^*}(x_0)\big\}\big(\Pi_{\mathfrak{R}_t;I^*}(v_{I^*;I_x},v_{I^*;I^*-I_x})\big)\\&=\big\{\widetilde{\Phi}_t(x')\big\}\big(\Pi_{\mathfrak{R}_t;I'}(\mathfrak{D}\Psi_{t;I^*;I'}(v_{I^*;I_x},v_{I^*;I^*-I_x}))\big).\end{aligned}$$

Thus,

$$\begin{aligned}&\big\{\widetilde{\Phi}_t(\Psi_{I;I'}(v_{I;I'}))\big\}\big(\Pi_{\mathfrak{R}_t;I'}\big(\mathfrak{D}\Psi_{t;I;I'}(v_{I;I'},v_{I;I-I'})\big)\big)\\&\qquad=\big\{\widetilde{\Phi}_t\big(\pi_I(v_{I;I'})\big)\big\}\big(\Pi_{\mathfrak{R}_t;I}\big(v_{I;I'},v_{I;I-I'}\big)\big)\\&\forall\,I'\subset I\subset I^*,\,|I'|\ge2,\quad(v_{I;I'},v_{I;I-I'})\in\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\Psi^{-1}_{t;I;I'}(W^\star_{I^*})|_{X_I\cap W^\star_{I^*}}}.\end{aligned}\tag{3.50}$$

We conclude that $\widetilde{\Phi}_{t;2}$ satisfies (3.27) over $W^\star_{I^*}$ whenever $I'\subset I\subset I^*$.

By the $\mathfrak{R}_t$-compatibility of Φ_t for all $t\in N(\partial B)$ and the $\mathfrak{R}_t$-compatibility of Φ'_t for all $t\in B$,

$$\begin{aligned}\Phi_t\big(\Pi_{\mathfrak{R};I}\big(\mathfrak{D}\Psi_{I^*;I}(v)\big)\big)&=\Phi_t\big(\Pi_{\mathfrak{R};I^*}(v)\big)\quad\forall\,v\in\pi^*_{I^*;I}\mathcal{N}_{I^*;I^*-I}\big|_{\mathrm{Dom}(\Psi_{t;I^*;I})},\,t\in N(\partial B),\\\Phi'_t\big(\Pi_{\mathfrak{R};I}\big(\mathfrak{D}\Psi_{I^*;I}(v)\big)\big)&=\Phi'_t\big(\Pi_{\mathfrak{R};I^*}(v)\big)\quad\forall\,v\in\pi^*_{I^*;I}\mathcal{N}_{I^*;I^*-I}\big|_{\Psi^{-1}_{t;I^*;I}(W)|_{X_{I^*}\cap W}},\,t\in B,\end{aligned}$$

whenever $I\subset I^*$ and $|I|\ge2$. Along with (3.48) and (3.49), these two statements imply that

$$\begin{aligned}\big(\widetilde{\Phi}_t|_{\Psi_{t;I^*;\emptyset}(\mathrm{Dom}(\Psi_{t;I^*;\emptyset})|_{X_{I^*}\cap W'''})\cap W^\star_{I^*}}\big)_{t\in B}&=\big(\Phi'_t|_{\Psi_{t;I^*;\emptyset}(\mathrm{Dom}(\Psi_{t;I^*;\emptyset})|_{X_{I^*}\cap W'''})\cap W^\star_{I^*}}\big)_{t\in B},\\\big(\widetilde{\Phi}_t|_{\mathrm{Im}(\Psi_{t;I^*;\emptyset})\cap W^\star_{I^*}}\big)_{t\in N(\partial B)}&=\big(\Phi_t|_{\mathrm{Im}(\Psi_{t;I^*;\emptyset})\cap W^\star_{I^*}}\big)_{t\in N(\partial B)}.\end{aligned}\tag{3.51}$$

Combining these identities with the first assumption in (3.47) and with (3.46), we obtain

$$\big(\widetilde{\Phi}_t|_{W''\cap W^\star_{I^*}}\big)_{t\in B}=\big(\Phi'_t|_{W''\cap W^\star_{I^*}}\big)_{t\in B},\quad \big(\widetilde{\Phi}_t|_{W^\star_{I^*}}\big)_{t\in N(\partial B)}=\big(\Phi_t|_{W^\star_{I^*}}\big)_{t\in N(\partial B)}. \tag{3.52}$$

By the first identity in (3.52), the isomorphism

$$\widetilde{\Phi}'_t(x)\colon \mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_x\longrightarrow\mathbb{C},\qquad \widetilde{\Phi}'_t(x)=\begin{cases}\Phi'_t(x), & \text{if } x\in W'';\\ \widetilde{\Phi}_t(x), & \text{if } x\in W^\star_{I^*};\end{cases} \tag{3.53}$$

is well-defined for every $t\in B$. By the second assumption in (3.47), $W''\cup W^\star_{I^*}=W''\cup W'_{I^*}$. Let

$$\widetilde{\Phi}'_t\colon \mathcal{O}_{\mathfrak{R}_t;X_\partial}(X_\emptyset)|_{W''\cup W'_{I^*}}\longrightarrow\mathbb{C},\quad \widetilde{\Phi}'_t(v)=\big\{\widetilde{\Phi}'_t(\pi(v))\big\}(v)\,. \tag{3.54}$$

By the first case in (3.53), this trivialization satisfies the second condition in (3.38). By (3.53), the first assumption in (3.37), and the second identity in (3.52), it also satisfies the first condition in (3.39).

We next verify that the restriction of (3.54) to $W'\cup W'_{I^*}$ is $\mathfrak{R}_t$-compatible. Suppose

$$\begin{aligned}&t\in B,\quad I'\subset I\subset[N],\quad |I'|\ge 2,\quad x\in\big(W'\cup W'_{I^*}\big)\cap X_I,\\ &\big(v_{I;I'},v_{I;I-I'}\big)\in\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_x,\quad x'\equiv\Psi_{t;I;I'}(v_{I;I'})\in W'\cup W'_{I^*}\,.\end{aligned}$$

If $x,x'\in W''$ or $x,x'\in W^\star_{I^*}$, then

$$\widetilde{\Phi}'_{t;2}\big(\Pi_{\mathfrak{R}_t;I'}\big(\mathfrak{D}\Psi_{t;I;I'}(v_{I;I'},v_{I;I-I'})\big)\big)=\widetilde{\Phi}'_{t;2}\big(\Pi_{\mathfrak{R}_t;I}(v_{I;I'},v_{I;I-I'})\big) \tag{3.55}$$

by the $\mathfrak{R}_t$-compatibility of Φ'_t in the first case and by (3.50) in the second case. If $x\in W^\star_{I^*}$ and $x'\in W'$, then (3.7) and the first assumption in (3.41) imply that

$$x'\in W'\cap\mathrm{Im}\big(\Psi_{t;I^*;\emptyset}\big)\subset\Psi_{t;I^*;\emptyset}\big(\mathrm{Dom}\big(\Psi_{t;I;\emptyset}\big)\big|_{X_{I^*}\cap W''}\big),\quad x\in\Psi_{t;I^*;\emptyset}\big(\mathrm{Dom}\big(\Psi_{t;I;\emptyset}\big)\big|_{X^\star_{I^*}\cap W''}\big)\cap W^\star_{I^*}.$$

The identity (3.55) in this case follows from (3.51) and the $\mathfrak{R}_t$-compatibility of Φ'_t. If $x\in W'$ and $x'\in W^\star_{I^*}$, then the second assumption in (3.41) implies that $x'\in W''$. The identity (3.55) in this case follows from the $\mathfrak{R}_t$-compatibility of Φ'_t.

Along with $\widetilde{\Phi}_{t;2}|_{X^\star_{I^*}}=\widetilde{\Phi}_{t;I^*}|_{X^\star_{I^*}}$, (3.53) and (3.44) imply that $\widetilde{\Phi}'_{t;2}|_{X_{I^*}}=\widetilde{\Phi}_{t;I^*}$. By the second statement in (3.45) and the compactness of B, there thus exists a neighborhood W_{I^*} of $X_{I^*}\subset W'_{I^*}$ such that

$$\big|\Phi_{t;2}(x)-\widetilde{\Phi}'_{t;2}(x)\big|<\big|\Phi_{t;2}(x)\big|\quad\forall\,x\in W_{I^*},\,t\in B.$$

Combining this with the first case in (3.53) and the second assumption in (3.37), we conclude that the isomorphism $\widetilde{\Phi}'_t$ satisfies the second condition in (3.39). □

Remark 3.12. Let $\mathrm{Symp}^+_\hbar(\mathbf{X})$ denote the space of pairs consisting of an element $(\omega_i)_{i\in[N]}$ of $\mathrm{Symp}^+(\mathbf{X})$ and a trivialization Φ of the associated line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ in a homotopy class $\hbar$. By our proof of Proposition 3.9, the projection

$$\widetilde{\mathrm{Aux}}_\hbar(\mathbf{X})\longrightarrow\mathrm{Symp}^+_\hbar(\mathbf{X}),\qquad\big((\omega_i)_{i\in[N]},\mathfrak{R},\Phi\big)\longrightarrow\big((\omega_i)_{i\in[N]},\Phi\big),$$

is a weak homotopy equivalence.

4 Main construction

Let $\mathbf{X}$ be an SC symplectic configuration as in (2.7) which satisfies (2.17) and $\hbar$ be a homotopy class of trivializations of the associated line bundle (2.14). By the $B=\{\mathrm{pt}\}$ case of Proposition 3.9, we can assume that this SC symplectic configuration admits a regularization $\mathfrak{R}$ as in (3.20) and an $\mathfrak{R}$-compatible trivialization Φ of the complex line bundle $\mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)$ as in Definition 3.8.

In Section 4.1, we rescale the diffeomorphisms $\Psi_{I;i}$ to increase their domains so that they contain balls of size at least 2^N in each fiber. In Section 4.2, we patch together open subsets of these domains to form a smooth manifold $\mathcal{Z}'$ with a smooth map $\pi_{\mathcal{C},\varepsilon}\colon\mathcal{Z}'\longrightarrow\mathbb{C}$. The latter is obtained by scaling the trivialization Φ so that the restriction of $\pi_{\mathcal{C},\varepsilon}$ to the preimage of the ball of radius 1 in $\mathbb{C}$ forms a nearly regular fibration with uniform smooth fiber so that the associated homomorphisms (2.16) are isomorphisms. In Section 4.3, we construct a closed two-form $\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)}$ on $\mathcal{Z}'$ and show that its restriction to a neighborhood $\mathcal{Z}$ of $X_\emptyset\subset\mathcal{Z}'$ is symplectic and that the associated isomorphisms (2.16) are orientation-preserving. If $X_\emptyset$ is compact, this implies that there exists a neighborhood Δ of 0 in $\mathbb{C}$ such that $\mathcal{Z}|_{\pi_{\mathcal{C},\varepsilon}^{-1}(\Delta)}$ is a nearly regular symplectic fibration and its fibers are compact. As the various pieces of $\mathcal{Z}'$ are patched together only along X_∂, the compactness of X_∂ suffices for the first conclusion; we comment in Remark 4.6 on obtaining this conclusion even if X_∂ is not compact.

The construction in Sections 4.2 and 4.3 works on compact families of the relevant data on $(X_I)_{I\in\mathcal{P}^*(N)}$. By the $B=\mathbb{I}$ case of Proposition 3.9, the deformation equivalence class of the output of this construction is thus determined by the deformation equivalence class of the original SC symplectic configuration $\mathbf{X}$ and the homotopy class $\hbar$ of trivializations of (2.14).

4.1 Setup and notation

We begin by setting up the relevant notation. We will need several smooth $\mathbb{R}^+$-valued functions on the strata X_I and their open subspaces. These will be denoted by ε or $\mathcal{C}$ with some decorations, depending on whether the function should be sufficiently small or sufficiently large. The former means that it is pointwise smaller than another pre-specified continuous function on the same space or on a neighborhood of its closure; the meaning of sufficiently large is similar. If X_∂ is compact, such functions can be chosen to be constant.

For each $I\in\mathcal{P}^*(N)$, let $\omega_I=\omega_i|_{X_I}$ for any $i\in I$; this symplectic form on X_I is independent of the choice of $i\in I$. For $i\in I\subset[N]$ with $|I|\ge 2$, let

$$\rho_{I;i}\colon\mathcal{N}_{X_{I-i}}X_I\longrightarrow\mathbb{R} \tag{4.1}$$

be as in (3.20) and

$$\alpha_{I;i}\equiv\alpha_{\rho_{I;i},\nabla^{(I;i)}}\in\Gamma\big(\mathcal{N}_{X_{I-i}}X_I-X_I;T^*\mathcal{N}_{X_{I-i}}X_I\big) \tag{4.2}$$

be the connection 1-form on $\mathcal{N}_{I;I-i}=\mathcal{N}_{X_{I-i}}X_I$ determined by the Hermitian structure $(\rho_{I;i},\nabla^{(I;i)})$.

We also denote by

$$\rho_{I;i}\colon\mathcal{N}X_I\longrightarrow\mathbb{R}\qquad\text{and}\qquad\alpha_{I;i}\in\Gamma\big(\mathcal{N}X_I-\mathcal{N}_{I;i};T^*\mathcal{N}X_I\big)$$

the function and the 1-form obtained by pulling back (4.1) and (4.2) by the projection map $\mathrm{pr}_{I;I-i}\colon \mathcal{N}X_I \longrightarrow \mathcal{N}_{X_{I-i}}X_I$. The 1-form $\rho_{I;i}\alpha_{I;i}$ is then smooth on $\mathcal{N}X_I$. Define

$$\rho_I\colon \mathcal{N}X_I \longrightarrow \mathbb{R}, \qquad \rho_I(v) = \max\big\{\rho_{I;i}(v)\colon\, i\in I\big\},$$

to be the square norm on $\mathcal{N}X_I$. Let

$$\widehat{\omega}_I \equiv \widehat{\omega}^{\bullet}_{(\rho_{I;i},\nabla^{(I;i)})_{i\in I}} = \pi_I^*\omega_I + \frac{1}{2}\sum_{i\in I}\mathrm{d}(\rho_{I;i}\alpha_{I;i}) = \pi_I^*\omega_I + \frac{1}{2}\sum_{i\in I}\mathrm{pr}_{I;I-i}^*\mathrm{d}(\rho_{I;i}\alpha_{I;i})$$

be the closed 2-form on the total space of $\mathcal{N}X_I$ as in (3.1).

For each $I\subset[N]$ with $|I|\ge 2$, the homomorphism

$$\begin{aligned}\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathcal{N}X_I \equiv \bigotimes_{i\in I}\mathcal{N}_{X_{I-i}}X_I \longrightarrow \mathcal{O}_{\mathfrak{R};X_\partial}(X_\emptyset)\big|_{X_I} \equiv \Lambda^{\mathrm{top}}_{\mathbb{C}}\mathcal{N}X_I \otimes \bigotimes_{i\notin I}\mathcal{O}_{\mathfrak{R};X_I}(X_i),\\ w \longrightarrow w\otimes\bigotimes_{i\notin I}s_{I;i}(x) \qquad \forall\, w\in\Lambda^{\mathrm{top}}_{\mathbb{C}}\mathcal{N}X_I\big|_x,\ x\in X_I,\end{aligned} \tag{4.3}$$

is an isomorphism over $X_I^\star$. Thus, there exists a smooth function $\mathcal{C}_{\Phi;I}\colon X_I^\star\longrightarrow\mathbb{R}^+$ such that

$$\prod_{i\in I}\rho_{I;i}(v) = \mathcal{C}_{\Phi;I}\big(\pi_I(v)\big)\big|\Phi\big(\Pi_{\mathfrak{R};I}(v)\big)\big|^2 \qquad \forall\, v\in\mathcal{N}X_I\big|_{X_I^\star}, \tag{4.4}$$

where $\Pi_{\mathfrak{R};I}$ is the bundle map defined in (3.26); it is surjective over $X_I^\star$.

Given $\varepsilon\colon X_\emptyset\longrightarrow\mathbb{R}^+$ and $I\subset[N]$ with $|I|\ge 2$, we also denote by ε the composition

$$\varepsilon\colon \mathcal{N}X_I \xrightarrow{\pi_I} X_I \xrightarrow{\varepsilon}\mathbb{R}^+.$$

Define

$$\begin{aligned}\mathcal{N}X_I(\varepsilon) = \big\{v\in\mathcal{N}X_I\colon\, \rho_I(v)<\varepsilon(v)\big\}, \qquad \mathcal{N}_{I;I'}(\varepsilon) = \mathcal{N}X_I(\varepsilon)\cap\mathcal{N}_{I;I'} \ \ \forall\, I'\subset I,\\ m_{\varepsilon;I}\colon \mathcal{N}X_I\longrightarrow\mathcal{N}X_I, \qquad m_{\varepsilon;I}(v) = \varepsilon(v)v \ \ \forall\, v\in\mathcal{N}X_I.\end{aligned}$$

For each $i\in I$, let

$$\rho^{(\varepsilon)}_{I;i}\colon \mathcal{N}X_I\longrightarrow\mathbb{R}^{\ge 0}, \qquad \rho^{(\varepsilon)}_{I;i} = m^*_{\varepsilon;I}\rho_{I;i} = \varepsilon^2\rho_{I;i}.$$

If $\varepsilon|_{X_I}$ is smooth, set

$$\widehat{\omega}^{(\varepsilon)}_I = m^*_{\varepsilon;I}\widehat{\omega}_I = \pi^*\omega_I + \frac{1}{2}\sum_{i\in I}\mathrm{d}(\rho^{(\varepsilon)}_{I;i}\alpha_{I;i}). \tag{4.5}$$

For $I'\subset I\subset[N]$ with $|I|\ge 2$ and $I'\neq\emptyset$, let

$$\mathcal{N}'_{I;I'}\subset\mathcal{N}_{I;I'}\subset\mathcal{N}X_I, \qquad \Psi_{I;I'}\colon\mathcal{N}'_{I;I'}\longrightarrow X_{I'}$$

be as in (3.15). If in addition $|I'|\ge 2$, let

$$\mathfrak{D}\Psi_{I;I'}\colon \pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}'_{I;I'}}\longrightarrow\mathcal{N}X_{I'}\big|_{\mathrm{Im}(\Psi_{I;I'})}$$

be as in (3.17). Since $\mathfrak{D}\Psi_{I;I'}$ is a product Hermitian isomorphism, (3.27) implies that

$$\mathcal{C}_{\Phi;I}\big(\pi_I(v)\big) = \mathcal{C}_{\Phi;I'}\big(\Psi_{I;I'}(v)\big)\prod_{i\in I-I'}\rho_{I;i}(v) \quad \forall\, v\equiv(v_i)_{i\in I-I'}\in\mathcal{N}'_{I;I'}\big|_{X_I^\star} \text{ s.t. } v_i\neq 0\ \forall\, i\in I-I', \tag{4.6}$$

whenever $I'\subset I\subset[N]$ and $|I'|\geq 2$.

As shown in the proof of [4, Lemma 5.8], there exists a continuous function $\varepsilon\colon X_\emptyset\longrightarrow\mathbb{R}^+$ such that $\varepsilon|_{X_i}$ is smooth for all $i\in[N]$,

$$\overline{\mathcal{N}_{I;i}(4^N\varepsilon^2)}\subset\mathcal{N}'_{I;i}, \qquad \varepsilon\big(\Psi_{I;i}(v)\big)=\varepsilon(v)\ \ \forall\, v\in\mathcal{N}_{I;i}(4^N\varepsilon^2) \tag{4.7}$$

for all $i\in I\subset[N]$ with $|I|\geq 2$, and

$$\Psi_{I_1;i}\big(\mathcal{N}_{I_1;i}(4^N\varepsilon^2)\big)\cap\Psi_{I_2;i}\big(\mathcal{N}_{I_2;i}(4^N\varepsilon^2)\big)=\Psi_{I_1\cup I_2;i}\big(\mathcal{N}_{I_1\cup I_2;i}(4^N\varepsilon^2)\big) \tag{4.8}$$

for all $i\in I_1,I_2\subset[N]$ with $|I_1|,|I_2|\geq 2$. Furthermore, ε can be chosen so that its restriction to X_∂ is smaller than any given continuous function $\varepsilon_\partial\colon X_\partial\longrightarrow\mathbb{R}^+$.

For $\varepsilon\colon X_\emptyset\longrightarrow\mathbb{R}^+$ as in (4.7) and $i\in I\subset[N]$ with $|I|\geq 2$,

$$\Psi^{(\varepsilon)}_{I;i}\colon\big(\mathcal{N}_{I;i}(4^N),\widehat{\omega}_I^{(\varepsilon)}\big)\longrightarrow(X_i,\omega_i), \quad \Psi^{(\varepsilon)}_{I;i}(v)=\Psi_{I;i}\big(m_{\varepsilon;I}(v)\big), \tag{4.9}$$

is a symplectomorphism onto a neighborhood of X_I in X_i. Since these symplectomorphisms satisfy the matching condition (3.14) with $\Psi_{I;i}$ replaced by $\Psi^{(\varepsilon)}_{I;i}$, we can define smooth maps

$$\Psi^{(\varepsilon)}_{I;I'}\colon\mathcal{N}_{I;I'}(4^N)\longrightarrow X_{I'}, \qquad \emptyset\neq I'\subset I,$$

as in (3.15). By (4.6) and the second assumption in (4.7),

$$\mathcal{C}_{\Phi;I}\big(\pi_I(v)\big)=\mathcal{C}_{\Phi;I'}\big(\Psi^{(\varepsilon)}_{I;I'}(v)\big)\,\varepsilon\big(\Psi^{(\varepsilon)}_{I;I'}(v)\big)^{2|I-I'|}\prod_{i\in I-I'}\rho_{I;i}(v) \tag{4.10}$$

$$\forall\, v\equiv(v_i)_{i\in I-I'}\in\mathcal{N}_{I;I'}(4^N\varepsilon)\big|_{X_I^\star},\ I'\subset I\subset[N] \text{ s.t. } v_i\neq 0\,\forall\, i\in I-I',\ |I'|\geq 2.$$

By (4.8),

$$\Psi^{(\varepsilon)}_{I_1;I'}\big(\mathcal{N}_{I_1;I'}(4^N)\big)\cap\Psi^{(\varepsilon)}_{I_2;I'}\big(\mathcal{N}_{I_2;I'}(4^N)\big)=\Psi^{(\varepsilon)}_{I_1\cup I_2;I'}\big(\mathcal{N}_{I_1\cup I_2;I'}(4^N)\big) \tag{4.11}$$

for all $I'\subset I_1,I_2\subset[N]$ with $I'\neq\emptyset$ and $|I_1|,|I_2|\geq 2$.

For $I'\subset I$ with $|I'|\geq 2$, define

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\colon\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}_{I;I'}(4^N)}\longrightarrow\mathcal{N}X_{I'}\big|_{\mathrm{Im}(\Psi^{(\varepsilon)}_{I;I'})} \qquad \text{by} \tag{4.12}$$

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big(v_{I;I'},v_{I;I-I'}\big)=\mathfrak{D}\Psi_{I;I'}\big(m_{\varepsilon;I}(v_{I;I'}),v_{I;I-I'}\big) \quad \forall\,\big(v_{I;I'},v_{I;I-I'}\big)\in\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}_{I;I'}(4^N)}.$$

By the second assumption in (4.7),

$$\begin{aligned}\varepsilon\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}&=\varepsilon\circ\pi^*_{I;I'}\pi_{I;I-I'}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'}(4^N)}},\\ m_{\varepsilon;I'}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}&=\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\circ\pi^*_{I;I'}m_{\varepsilon;I}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'}(4^N)}}.\end{aligned} \tag{4.13}$$

Since $\mathfrak{D}\Psi_{I;I'}$ lifts $\Psi_{I;I'}$ to a product Hermitian isomorphism with respect to the product Hermitian structures $(\rho_{I;i},\nabla^{(I;i)})_{i\in I'}$ and $(\rho_{I';i},\nabla^{(I';i)})_{i\in I'}$, $\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}$ lifts $\Psi^{(\varepsilon)}_{I;I'}$ to a product Hermitian isomorphism with respect to these structures. In particular,

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big(\mathcal{N}X_I(4^N)\big)=\mathcal{N}X_{I'}(4^N)\big|_{\mathrm{Im}(\Psi^{(\varepsilon)}_{I;I'})}.$$

By (4.13) and (3.19),

$$\Psi^{(\varepsilon)}_{I;I''}=\Psi^{(\varepsilon)}_{I';I''}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big|_{\mathcal{N}_{I;I''}(4^N)},\quad \mathfrak{D}\Psi^{(\varepsilon)}_{I;I''}=\mathfrak{D}\Psi^{(\varepsilon)}_{I';I''}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big|_{\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}_{I;I''}(4^N)}} \tag{4.14}$$

for all $I''\subset I'\subset I\subset[N]$ with $|I'|\ge 2$ in both cases, $|I''|\ge 1$ in the first case, and $|I''|\ge 2$ in the second case.

Let $\mathcal{C}\colon X_\emptyset\longrightarrow\mathbb{R}^+$ be a continuous function such that

$$\mathcal{C}\big(\Psi_{I;i}(v)\big)=\mathcal{C}\big(\pi_{I;i}(v)\big)\quad\forall\, v\in\mathcal{N}_{I;i}(4^N\varepsilon),\ i\in I\subset[N],\ |I|\ge 2. \tag{4.15}$$

For $I\subset[N]$ with $|I|\ge 2$, define

$$\Phi_{\mathcal{C},\varepsilon;I}\colon\mathcal{N}X_I\longrightarrow\mathbb{C},\quad \Phi_{\mathcal{C},\varepsilon;I}(v)=\mathcal{C}\big(\pi_I(v)\big)\varepsilon\big(\pi_I(v)\big)^{|I|-1}\Phi\big(\Pi_{\mathfrak{R};I}(v)\big)\quad\forall\, v\in\mathcal{N}X_I. \tag{4.16}$$

By (4.4),

$$\mathcal{C}\big(\pi_I(v)\big)^2\varepsilon(v)^{2(|I|-1)}\prod_{i\in I}\rho_{I;i}(v)=\mathcal{C}_{\Phi;I}\big(\pi_I(v)\big)\big|\Phi_{\mathcal{C},\varepsilon;I}(v)\big|^2\quad\forall\, v\in\mathcal{N}X_I\big|_{X^\star_I}, \tag{4.17}$$

whenever $I\subset[N]$ and $|I|\ge 2$. By (4.15), (4.13), and (3.27),

$$\Phi_{\mathcal{C},\varepsilon;I'}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}=\Phi_{\mathcal{C},\varepsilon;I}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'}(4^N)}} \tag{4.18}$$

for all $I'\subset I\subset[N]$ and $|I'|\ge 2$.

If $I=\{i\}$ with $i\in[N]$, let

$$\Phi_{\mathcal{C},\varepsilon;I}=\pi_2:\mathcal{N}X_I\equiv X_i\times\mathbb{C}\longrightarrow\mathbb{C} \tag{4.19}$$

be the projection to the second component. For $I'\equiv\{i\}\subsetneq I\subset[N]$, define

$$\begin{aligned}\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\colon\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}_{I;I'}(4^N)}&\equiv\pi^*_{I;I'}\mathcal{N}_{X_{I-i}X_I}\big|_{\mathcal{N}_{I;I'}(4^N)}\longrightarrow\mathcal{N}X_{I'}\big|_{\mathrm{Im}(\Psi^{(\varepsilon)}_{I;I'})},\\ \mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big((v_{I;j})_{j\in I-I'},v_{I;i}\big)&=\big(\Psi^{(\varepsilon)}_{I;I'}\big((v_{I;j})_{j\in I-I'}\big),\Phi_{\mathcal{C},\varepsilon;I}\big((v_{I;j})_{j\in I}\big)\big).\end{aligned} \tag{4.20}$$

The restriction of this map to the subspace

$$\big\{((v_{I;j})_{j\in I-I'},v_{I;i})\in\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'}(4^N)|_{X^\star_I}}:\ v_{I;j}\neq 0\ \forall\, j\in I-I'\big\} \tag{4.21}$$

is a diffeomorphism onto an open subspace of $\mathcal{N}X_{I'}=\mathcal{N}X_i$.

For $I=\{i\}$, let

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I;I}=\mathrm{id}\colon\pi^*_{I;I}\mathcal{N}_{I;\emptyset}\big|_{\mathcal{N}_{I;I}(4^N)}=\mathcal{N}X_I\longrightarrow\mathcal{N}X_I;$$

for $I\subset[N]$ with $|I|\ge 2$, this is already the case by (3.17) and the $I'=I$ case of (3.6). By (4.18), (4.19), and (4.20),

$$\Phi_{\mathcal{C},\varepsilon;I'}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}=\Phi_{\mathcal{C},\varepsilon;I}\big|_{\pi^*_{I;I'}\mathcal{N}_{I;I-I'}|_{\mathcal{N}_{I;I'}(4^N)}} \tag{4.22}$$

for all $I'\subsetneq I\subset[N]$ with $I'\neq\emptyset$. By (4.14) and (4.18),

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I;I''}=\mathfrak{D}\Psi^{(\varepsilon)}_{I';I''}\circ\mathfrak{D}\Psi^{(\varepsilon)}_{I;I'}\big|_{\pi^*_{I;I''}\mathcal{N}_{I;I-I''}|_{\mathcal{N}_{I;I''}(4^N)}} \tag{4.23}$$

for all $I''\subset I'\subset I\subset[N]$ with $I''\neq\emptyset$.

For $\varepsilon\colon X_\emptyset\longrightarrow\mathbb{R}^+$ as in (4.7) and $I\in\mathcal{P}^*(N)$, let

$$X^\circ_I=X_{I;I}=X_I-\bigcup_{I\subsetneq I'\subset[N]}\overline{\Psi^{(\varepsilon)}_{I';I}\big(\mathcal{N}_{I';I}(2^{|I'|-1})\big)}; \tag{4.24}$$

see the left diagram in Figure 4. Since $\mathfrak{D}\Psi^{(\varepsilon)}_{I_1\cup I_2;I_1}$ is a product Hermitian isomorphism, (4.11) and the first equality in (4.14) imply that

$$\begin{aligned}
\Psi^{(\varepsilon)}_{I_1;I'}\big(\mathcal{N}_{I_1;I'}(2^{|I_1|})|_{X^\circ_{I_1}}\big)\cap\Psi^{(\varepsilon)}_{I_2;I'}\big(\mathcal{N}_{I_2;I'}(2^{|I_2|})|_{X^\circ_{I_2}}\big)\neq\emptyset \quad&\Longrightarrow\quad I_1\supset I_2 \ \text{ or } \ I_1\subset I_2\,,\\
\overline{\Psi^{(\varepsilon)}_{I_1;I'}\big(\mathcal{N}_{I_1;I'}(2^{|I_1|})|_{X^\circ_{I_1}}\big)}\cap\overline{\Psi^{(\varepsilon)}_{I_2;I'}\big(\mathcal{N}_{I_2;I'}(2^{|I_2|-1})\big)}\neq\emptyset \quad&\Longrightarrow\quad I_1\supset I_2\,,
\end{aligned} \tag{4.25}$$

whenever $I'\subset I_1,I_2\subset[N]$ with $I'\neq\emptyset$ and $|I_1|,|I_2|\ge 2$.

Let $i\in[N]$. By (4.11), the first identity in (4.14), and (4.10), the function

$$\widehat{\varepsilon}_i\colon X^\circ_i\cap\bigcup_{\{i\}\subsetneq I\subset[N]}\Psi^{(\varepsilon)}_{I;i}\big(\mathcal{N}_{I;i}(4^{|I|})\big)\longrightarrow\mathbb{R}^+,$$

$$\widehat{\varepsilon}_i\big(\Psi^{(\varepsilon)}_{I;i}(v)\big)^2=\frac{\varepsilon(v)^{2(|I|-1)}}{\mathcal{C}_{\Phi;I}(\pi_I(v))}\prod_{j\in I-i}\rho_{I;j}(v)\quad\forall\,v\in\{\Psi^{(\varepsilon)}_{I;i}\}^{-1}(X^\circ_i)\cap\mathcal{N}_{I;i}(4^N),\ \{i\}\subsetneq I\subset[N],$$

is well-defined and smooth. Thus, there is a smooth function $\widetilde{\varepsilon}_i\colon X^\circ_i\longrightarrow\mathbb{R}^+$ such that

$$\widetilde{\varepsilon}_i\big(\Psi^{(\varepsilon)}_{I;i}(v)\big)^2=\frac{\varepsilon(v)^{2(|I|-1)}}{\mathcal{C}_{\Phi;I}(\pi_I(v))}\prod_{j\in I-i}\rho_{I;j}(v)\quad\forall\,v\in\{\Psi^{(\varepsilon)}_{I;i}\}^{-1}(X^\circ_i)\cap\mathcal{N}_{I;i}(2^N),\ \{i\}\subsetneq I\subset[N].$$

By (4.17) and (4.15),

$$\begin{aligned}
\big|\Phi_{\mathcal{C},\varepsilon;I}(v)\big|^2&=\mathcal{C}\big(\Psi^{(\varepsilon)}_{I;i}((v_{I;j})_{j\in I-i})\big)^2\widetilde{\varepsilon}_i\big(\Psi^{(\varepsilon)}_{I;i}((v_{I;j})_{j\in I-i})\big)^2\rho_{I;i}(v)\\
\forall\ v\equiv\big((v_{I;j})_{j\in I-i},v_{I;i}\big)&\in\pi^*_{I;i}\mathcal{N}_{I;I-i}\big|_{\{\Psi^{(\varepsilon)}_{I;i}\}^{-1}(X^\circ_i)\cap\mathcal{N}_{I;i}(2^N)},\ \{i\}\subsetneq I\subset[N].
\end{aligned} \tag{4.26}$$

4.2 Construction of fibration

With $\mathcal{C}$ as in (4.15), define

$$\mathcal{Z}_I=\begin{cases}\{v_I\in\mathcal{N}X_I|_{X^\circ_I}\colon\ \rho_I(v_I)<2^{|I|}\}, & \text{if } |I|\ge 2;\\ \{(x,\lambda)\in\mathcal{N}X_i|_{X^\circ_i}\colon\ |\lambda|^2<2\,\mathcal{C}(x)^2\widetilde{\varepsilon}_i(x)^2\}, & \text{if } I=\{i\}.\end{cases}$$

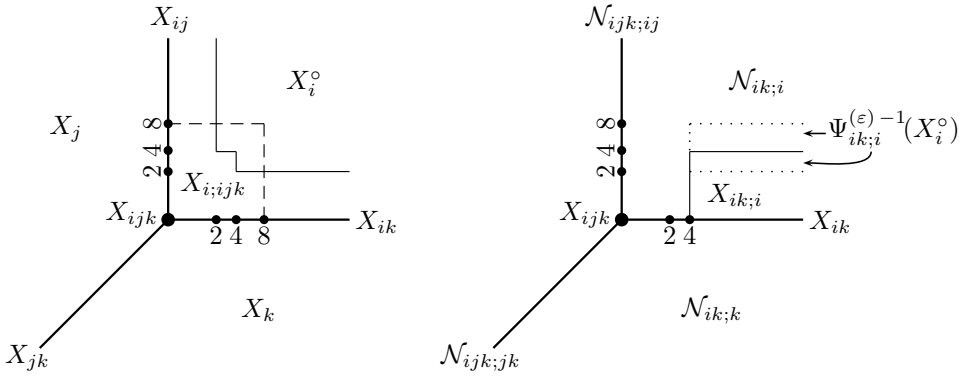

Figure 4: The subspaces $X_i^\circ, X_{i;ijk} \subset X_i$, and $X_{ik;i} \subset \mathcal{N}_{ik;i}$ appearing in the smoothing construction for a triple configuration as in Figure 1; the numbers indicate the value of the square distance along each axis in $\mathcal{N}X_{ijk}$.

For $I \subsetneq I' \subset [N]$ with $I \neq \emptyset$, let

$$X_{I';I} = \mathcal{Z}_{I'} \cap \mathcal{N}_{I';I}, \qquad X_{I;I'} = \Psi_{I';I}^{(\varepsilon)}\big(X_{I';I}\big) \subset X_I, \tag{4.27}$$

$$\mathcal{Z}_{I';I} = \big\{v \in \mathcal{Z}_{I'} \colon \rho_{I';i}(v) < 2^{|I|} \ \forall\, i \in I\big\} - \bigcup_{I \subsetneq J \subset I'} \big\{v \in \mathcal{Z}_{I'} \colon \rho_{I';i}(v) \le 2^{|J|-1} \ \forall\, i \in J - I\big\},$$

$$\mathcal{Z}_{I;I'} = \mathfrak{D}\Psi_{I';I}^{(\varepsilon)}\big(\mathcal{Z}_{I';I}\big) = \mathcal{Z}_I\big|_{X_I^\circ \cap X_{I;I'}}. \tag{4.28}$$

The first two subspaces above are illustrated in Figure 4. For $|I| \ge 2$, the last equality in (4.28) holds because $\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}$ is a product Hermitian isomorphism; for $|I| = 1$, it holds by (4.20) and (4.26). This crucial equality is used in the proof of Hausdorffness in Proposition 4.1 below.

Define

$$\mathcal{Z}' = \Big(\bigsqcup_{I \in \mathcal{P}^*(N)} \mathcal{Z}_I\Big)\Big/\!\sim,$$

$$\mathcal{Z}_{I'} \supset \mathcal{Z}_{I';I} \ni v_{I'} \sim \mathfrak{D}\Psi_{I';I}^{(\varepsilon)}(v_{I'}) \in \mathcal{Z}_{I;I'} \subset \mathcal{Z}_I \qquad \forall\, I \subsetneq I' \subset [N],\ I \neq \emptyset. \tag{4.29}$$

By the first statement in (4.25) and (4.23), $\sim$ is an equivalence relation. The restriction of the quotient map

$$q \colon \bigsqcup_{I \in \mathcal{P}^*(N)} \mathcal{Z}_I \longrightarrow \mathcal{Z}' \tag{4.30}$$

to each $\mathcal{Z}_I$ is injective; we thus identify $\mathcal{Z}_I$ with $q(\mathcal{Z}_I)$ as sets. By Proposition 4.1 below, this identification respects the smooth structures.

By (4.22) with I and I' interchanged, the map

$$\pi_{\mathcal{C},\varepsilon} \colon \mathcal{Z}' \longrightarrow \mathbb{C}, \qquad \pi_{\mathcal{C},\varepsilon}\big(q(v)\big) = \Phi_{\mathcal{C},\varepsilon;I}(v) \quad \forall\, v \in \mathcal{Z}_I,\, I \in \mathcal{P}^*(N), \tag{4.31}$$

is well-defined. It is continuous, since the maps $\Phi_{\mathcal{C},\varepsilon;I}$ are continuous. For each $i \in [N]$, the map

$$\iota_{\mathcal{C},\varepsilon;i} \colon X_i \longrightarrow \mathcal{Z}', \quad \iota_{\mathcal{C},\varepsilon;i}(x) = \begin{cases} q(x,0), & \text{if } x \in X_i^\circ; \\ q(v), & \text{if } x = \Psi_{I;i}^{(\varepsilon)}(v),\ v \in X_{I;i},\ i \in I \subset [N]; \end{cases} \tag{4.32}$$

is well-defined, injective, and continuous. Since $\iota_{\mathcal{C},\varepsilon;i}|_{X_{ij}} = \iota_{\mathcal{C},\varepsilon;j}|_{X_{ij}}$ for all $i,j\in[N]$, we obtain an injective continuous map

$$\iota_{\mathcal{C},\varepsilon}\colon X_\emptyset \longrightarrow \mathcal{Z}'. \tag{4.33}$$

The image of this map is $\mathcal{Z}'_0 \equiv \pi^{-1}_{\mathcal{C},\varepsilon}(0)$. The substance of the last statement in Proposition 4.1 is that the fibers $\mathcal{Z}'_\lambda \equiv \pi^{-1}_{\mathcal{C},\varepsilon}(\lambda)$ are compact if $X_\emptyset$ is compact, $\mathcal{C}$ is sufficiently large, and $|\lambda|<1$.

Proposition 4.1. *If $\varepsilon|_{X_i}$ and $\mathcal{C}|_{X_i}$ are smooth for all $i\in[N]$, then $\mathcal{Z}'$ is a Hausdorff topological space with a smooth structure so that*

(1) the restriction of (4.30) to each $\mathcal{Z}_I$ is a diffeomorphism onto $q(\mathcal{Z}_I)$;

(2) the map (4.31) is smooth on $\mathcal{Z}'$ and a submersion outside of $\iota_{\mathcal{C},\varepsilon}(X_\partial)$;

(3) the maps (4.32) are smooth embeddings, their images form a transverse collection of closed submanifolds of $\mathcal{Z}'$ of codimension 2, and the associated homomorphisms (2.16) are $\mathbb{C}$-linear isomorphisms.

If $\mathcal{C}$ is sufficiently large (depending on ε), every sequence $v_I^{(k)}$ in $\mathcal{Z}_I\cap\pi^{-1}_{\mathcal{C},\varepsilon}(\lambda)$ with $|\lambda|<1$ and with the sequence $\pi_I(v_I^{(k)})$ converging in X_I has a limit point in $\mathcal{Z}'$.

Proof. For all $I\subset I'\subset[N]$ with $I\neq\emptyset$, the subspace $\mathcal{Z}_{I';I}\subset\mathcal{N}X_{I'}$ is open by definition. Since each map $\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}$ is a diffeomorphism onto an open subspace of $\mathcal{N}X_I$, everywhere on its domain if $|I|\ge2$ and on the subspace (4.21) with I and I' interchanged if $|I|=1$, the subspace $\mathcal{Z}_{I;I'}\subset\mathcal{N}X_I$ is also open. Since the identification maps are diffeomorphisms between open subspaces of manifolds, the quotient map (4.30) is open. Assuming $\mathcal{Z}'$ is Hausdorff (as shown below), q thus induces a smooth structure on $\mathcal{Z}'$. Since $\Phi_{\mathcal{C},\varepsilon;I}:\mathcal{Z}_I\longrightarrow\mathbb{C}$ is a submersion outside of the subspaces $\mathcal{Z}_I\cap\mathcal{N}_{I;I'}$ with $I'\subset I$ such that $|I'|\ge2$, the map (4.31) is a submersion outside of $\iota_{\mathcal{C},\varepsilon}(X_\partial)$. The maps (4.32) are smooth embeddings because their restrictions to the preimages of $\mathcal{Z}_I$ correspond to the inclusions of the hyperplane subbundles $\mathcal{N}_{I;i}$ of $\mathcal{N}X_I$. For the same reason, their images form a transverse collection of submanifolds of $\mathcal{Z}'$ of codimension 2. These submanifolds X_i are closed in $\mathcal{Z}'$ because X_i is closed in $X_\emptyset$ and $\iota_{\mathcal{C},\varepsilon}(X_\emptyset)=\pi^{-1}_{\mathcal{C},\varepsilon}(0)$ is closed in $\mathcal{Z}'$. For each $I\in\mathcal{P}^*(N)$, the homomorphism $D_I\pi_{\mathcal{C},\varepsilon}$ corresponds via the diffeomorphism $q|_{\mathcal{Z}_I}$ to the linear map induced by $\Phi_{\mathcal{C},\varepsilon;I}$. Since Φ is a $\mathbb{C}$-linear isomorphism, the associated homomorphisms (2.16) with $\pi=\pi_{\mathcal{C},\varepsilon}$ are also $\mathbb{C}$-isomorphisms. This establishes (1)-(3).

Let $[v],[w]\in\mathcal{Z}'$ be distinct points and $I,J\subset[N]$ be the maximal subsets so that $[v]$ lies in the image of some $v\in\mathcal{Z}_I$ under q and $[w]$ lies in the image of some $w\in\mathcal{Z}_J$; I and J are well-defined by the first statement in (4.25). If $I=J$, let $V,W\subset\mathcal{Z}_I$ be disjoint open subsets around v and w. Since q is an open map which is injective on $\mathcal{Z}_I$, $q(V),q(W)\subset\mathcal{Z}'$ are disjoint open subsets containing $[v]$ and $[w]$, respectively. If $I\not\subset J$ and $J\not\subset I$, then the open neighborhoods $q(\mathcal{Z}_I),q(\mathcal{Z}_J)\subset\mathcal{Z}'$ of $[v]$ and $[w]$, respectively, are disjoint by the first statement in (4.25).

Suppose $I\subsetneq J$. Let $\delta>0$ be such that

$$w\in W\equiv\big\{v_J\in\mathcal{Z}_J\colon\ \rho_{J;j}(v_J)<2^{|J|}-\delta\ \forall\, j\in J-I\big\}.$$

Since $v\notin\mathcal{Z}_{I;J}$ (by the maximality assumption on I),

$$v\in V\equiv\mathcal{Z}_I-\mathcal{N}X_I\big|_{\Psi^{(\varepsilon)}_{J;I}(\{v_{J;I}\in\mathcal{N}_{J;I}\colon\rho_J(v_{J;I})\le2^{|J|}-\delta\})}$$

by (4.28) with $I'=J$. Since (4.29) is an equivalence relation, $q(V), q(W) \subset \mathcal{Z}'$ are disjoint open subsets containing $[v]$ and $[w]$, respectively. Thus, $\mathcal{Z}'$ is Hausdorff.

We now verify the last claim. By (4.17), there exists a continuous function $\mathcal{C}_{\Phi,\varepsilon}\colon X_\emptyset \longrightarrow \mathbb{R}^+$ such that

$$\mathcal{C}\big(\pi_I(v)\big)^2 \prod_{i\in I} \rho_{I;i}(v) \le \mathcal{C}_{\Phi,\varepsilon}\big(\pi_I(v)\big)\big|\Phi_{\mathcal{C},\varepsilon;I}(v)\big|^2 \quad \forall\, v\in \mathcal{N}X_I\big|_{X_I^\circ},\, I\subset[N],\, |I|\ge 2; \tag{4.34}$$

the function $\mathcal{C}_{\Phi,\varepsilon}$ depends on ε, but not on $\mathcal{C}$. The inequality (4.34) provides a bound on the product of the norms $\rho_{I;i}(v)$ with $i\in I$ in terms of $|\lambda|=|\Phi_{\mathcal{C},\varepsilon;I}(v)|$; this bound becomes stronger as $\mathcal{C}$ increases.

Suppose $\lambda\in\mathbb{C}^*$, $I\in\mathcal{P}^*(N)$, and

$$v_I^{(k)} \equiv \big(v_{I;i}^{(k)}\big)_{i\in I} \in \mathcal{Z}_I \cap \Phi_{\mathcal{C},\varepsilon;I}^{-1}(\lambda) \subset \mathcal{N}X_I \xrightarrow{\pi_I} X_I^\circ \subset X_I$$

is a sequence such that $\pi_I(v_I^{(k)})$ converges to a point $x_I\in X_I$. A subsequence of $v_I^{(k)}$ then converges to some

$$v_I \equiv (v_{I;i})_{i\in I} \in \mathcal{N}X_I\big|_{x_I} \qquad \text{s.t.} \qquad \Phi_{\mathcal{C},\varepsilon;I}(v_I)=\lambda, \quad \rho_{I;i}(v_I)\le 2^{|I|}\ \forall\, i\in I \ \text{ if } \ |I|\ge 2.$$

If $x_I\in X_I^\circ$ and $|I|=\{i\}$ for some $i\in[N]$, then $v_I=(x_I,\lambda)\in\mathcal{Z}_I$ if $|\lambda|<1$ and $2\mathcal{C}^2\widetilde{\varepsilon}_i^2>1$ (i.e. $\mathcal{C}$ is sufficiently large). The sequence $v_I^{(k)}$ then has a limit point in $\mathcal{Z}'$.

Suppose $x_I\in X_I^\circ$ and $|I|\ge 2$. Let $J\subset I$ be a maximal subset such that $\rho_{I;i}(v_I)<2^{|J|}$ for all $i\in J$. If $J\neq\emptyset$, define

$$x_J = \Psi_{I;J}^{(\varepsilon)}\big((v_{I;i})_{i\in I-J}\big) \in X_J.$$

If $x_J=\Psi_{I';J}^{(\varepsilon)}(v_{I'})$ for some $v_{I'}\in\mathcal{N}_{I';J}$ with $J\subsetneq I'\subset[N]$ and $\rho_{I'}(v_{I'})\le 2^{|I'|-1}$, then $I\supset I'$ by the second statement in (4.25) with (I_1,I_2,I') replaced by (I,I',J). By the first identity in (4.14) and $\mathfrak{D}\Psi_{I;I'}^{(\varepsilon)}$ being a product Hermitian isomorphism,

$$v_{I'} = \mathfrak{D}\Psi_{I;I'}^{(\varepsilon)}\big((v_{I;i})_{i\in I-I'}, (v_{I;i})_{i\in I'-J}\big), \qquad \rho_{I;i}(v_I)\le 2^{|I'|-1} \ \ \forall\, i\in I'-J.$$

This would contradict maximality of J. Thus, $x_J\in X_J^\circ$. If $J=\emptyset$, replace it with any single-element subset of I and define x_J in the same way. As in the first case, $x_J\in X_J^\circ$. If $|J|=1$,

$$\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}(v_I) = (x_J,\lambda) \in \mathcal{Z}_J\,,$$

as in the previous paragraph. If $|J|\ge 2$, $\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}(v_I)$ lies in $\mathcal{Z}_J$ as well, since $\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}$ is a product Hermitian isomorphism and $\rho_{I;i}(v_I)<2^{|J|}$ for all $i\in J$. It follows that $\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}(v_I^{(k)})\in\mathcal{Z}_J$ for all k large and $\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}(v_I)$ is a limit point of this set. Thus, $\mathfrak{D}\Psi_{I;J}^{(\varepsilon)}(v_I)$ is a limit point of the sequence $v_I^{(k)}$ in $\mathcal{Z}'$.

Suppose $x_I\notin X_I^\circ$. Let $I'\subset[N]$ be the maximal subset so that $I'\supset I$ and $x_I=\Psi_{I';I}^{(\varepsilon)}(v_{I'})$ for some $v_{I'}\in\mathcal{N}_{I';I}$ with $\rho_{I'}(v_{I'})=2^{|I'|-1}$. By (4.24), (4.11), and the first identity in (4.14), I' is well-defined. Furthermore,

$$\rho_{I';j}(v_{I'}) \ge 2^{|I|} \ge 2 \ \ \forall\, j\in I'-I. \tag{4.35}$$

If $\pi_{I'}(v_{I'})=\Psi^{(\varepsilon)}_{J;I'}(v_J)$ for some $v_J\in\mathcal{N}_{J;I'}$ with $I'\subsetneq J\subset[N]$ and $\rho_J(v_J)\le 2^{|J|-1}$, then either

$$x_I \in \Psi^{(\varepsilon)}_{J;I}\big(\mathcal{N}_{J;I}(2^{|J|-1})\big) \qquad \text{or} \qquad \rho_J(v_J)=2^{|J|-1}\,.$$

The first possibility would contradict to x_I being in the closure of X°_I; the second possibility would contradict the maximality of I'. Thus, $\pi_{I'}(v_{I'})\in X^\circ_{I'}$.

For all $k\in\mathbb{Z}^+$ sufficiently large (so that $\pi_I(v_I^{(k)})$ lies in $X_{I;I'}$), let

$$v^{(k)}_{I'}\equiv\big(v^{(k)}_{I';I},v^{(k)}_{I';I'-I}\big)=\big\{\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big\}^{-1}\big(v^{(k)}_I\big)\in\pi^*_{I';I}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}_{I';I}(2^{|I'|})}\,.$$

Since $\pi_I(v_I^{(k)})$ converges to x_I, the sequence $v^{(k)}_{I';I}$ converges to $v_{I'}$. If $|I|\ge 2$,

$$\rho_{I';i}(v^{(k)}_{I'})=\rho_{I;i}(v^{(k)}_I)\le 2^{|I|}<2^{|I'|}\qquad\forall\, i\in I;$$

thus, $v^{(k)}_{I'}$ has a limit point in $\mathcal{Z}_{I'}$. If $I=\{i\}$,

$$2^{|I'|-1}\mathcal{C}\big(\pi_{I'}(v^{(k)}_{I'})\big)^2\rho_{I';i}(v^{(k)}_{I'})\le\mathcal{C}\big(\pi_{I'}(v^{(k)}_{I'})\big)^2\prod_{j\in I'}\rho_{I';j}(v^{(k)}_{I'})\le\mathcal{C}_{\Phi,\varepsilon}\big(\pi_I(v^{(k)}_{I'})\big)|\lambda|^2$$

by (4.35) and (4.34). Thus, the sequence $v^{(k)}_{I'}$ converges in $\mathcal{Z}_{I'}$, provided

$$\mathcal{C}_{\Phi,\varepsilon}(x)|\lambda|^2<2^{|I'|-1}\mathcal{C}(x)^2\cdot 2^{|I'|}\qquad\forall\, x\in X_{I'}.$$

Therefore, the last claim of the proposition holds if $\mathcal{C}^2>\mathcal{C}_{\Phi,\varepsilon}$. □

4.3 Symplectic structure

We next define a closed 2-form $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}$ on $\mathcal{Z}_I$ for each $I\in\mathcal{P}^*(N)$. Let $\eta:\mathbb{R}\longrightarrow[0,1]$ be a smooth function such that

$$\eta(r)=\begin{cases}0, & \text{if } r\le\frac{3}{4};\\ 1, & \text{if } r\ge 1.\end{cases}$$

For $i\in I\subset[N]$ with $|I|\ge 2$, let $\eta_{I;i}=\eta\circ\rho_{I;i}:\mathcal{N}X_I\longrightarrow\mathbb{R}$ and

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}=\left(\pi_I^*\omega_I+\frac{1}{2}\mathrm{d}\left(\sum_{i\in I}\left(1-\prod_{j\in I-\{i\}}\eta_{I;j}\right)\varepsilon^2\rho_{I;i}\alpha_{I;i}+\left(\sum_{i\in I}\prod_{j\in I-\{i\}}\eta_{I;j}\right)\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I}\big(r^2\mathrm{d}\theta\big)\right)\right)\Big|_{\mathcal{Z}_I}. \tag{4.36}$$

For $I=\{i\}$, define

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}=\Big(\pi_1^*\omega_i+\frac{1}{2}\mathrm{d}\big(\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I}(r^2\mathrm{d}\theta)\big)\Big)\Big|_{\mathcal{Z}_I}=\Big(\pi_1^*\omega_i+\frac{1}{2}\mathrm{d}\big(\varepsilon^2\pi_2^*(r^2\mathrm{d}\theta)\big)\Big)\Big|_{\mathcal{Z}_I}, \tag{4.37}$$

where $\pi_1,\pi_2:X_i\times\mathbb{C}\longrightarrow X_i,\mathbb{C}$ are the two projections and (r,θ) are the polar coordinates on $\mathbb{C}$. Let $\Delta_r\subset\mathbb{C}$ denote the disk of radius r centered at the origin.

Lemma 4.2. *If $\varepsilon|_{X_i}$ and $\mathcal{C}|_{X_i}$ are smooth for all $i\in[N]$ and $\mathcal{C}$ is sufficiently large (depending on ε), then*

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I'}\big|_{\Phi^{-1}_{\mathcal{C},\varepsilon;I'}(\Delta_1)\cap\mathcal{Z}_{I';I}}=\big\{\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big\}^*\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}\big|_{\Phi^{-1}_{\mathcal{C},\varepsilon;I'}(\Delta_1)\cap\mathcal{Z}_{I';I}} \tag{4.38}$$

for all $I\subsetneq I'\subset[N]$ with $I\neq\emptyset$.

Proof. We show that the claim holds with $\mathcal{C}^2 > 2^N \mathcal{C}_{\Phi,\varepsilon}$, with $\mathcal{C}_{\Phi,\varepsilon}$ as in (4.34). This assumption implies that

$$\big\{i \in I\colon \rho_{I;i}(v_I) \le \frac{3}{4}\big\} \neq \emptyset \qquad \forall\ v_I \in \mathcal{Z}_I \cap \Phi_{\mathcal{C},\varepsilon;I}^{-1}(\Delta_1),\ I \subset N,\ |I| \ge 2\,. \tag{4.39}$$

In particular, there exists at most one i with a nonzero product in (4.36). By the definition of $\mathcal{Z}_{I';I}$, $\rho_{I';j} \ge 2$ on $\mathcal{Z}_{I';I}$ for all $j \in I'-I$ and so

$$\eta_{I';j}\big|_{\mathcal{Z}_{I';I}} \equiv 1 \qquad \forall\ j \in I'-I\,. \tag{4.40}$$

By (4.39) and (4.40),

$$\begin{aligned} \sum_{i\in I'-I} \rho_{I';i}\alpha_{I';i} + \sum_{i\in I}\bigg(1 - \prod_{j\in I-\{i\}}\eta_{I';j}\bigg)\rho_{I';i}\alpha_{I';i} &= \sum_{i\in I'}\bigg(1 - \prod_{j\in I'-\{i\}}\eta_{I';j}\bigg)\rho_{I';i}\alpha_{I';i} \\ \text{and} \qquad \sum_{i\in I}\prod_{j\in I-\{i\}}\eta_{I';j} &= \sum_{i\in I'}\prod_{j\in I'-\{i\}}\eta_{I';j} \end{aligned} \tag{4.41}$$

on $\Phi_{\mathcal{C},\varepsilon;I'}^{-1}(\Delta_1) \cap \mathcal{Z}_{I';I}$.

By (4.9) with I replaced by I' and (4.5),

$$\Psi_{I';I}^{(\varepsilon)*}\omega_I = \bigg(\pi_{I';I}^*\omega_{I'} + \frac{1}{2}\mathrm{d}\sum_{i\in I'-I}\varepsilon^2\rho_{I';i}\alpha_{I';i}\bigg)\bigg|_{\mathcal{N}_{I';I}(4^N)}\,. \tag{4.42}$$

Suppose $|I| \ge 2$. The product Hermitian isomorphism $\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}$ satisfies

$$\pi_I \circ \mathfrak{D}\Psi_{I';I}^{(\varepsilon)} = \Psi_{I';I}^{(\varepsilon)} \circ \pi_{I';I}^*\pi_{I';I'-I}\big|_{\pi_{I';I}^*\mathcal{N}_{I';I'-I}|_{\mathcal{N}_{I';I}(4^N)}}\,.$$

Along with (4.42), this implies that

$$\big\{\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}\big\}^*\pi_I^*\omega_I = \bigg(\pi_{I'}^*\omega_{I'} + \frac{1}{2}\mathrm{d}\sum_{i\in I'-I}\varepsilon^2\rho_{I';i}\alpha_{I';i}\bigg)\bigg|_{\pi_{I';I}^*\mathcal{N}_{I';I'-I}|\mathcal{N}_{I';I}(4^N)}\,. \tag{4.43}$$

Combining this with the second assumption in (4.7) with I replaced by I' and (4.22), we obtain

$$\begin{aligned} \big\{\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}\big\}^*\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)} = \pi_{I'}^*\omega_{I'} + \frac{1}{2}\mathrm{d}\bigg(&\sum_{i\in I'-I}\varepsilon^2\rho_{I';i}\alpha_{I';i} + \sum_{i\in I}\bigg(1 - \prod_{j\in I-\{i\}}\eta_{I';j}\bigg)\varepsilon^2\rho_{I';i}\alpha_{I';i} \\ &+ \sum_{i\in I}\bigg(\prod_{j\in I-\{i\}}\eta_{I';j}\bigg)\varepsilon^2\Phi_{\mathcal{C},\varepsilon;I'}^*\big(r^2\mathrm{d}\theta\big)\bigg). \end{aligned} \tag{4.44}$$

By (4.41), the right-hand side of (4.44) equals the right-hand side of (4.36) with I replaced by I' on $\Phi_{\mathcal{C},\varepsilon;I'}^{-1}(\Delta_1) \cap \mathcal{Z}_{I';I}$. By (4.20) and (4.42), (4.44) also holds if $|I| = 1$. Thus, (4.38) holds in this case as well. □

By Lemma 4.2, the 2-forms $\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}$ induce a closed 2-form $\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)}$ on $\mathcal{Z}'$. It remains to show that its restrictions to a neighborhood $\mathcal{Z}$ of $\mathcal{Z}_0'$ in $\mathcal{Z}'$ and to the fibers $\mathcal{Z} \cap \pi_{\mathcal{C},\varepsilon}^{-1}(\lambda)$ with $\lambda \in \mathbb{C}^*$ are nondegenerate. The next lemma, which follows immediately from Corollary A.3 and (4.16), is used to verify that this is the case in the "middle" region of each domain.

Lemma 4.3. *For each $I\subset[N]$ with $|I|\ge 2$, there exist $f_I\in C^\infty(X_I^\star;\mathbb{R}^+)$ and an $\mathbb{R}$-valued 1-form μ_I on $X_I^\star$ such that*

$$\Phi_{\mathcal{C},\varepsilon;I}^*r^2=\big(\mathcal{C}\varepsilon^{|I|-1}\big)^2 f_I\prod_{i\in I}\rho_{I;i},\quad \Phi_{\mathcal{C},\varepsilon;I}^*\mathrm{d}\theta=\sum_{i\in I}\alpha_{I;i}+\pi_I^*\mu_I\quad \text{on } \mathcal{N}X_I|_{X_I^\star} \tag{4.45}$$

for all functions $\varepsilon,\mathcal{C}\colon X_\emptyset\longrightarrow\mathbb{R}^+$ that restrict to smooth functions on X_I.

Proposition 4.4. *Let $\varepsilon,\mathcal{C}\colon X_\emptyset\longrightarrow\mathbb{R}^+$ be as in (4.7) and (4.15). Suppose $\varepsilon|_{X_i},\mathcal{C}|_{X_i}$ are smooth for all $i\in[N]$, ε is sufficiently small, and $\mathcal{C}$ is sufficiently large (depending on ε). Then there exists a neighborhood $\mathcal{Z}$ of $\mathcal{Z}_0'$ in $\mathcal{Z}'$ such that the restrictions of $\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)}$ to $\mathcal{Z}$ and to $\mathcal{Z}\cap\pi_{\mathcal{C},\varepsilon}^{-1}(\lambda)$ with $\lambda\in\mathbb{C}^*$ are nondegenerate.*

Proof. We need to show that every $x\in X_I$ with $I\in\mathcal{P}^*(N)$ has a neighborhood $\mathcal{Z}_x$ in $\mathcal{Z}'$ such that the restrictions of $\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)}$ to $\mathcal{Z}_x$ and to $\mathcal{Z}_x\cap\pi_{\mathcal{C},\varepsilon}^{-1}(\lambda)$ with $\lambda\in\mathbb{C}^*$ are nondegenerate. Since X_I is contained in $\mathcal{Z}_0'$, it is sufficient to show that for each $x\in X_I^\circ$ there exist a neighborhood U_x of x in X_I° and $r_x\in\mathbb{R}^+$ such that the restrictions of $\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}$ to $\mathcal{Z}_I|_{U_x}\cap\Phi_{\mathcal{C},\varepsilon;I}^{-1}(\Delta_{r_x})$ and to $\mathcal{Z}_I|_{U_x}\cap\Phi_{\mathcal{C},\varepsilon;I}^{-1}(\lambda)$ with $\lambda\in\Delta_{r_x}$ are nondegenerate. Since the 2-form (4.37) and its restriction to a fiber of $\Phi_{\mathcal{C},\varepsilon;i}$ are symplectic over $X_i\times\mathbb{C}$ (even if ε is not constant), it remains to consider the case $|I|\ge 2$.

For each $i\in I$ with $|I|\ge 2$, let $\kappa_{I;i}\in\Omega^2(TX_I)$ be the curvature form of $\alpha_{I;i}$. Thus,

$$\widehat{\omega}_I=\pi_I^*\omega_I+\frac{1}{2}\sum_{i\in I}\big(\rho_{I;i}\pi_I^*\kappa_{I;i}+\mathrm{pr}_{I;I-i}^*(\mathrm{d}\rho_{I;i}\wedge\alpha_{I;i})\big).$$

Let $\widetilde{J}_I$ denote the almost complex structure on the total space of $\mathcal{N}X_I\longrightarrow X_I$ induced by an ω-tame almost complex structure J_I on X_I via the product Hermitian structure $(\rho_{I;i},\nabla^{(I;i)})_{i\in I}$; see the paragraph above Lemma A.4. If ε as in (4.7) is sufficiently small, then $\widehat{\omega}_I$ tames $\widetilde{J}_I$ over $\mathcal{N}X_I(4^N\varepsilon^2)$. Thus, $\widehat{\omega}_I^{(\varepsilon)}$ tames $\widetilde{J}_{\varepsilon;I}\equiv m_{\varepsilon;I}^*\widetilde{J}_I$ over $\mathcal{N}X_I(4^N)$.

Let $\lambda\in\mathbb{C}^*$ and $v_I\in\mathcal{Z}_I\cap\Phi_{\mathcal{C},\varepsilon;I}^{-1}(\lambda)$. Define

$$I_0=\big\{i\in I\colon\,\rho_{I;i}(v_I)\le\frac{3}{4}\big\},\quad I_{01}=\big\{i\in I\colon\,\frac{3}{4}<\rho_{I;i}(v_I)<1\big\},\quad I_1=\big\{i\in I\colon\,\rho_{I;i}(v_I)\ge 1\big\}.$$

If $|\lambda|<1$ and $\mathcal{C}^2>2^N\mathcal{C}_{\Phi,\varepsilon}$, with $\mathcal{C}_{\Phi,\varepsilon}$ as in (4.34), then $I_0\ne\emptyset$; see (4.39).

Suppose $|I_0|\ge 2$. By (4.36), $\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}$ at v_I is then given by

$$\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}=\pi_I^*\omega_I+\frac{1}{2}\sum_{i\in I}\mathrm{d}\big(\varepsilon^2\rho_{I;i}\alpha_{I;i}\big)\equiv\widehat{\omega}_I^{(\varepsilon)}.$$

This form tames $\widetilde{J}_{\varepsilon;I}$ on $T_{v_I}\mathcal{Z}_I$ and thus is nondegenerate on this space. By Corollary A.5, the restriction of this form to $T_{v_I}\Phi_{\mathcal{C},\varepsilon;I}^{-1}(\lambda)$ is also nondegenerate if $|\lambda|$ is sufficiently small (depending only on $\pi_I(v_I)$).

Suppose $|I_0|=1$ and $I_{01}=\emptyset$. By (4.36), $\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}$ at v_I is then given by

$$\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}=\pi_I^*\omega_I+\frac{1}{2}\mathrm{d}\sum_{i\in I_1}\varepsilon^2\rho_{I;i}\alpha_{I;i}+\varepsilon^2\Phi_{\mathcal{C},\varepsilon;I}^*\big(r^2\mathrm{d}\theta\big).$$

Along with (4.43) with $I\subset I'$ replaced by $I_0\subset I$, the second assumption in (4.7), (4.22) with $I'=I_0$, and (4.19), this implies that

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}\big|_{v_I}=\bigg(\big\{\mathfrak{D}\Psi^{(\varepsilon)}_{I;I_0}\big\}^*\Big(\pi_1^*\omega_{I_0}+\frac{1}{2}\mathrm{d}\big(\varepsilon^2\pi_2^*(r^2\mathrm{d}\theta)\big)\Big)\bigg)_{v_I}.$$

This form is nondegenerate, since $\mathfrak{D}\Psi^{(\varepsilon)}_{I;I_0}$ is a diffeomorphism on a neighborhood of v_I contained in the set (4.21) with $I'=I_0$. By (4.22) with $I'=I_0$, the restriction of this form to $T_{v_I}\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\lambda)$ is also nondegenerate.

Suppose $I_0=\{i_0\}$ is a single-element set and $I_{01}\neq\emptyset$. By (4.36), $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}$ at v_I is then given by

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}=\pi_I^*\omega_I+\frac{1}{2}\mathrm{d}\bigg(\sum_{i\in I_{01}\cup I_1}\varepsilon^2\rho_{I;i}\alpha_{I;i}+\Big(1-\prod_{i\in I_{01}}\eta_{I;i}\Big)\varepsilon^2\rho_{I;i_0}\alpha_{I;i_0}+\Big(\prod_{i\in I_{01}}\eta_{I;i}\Big)\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I}\big(r^2\mathrm{d}\theta\big)\bigg).$$

Since $\mathfrak{D}\Psi^{(\varepsilon)}_{I;I_0\cup I_{01}}$ is a product Hermitian isomorphism, (4.43) with $I\subset I'$ replaced by $I_0\cup I_{01}\subset I$, the second assumption in (4.7), and (4.22) with $I'=I_0\cup I_{01}$ thus imply that

$$\begin{aligned}\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}\big|_{v_I}=\big\{\mathfrak{D}\Psi^{(\varepsilon)}_{I;I_0\cup I_{01}}\big\}^*\bigg(&\pi^*_{I_0\cup I_{01}}\omega_{I_0\cup I_{01}}+\frac{1}{2}\mathrm{d}\bigg(\sum_{i\in I_{01}}\rho^{(\varepsilon)}_{I_0\cup I_{01};i}\alpha_{I_0\cup I_{01};i}\\&+\Big(1-\prod_{i\in I_{01}}\eta_{I_0\cup I_{01};i}\Big)\rho^{(\varepsilon)}_{I_0\cup I_{01};i_0}\alpha_{I_0\cup I_{01};i_0}+\Big(\prod_{i\in I_{01}}\eta_{I_0\cup I_{01};i}\Big)\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I_0\cup I_{01}}\big(r^2\mathrm{d}\theta\big)\bigg)\bigg)_{v_I}.\end{aligned}$$

Since $\mathfrak{D}\Psi^{(\varepsilon)}_{I;I_0\cup I_{01}}$ is a diffeomorphism satisfying (4.22) with $I'=I_0\cup I_{01}$, we can therefore assume that $I_1=\emptyset$.

With f_I and μ_I as in Lemma 4.3, let

$$\begin{aligned}f_{\mathcal{C};I}=\mathcal{C}^2f_I,\quad \widetilde{f}_{I;i_0}=\Big(1-\prod_{i\in I_{01}}\eta_{I;i}\Big)+f_{\mathcal{C};I}\Big(\prod_{i\in I_{01}}\eta_{I;i}\rho^{(\varepsilon)}_{I;i}\Big),\quad \widetilde{f}_{I;i}=f_{\mathcal{C};I}\eta_{I;i}\prod_{j\in I_{01}-i}\eta_{I;j}\rho^{(\varepsilon)}_{I;j}\ \forall\, i\in I_{01},\\ \beta_{I;i_0}=\mathrm{d}\bigg(f_{\mathcal{C};I}\prod_{i\in I_{01}}\eta_{I;i}\rho^{(\varepsilon)}_{I;i}-\prod_{i\in I_{01}}\eta_{I;i}\bigg),\qquad \widetilde{\mu}_I=f_{\mathcal{C};I}\Big(\prod_{i\in I_{01}}\eta_{I;i}\rho^{(\varepsilon)}_{I;i}\Big)\pi_I^*\mu_I\,.\end{aligned}$$

Fix a Riemannian metric on X_I. Via the Hermitian data $(\rho_{I;i},\nabla^{(I;i)})_{i\in I}$, it lifts to a Riemannian metric on $\mathcal{N}X_I$. By the first statement in (4.45) and the definition of I_{01}, there exists a smooth function $\mathcal{C}_I\colon X_I\longrightarrow\mathbb{R}^+$, dependent only on ε and $\mathcal{C}$, such that

$$\frac{1}{\mathcal{C}_I(\pi_I(v_I))}\le\big|\widetilde{f}_{I;i_0}\big|_{v_I}\le\mathcal{C}_I\big(\pi_I(v_I)\big),\tag{4.46}$$

$$\rho^{(\varepsilon)}_{I;i_0}(v_I)\le\mathcal{C}_I\big(\pi_I(v_I)\big)|\lambda|^2,\quad \big|\mathrm{d}\rho^{(\varepsilon)}_{I;i_0}\big|_{v_I},\big|\rho^{(\varepsilon)}_{I;i_0}\alpha_{I;i_0}\big|_{v_I}\le\mathcal{C}_I\big(\pi_I(v_I)\big)|\lambda|,\tag{4.47}$$

$$\big|\mathrm{d}\rho^{(\varepsilon)}_{I;i}\big|_{v_I},\big|\rho^{(\varepsilon)}_{I;i}\alpha_{I;i}\big|_{v_I},\big|\beta_{I;i_0}(v_I)\big|,\big|\widetilde{f}_{I;i}(v_I)\big|,\big|\mathrm{d}\widetilde{f}_{I;i}\big|_{v_I},\big|\widetilde{\mu}_I\big|_{v_I},\big|\mathrm{d}\widetilde{\mu}_I\big|_{v_I}\le\mathcal{C}_I\big(\pi_I(v_I)\big);\tag{4.48}$$

these bounds apply to any $v_I\in\mathcal{Z}_I$ with $I_0=\{i_0\}$ and $I_1=\emptyset$.

By (4.45),

$$\Big(\prod_{i\in I_{01}}\eta_{I;i}\Big)\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I}(r^2\mathrm{d}\theta)=f_{\mathcal{C};I}\Big(\prod_{i\in I_{01}}\eta_{I;i}\rho^{(\varepsilon)}_{I;i}\Big)\big(\rho^{(\varepsilon)}_{I;i_0}\alpha_{I;i_0}\big)+\sum_{i\in I_{01}}\rho^{(\varepsilon)}_{I;i_0}\widetilde{f}_{I;i}\big(\rho^{(\varepsilon)}_{I;i}\alpha_{I;i}\big)+\rho^{(\varepsilon)}_{I;i_0}\widetilde{\mu}_I\,.$$

Thus, $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}$ at v_I is given by

$$\begin{aligned}\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}=&\pi_I^*\omega_I+\frac{1}{2}\sum_{i\in I_{01}}\mathrm{d}(\rho^{(\varepsilon)}_{I;i}\alpha_{I;i})+\frac{1}{2}\widetilde{f}_{I;i_0}\mathrm{d}(\rho^{(\varepsilon)}_{I;i_0}\alpha_{I;i_0})\\&+\frac{1}{2}\sum_{i\in I_{01}}\Big(\rho^{(\varepsilon)}_{I;i_0}\widetilde{f}_{I;i}\mathrm{d}(\rho^{(\varepsilon)}_{I;i}\alpha_{I;i})+\big(\widetilde{f}_{I;i}\mathrm{d}\rho^{(\varepsilon)}_{I;i_0}+\rho^{(\varepsilon)}_{I;i_0}\mathrm{d}\widetilde{f}_{I;i}\big)\wedge(\rho^{(\varepsilon)}_{I;i}\alpha_{I;i})\Big)\\&+\frac{1}{2}\beta_{I;i_0}\wedge(\rho^{(\varepsilon)}_{I;i_0}\alpha_{I;i_0})+\frac{1}{2}\big(\mathrm{d}\rho^{(\varepsilon)}_{I;i_0}\wedge\widetilde{\mu}_I+\rho^{(\varepsilon)}_{I;i_0}\mathrm{d}\widetilde{\mu}_I\big)\,.\end{aligned}\tag{4.49}$$

The 2-form $\widetilde{\omega}^{(\varepsilon)}_{\mathrm{top}}$ on the first line on the right-hand side of (4.49) is the pullback by $m_{\varepsilon;I}$ of the 2-form

$$\pi_I^*\omega_I+\frac{1}{2}\sum_{i\in I_{01}}\mathrm{d}(\rho_{I;i}\alpha_{I;i})+\frac{1}{2}\widetilde{f}_{I;i_0}\circ m^{-1}_{\varepsilon;I}\,\mathrm{d}\big(\rho_{I;i_0}\alpha_{I;i_0}\big)$$

at $m_{\varepsilon;I}(v_I)$. By the $\widehat{\omega}_I$-tameness of $\widetilde{J}_I$ on $\mathcal{N}X_I(4^N)$ and (4.46), there exists a smooth function $\varepsilon_{I;i_0}\colon X_I\longrightarrow\mathbb{R}^+$, dependent only on ε and $\mathcal{C}$, so that this form tames $\widetilde{J}_I$ on

$$\big\{v\in\mathcal{N}X_I(3^N\varepsilon^2)\colon\rho_{I;i_0}(v)\le\varepsilon_{I;i_0}\big(\pi_I(v)\big)\big\}\,.$$

The 2-form $\widetilde{\omega}^{(\varepsilon)}_{\mathrm{top}}$ then tames $\widetilde{J}_{\varepsilon;I}$ on

$$W_{I;i_0}\equiv\big\{v\in\mathcal{N}X_I(3^N)\colon\rho^{(\varepsilon)}_{I;i_0}(v)\le\varepsilon_{I;i_0}\big(\pi_I(v)\big)\big\}\,.$$

By Corollary A.5, for every $x\in X_I^\circ$ there exist a precompact neighborhood U_x and $r_x\in\mathbb{R}^+$ such that the restriction of $\widetilde{\omega}^{(\varepsilon)}_{\mathrm{top}}$ to $T_v\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\lambda)$ is nondegenerate for all $v\in W_{I;i_0}|_{\overline{U_x}}\cap\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\lambda)$ and $\lambda\in\Delta_{r_x}$. There thus exists $r'_x\in(0,r_x)$ such that the right-hand side of (4.49) is nondegenerate on $T_v\mathcal{Z}_I$ and on $T_v\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\lambda)$ for all $v\in W_{I;i_0}|_{\overline{U_x}}\cap\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\Delta_{r'_x})$ and $\lambda\in\Delta_{r'_x}$ satisfying the bounds in (4.47) and (4.48). We conclude that $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{v_I}$ and the restriction $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{v_I}$ to $T_{v_I}\Phi^{-1}_{\mathcal{C},\varepsilon;I}(\lambda)$ are nondegenerate if $\lambda\in\Delta_{r'_x}$. □

Corollary 4.5. *Let $\varepsilon,\mathcal{C}\colon X\longrightarrow\mathbb{R}^+$ be as in (4.7) and (4.15). Suppose $\varepsilon|_{X_i},\mathcal{C}|_{X_i}$ are smooth for all $i\in[N]$, ε is sufficiently small, and $\mathcal{C}$ is sufficiently large (depending on ε). Then $\iota^*_{\mathcal{C},\varepsilon;i}\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C}}=\omega_i$ for all $i\in[N]$ and $\{\iota_{\mathcal{C},\varepsilon;i}(X_i)\}_{i\in[N]}$ is an SC symplectic divisor in $(\mathcal{Z},\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C}})$ for some neighborhood $\mathcal{Z}$ of $\mathcal{Z}'_0$ in $\mathcal{Z}'$. Furthermore, the associated homomorphisms (2.16) are orientation-preserving isomorphisms.*

Proof. By the first case in (4.32) and (4.37),

$$\iota^*_{\mathcal{C},\varepsilon;i}\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C}}\big|_{X_i^\circ}=\omega_i|_{X_i^\circ}\quad\forall\,i\in[N]\,.$$

By the second in (4.32), (4.36), and the vanishing of $\eta_{I;i}$, $\rho_{I;i}\alpha_{I;i}$, and $\Phi_{\mathcal{C},\varepsilon;I}$ along $X_{I;i}\subset\mathcal{N}_{I;i}$,

$$\big\{\Psi_{I;i}^{(\varepsilon)}\big\}^*\iota_{\mathcal{C},\varepsilon;i}^*\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)}|_{X_{I;i}}=\bigg(\pi_I^*\omega_I+\frac{1}{2}\mathrm{d}\bigg(\sum_{j\in I-i}\varepsilon^2\rho_{I;j}\alpha_{I;j}\bigg)\bigg)\bigg|_{X_{I;i}}\quad\forall\,\{i\}\subsetneq I\subset[N].$$

The first claim of the corollary now follows from (4.9).

By Proposition 4.1, $\{\iota_{\mathcal{C},\varepsilon;i}(X_i)\}_{i\in[N]}$ is a transverse collection of closed submanifolds of $\mathcal{Z}'$ of codimension 2. By Proposition 4.4 and the first claim of the corollary, the intersections of these submanifolds are symplectic submanifolds of $(\mathcal{Z},\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)})$ for some neighborhood $\mathcal{Z}$ of $\mathcal{Z}_0'$ in $\mathcal{Z}'$. The preimages of these intersections in $\mathcal{Z}_I$ correspond to the intersections of the hyperplane subbundles $\mathcal{N}_{I;i}\subset\mathcal{N}X_I$ on a neighborhood of X_I°. These hyperplanes are $\widetilde{\omega}_{\mathcal{C};I}^{(\varepsilon)}$-orthogonal; see (5.34). Thus, the symplectic orientations on neighborhoods of X_I° in the corresponding subbundles $\mathcal{N}_{I;I'}\subset\mathcal{N}X_I$ are the same as the intersection orientations induced from the hyperplane subbundles $\mathcal{N}_{I;i}\subset\mathcal{N}X_I$. This establishes the second claim of the corollary and implies the last one, in light of the last claim of Proposition 4.1(3). □

Remark 4.6. We believe that the dependence of the maximal norm r_x for the acceptable values of $\lambda\in\mathbb{C}$ on x in the proof of Proposition 4.4 can be dropped by constructing the trivializations $(\widetilde{\Phi}_t)_{t\in B}$ in Proposition 3.9 along with constructing the regularizations $(\mathfrak{R}_t)_{t\in B}$ in the proof of [4, Theorem 2.17]. This would then lead to a fibration $\mathcal{Z}$ with uniform smooth fibers over a neighborhood Δ of 0 in $\mathbb{C}$ and remove the compactness assumption from the last statement of Theorem 2.8.

5 The smoothability criterion

We show in Section 5.1 that a nearly regular symplectic fibration $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ as in Definition 2.6 determines a homotopy class of trivializations of the associated complex line bundle $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$; see Proposition 5.1. If $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ is a one-parameter family of smoothings of an SC symplectic variety $(X_\emptyset,(\omega_i)_{i\in[N]})$ as in Definition 2.5, this line bundle restricts to (2.14) over the singular locus X_∂. Proposition 5.1 then determines a homotopy class of trivializations of (2.14). In particular, (2.17) is a necessary condition for an SC symplectic variety to be smoothable. By Proposition 5.5 proved in Section 5.2, the homotopy class of trivializations determined by a one-parameter family $(\mathcal{Z},\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)},\pi_{\mathcal{C},\varepsilon})$ of smoothings constructed in Section 4 is the homotopy class used to construct this family.

Proposition 5.1 and its proof readily extend to families of nearly regular symplectic fibrations parametrized by a manifold B. They endow a natural complex line bundle $\mathcal{O}_{B;\mathcal{Z}}(\mathcal{Z}_0)$ over the total space of such a family with a canonical homotopy class of trivializations. Proposition 5.5 and its proof extend directly to compact families of the relevant data on $(X_I)_{I\in\mathcal{P}^*(N)}$. They ensure that the family of one-parameter families of smoothings then arising from the construction of Section 4 encodes the homotopy class of trivializations used to obtain this family.

5.1 The necessity of (2.17)

If $V\subset X$ is a smooth symplectic divisor, the line bundle $\mathcal{O}_X(V)$ has a canonical section s_V with zero set V; it is described similarly to (3.24). Thus, any tensor product of such line bundles also

has a canonical section; its zero set is the union of the associated symplectic divisors.

Proposition 5.1. *Let $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ be a nearly regular symplectic fibration over Δ as in Definition 2.6 and $s_\emptyset$ be the canonical section of the complex line bundle*

$$\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)\equiv\bigotimes_{i\in[N]}\mathcal{O}_{\mathcal{Z}}(X_i)\longrightarrow\mathcal{Z}.$$

Then there exists a trivialization $\Phi_{\mathcal{Z}}$ of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ such that the smooth maps

$$\Phi_{\mathcal{Z}}\circ s_\emptyset|_{\mathcal{Z}-\mathcal{Z}_0},\pi|_{\mathcal{Z}-\mathcal{Z}_0}:\mathcal{Z}-\mathcal{Z}_0\longrightarrow\mathbb{C}^*$$

are homotopic. Furthermore, any two such trivializations $\Phi_{\mathcal{Z}}$ are homotopic.

Lemma 5.2. *Let $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ be as in Proposition 5.1 and $\mathcal{Z}_{0;\partial}\subset\mathcal{Z}_0$ be the singular locus of $\mathcal{Z}_0$. If $\mathcal{Z}_0$ is connected, there exists a smooth embedding $(\Delta,0)\hookrightarrow(\mathbb{C},0)$ such that the linear map*

$$D\pi:\mathcal{N}_{\mathcal{Z}}\big(\mathcal{Z}_0-\mathcal{Z}_{0;\partial}\big)\longrightarrow T_0\Delta\cong T_0\mathbb{C}\cong\mathbb{C}\tag{5.1}$$

induced by $\mathrm{d}\pi$ is an orientation-preserving trivialization of $\mathcal{N}_{\mathcal{Z}}(\mathcal{Z}_0-\mathcal{Z}_{0;\partial})$.

Proof. Choose a point $p\in\mathcal{Z}_0-\mathcal{Z}_{0;\partial}$ and a smooth embedding $(\Delta,0)\hookrightarrow(\mathbb{C},0)$ such that the isomorphism

$$D_p\pi:\mathcal{N}_{\mathcal{Z}}(\mathcal{Z}_0-\mathcal{Z}_{0;\partial})\big|_p\longrightarrow T_0\Delta\cong T_0\mathbb{C}\cong\mathbb{C}$$

is orientation-preserving. Let $\mathcal{Z}^+,\mathcal{Z}^-\subset\mathcal{Z}-\mathcal{Z}_{0;\partial}$ be the subspaces of points x such that the isomorphism

$$D_x\pi:\mathcal{N}_{\mathcal{Z}}\big(\mathcal{Z}_{\pi(x)}-\mathcal{Z}_{0;\partial}\big)\big|_x\longrightarrow T_{\pi(x)}\Delta\cong T_{\pi(x)}\mathbb{C}\tag{5.2}$$

is orientation-preserving and orientation-reversing, respectively. By assumption, $p\in\mathcal{Z}^+$. Since $\mathcal{Z}_0$ is connected and $\mathcal{Z}_{0;\partial}$ consists of codimension 4 submanifolds of $\mathcal{Z}$, the complement $\mathcal{Z}^*$ of $\mathcal{Z}_{0;\partial}$ in the topological component of $\mathcal{Z}$ containing $\mathcal{Z}_0$ is connected as well. Since the disjoint open subsets $\mathcal{Z}^+$ and $\mathcal{Z}^-$ cover $\mathcal{Z}^*$, $\mathcal{Z}^*\subset\mathcal{Z}^+$ and thus (5.2) is orientation-preserving for all $x\in\mathcal{Z}_0-\mathcal{Z}_{0;\partial}$. □

Proof of Proposition 5.1. If s_1 and s_2 are non-vanishing sections of a complex line bundle L over some space X, we write $s_1\sim s_2$ if s_1 and s_2 are homotopic through non-vanishing sections of L. A trivialization Φ of L corresponds to a non-vanishing section of L. The equivalence classes of non-vanishing sections of L correspond to the homotopy classes of trivializations of L. The first claim of the proposition is equivalent to the existence of a non-vanishing section s of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ so that the smooth maps

$$(s_\emptyset/s)|_{\mathcal{Z}-\mathcal{Z}_0},\pi|_{\mathcal{Z}-\mathcal{Z}_0}:\mathcal{Z}-\mathcal{Z}_0\longrightarrow\mathbb{C}^*\tag{5.3}$$

are homotopic.

We can assume that $\mathcal{Z}_0$ is connected. Let $(\Delta,0)\hookrightarrow(\mathbb{C},0)$ be an embedding as in Lemma 5.2,

$$\mathcal{N}=\mathcal{N}_{\mathcal{Z}}(\mathcal{Z}_0-\mathcal{Z}_{0;\partial}),\qquad\mathcal{N}_i=\mathcal{N}_{\mathcal{Z}}X_i\,,$$

and $U\subset\mathcal{Z}$ be an open subset such that the inclusions

$$\mathcal{Z}-U\longrightarrow\mathcal{Z}-\mathcal{Z}_{0;\partial}\qquad\text{and}\qquad\mathcal{Z}-(\mathcal{Z}_0\cup U)\longrightarrow\mathcal{Z}-\mathcal{Z}_0$$

are homotopy equivalences. Denote by $\Psi_\Delta : \widetilde{\Delta} \longrightarrow \Delta$ the canonical identification of a neighborhood $\widetilde{\Delta}$ of 0 in $T_0\Delta = T_0\mathbb{C}$ with $\Delta \subset \mathbb{C}$ and by $\mathfrak{i}$ the standard complex structure on $T_0\Delta$. Since the restriction of (5.1) to each fiber of $\mathcal{N}$ is orientation-preserving, the complex structure $\widetilde{\mathfrak{i}} \equiv \{D\pi\}^*\mathfrak{i}$ is tamed by $\omega_{\mathcal{Z}}|_{\mathcal{N}}$.

Let $\Psi_i : \mathcal{N}_i' \longrightarrow \mathcal{Z}$, with $i \in [N]$, be regularizations of X_i in $\mathcal{Z}$ in the sense of Definition 3.1 such that

$$\begin{aligned} \mathrm{Im}\big(\Psi_i|_{\mathcal{N}_i'|_{X_i - U}}\big) \cap \mathrm{Im}\big(\Psi_j|_{\mathcal{N}_j'|_{X_j - U}}\big) &= \emptyset \quad \forall\, i,j \in [N],\, i \neq j, \\ \pi \circ \Psi_i = \Psi_\Delta \circ D\pi\big|_{\mathcal{N}_i'|_q} : \mathcal{N}_i'|_q &\longrightarrow \Delta \subset \mathbb{C} \quad \forall\, q \in X_i - U. \end{aligned} \tag{5.4}$$

We can use these regularizations and the complex structure $\widetilde{\mathfrak{i}}$ to construct the line bundles $\mathcal{O}_{\mathcal{Z}}(X_i)$ as in (2.8). The canonical regularization Ψ_Δ of 0 in $\mathbb{C}$ and the standard complex structure $\mathfrak{i}$ on $T_0\Delta$ similarly determine a line bundle $\mathcal{O}_\Delta(0) \longrightarrow \Delta$. By (5.4), $D\pi$ induces an isomorphism

$$\Phi : \mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)\big|_{\mathcal{Z}-U} \longrightarrow \pi^*\mathcal{O}_\Delta(0)\big|_{\mathcal{Z}-U}, \qquad \begin{aligned} [x,c] &\longrightarrow [\pi(x), c]; \\ [\Psi_i(v), v, w] &\longrightarrow [\pi(\Psi_i(v)), D\pi(v), D\pi(w)]; \end{aligned}$$

see (2.8) for the notation.

Let s_0 be the canonical section of $\mathcal{O}_\Delta(0)$ and s_1 be the constant section 1 of $\mathcal{O}_\Delta(0)$, i.e.

$$s_1\big(\Psi_\Delta(u)\big) = \big[\Psi_\Delta(u), u, 1\big] \qquad \forall\, u \in \widetilde{\Delta}.$$

In particular, $s_0/s_1 : \Delta \longrightarrow \mathbb{C}$ is the inclusion map. They lift to sections π^*s_0 and π^*s_1 of $\pi^*\mathcal{O}_\Delta(0)$ so that

$$\pi^*s_0/\pi^*s_1 = \pi : \mathcal{Z} \longrightarrow \mathbb{C}. \tag{5.5}$$

By (5.4),

$$s_\emptyset|_{\mathcal{Z}-U} = \Phi^{-1} \circ \pi^*s_0\big|_{\mathcal{Z}-U}. \tag{5.6}$$

Since the inclusion $\mathcal{Z} - U \longrightarrow \mathcal{Z} - \mathcal{Z}_{0;\partial}$ is a homotopy equivalence, there exists a non-vanishing section s' of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)|_{\mathcal{Z}-\mathcal{Z}_{0;\partial}}$ such that

$$s'|_{\mathcal{Z}-U} \sim \Phi^{-1} \circ \pi^*s_1\big|_{\mathcal{Z}-U}. \tag{5.7}$$

By Corollary 5.4(2) below, there exists a non-vanishing section s of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ so that

$$s|_{\mathcal{Z}-\mathcal{Z}_{0;\partial}} \sim s'. \tag{5.8}$$

By (5.6), (5.8), (5.7), and (5.5),

$$\big(s_\emptyset/s\big)\big|_{\mathcal{Z}-(\mathcal{Z}_0 \cup U)} \sim \Phi^{-1} \circ \pi^*s_0 \big/ \Phi^{-1} \circ \pi^*s_1\big|_{\mathcal{Z}-(\mathcal{Z}_0 \cup U)} = \pi\big|_{\mathcal{Z}-(\mathcal{Z}_0 \cup U)}.$$

Since the inclusion $\mathcal{Z} - (\mathcal{Z}_0 \cup U) \longrightarrow \mathcal{Z} - \mathcal{Z}_0$ is a homotopy equivalence, the maps (5.3) are homotopic. This establishes the first claim of the proposition.

The section $s_\emptyset$ of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ does not vanish on $\mathcal{Z} - \mathcal{Z}_0$. If $\Phi_{\mathcal{Z}}$ and $\Phi'_{\mathcal{Z}}$ are trivializations of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ satisfying the homotopy condition in the proposition, then $\Phi_{\mathcal{Z}}|_{\mathcal{Z}-\mathcal{Z}_0}$ and $\Phi'_{\mathcal{Z}}|_{\mathcal{Z}-\mathcal{Z}_0}$ are homotopic trivializations of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)|_{\mathcal{Z}-\mathcal{Z}_0}$. By Corollary 5.4(1), $\Phi_{\mathcal{Z}}$ and $\Phi'_{\mathcal{Z}}$ are thus homotopic trivializations of $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$. □

It remains to establish the two statements of Corollary 5.4 used above.

Lemma 5.3. *Let B be a paracompact topological space and $(E,\dot{E})\longrightarrow B$ be a relative bundle pair with fiber pair $(F,\dot{F})$ in the sense of [24, Section 5.7]. If $\mathfrak{c}\in\mathbb{Z}^+$ and $H_i(F,\dot{F};\mathbb{Z})=0$ for all $i<\mathfrak{c}$, then $H^i(E,\dot{E};\mathbb{Z})=0$ for all $i<\mathfrak{c}$.*

Proof. By our assumptions and the Kunneth formula [24, Theorem 5.3.10],

$$H_i\big(B\times F, B\times\dot{F};\mathbb{Z}\big)=0 \qquad \forall\, i<\mathfrak{c}.$$

By Mayer-Vietoris [24, Corollary 5.1.14] and induction, this implies that $H_i(E,\dot{E};\mathbb{Z})=0$ for all $i<\mathfrak{c}$ if $(E,\dot{E})\longrightarrow B$ admits a finite trivializing open cover. Taking the direct limit over compact subsets of B, we find that $H_i(E,\dot{E};\mathbb{Z})=0$ for all $i<\mathfrak{c}$ for any paracompact space B. The claim now follows from the Universal Coefficients Theorem [18, Theorem 53.1]. □

Corollary 5.4. *Suppose M is a manifold, $\pi\colon L\longrightarrow M$ is a Hermitian line bundle, and $M'\subset M$ is the complement of closed submanifolds $V_1,\cdots,V_\ell\subset M$ of real codimension $\mathfrak{c}\in\mathbb{Z}^+$ or higher.*

(1) If $\mathfrak{c}\ge 2$, any two trivializations of L over M that restrict to homotopic trivializations of $L|_{M'}$ are homotopic as trivializations of L over M.

(2) If $\mathfrak{c}\ge 3$, every smooth trivialization of $L|_{M'}$ is homotopic through trivializations of $L|_{M'}$ to the restriction of a smooth trivialization of L over M.

Proof. By induction, we can assume that $\ell=1$ and $V\equiv V_1$ is a closed submanifold of M of codimension $\mathfrak{c}$. By the Tubular Neighborhood Theorem [2, (12.11)] and excision [24, Corollary 4.6.5],

$$H^i(M,M';\mathbb{Z})\approx H^i(\mathcal{N}_XV,\mathcal{N}_XV-V;\mathbb{Z}) \qquad \forall\, i\in\mathbb{Z}.$$

Since $H_i(\mathbb{R}^{\mathfrak{c}},\mathbb{R}^{\mathfrak{c}-1})=0$ for $i<\mathfrak{c}$,

$$H^i(M,M';\mathbb{Z})\approx H^i(\mathcal{N}_XV,\mathcal{N}_XV-V;\mathbb{Z})=0 \qquad \forall\, i\le\mathfrak{c}-1$$

by Lemma 5.3. By the cohomology long exact sequence for the pair (M,M'), the sequences

$$0\longrightarrow H^i(M;\mathbb{Z})\longrightarrow H^i(M';\mathbb{Z})\longrightarrow 0,\ i\le\mathfrak{c}-2, \quad 0\longrightarrow H^{\mathfrak{c}-1}(M;\mathbb{Z})\longrightarrow H^{\mathfrak{c}-1}(M';\mathbb{Z}), \qquad (5.9)$$

where the second arrows are the restriction homomorphisms, are thus exact.

(1) Suppose $\mathfrak{c}\ge 2$. If $\Phi,\Phi'\colon L\longrightarrow\mathbb{C}$ are trivializations of L, there exists a smooth map $f\colon M\longrightarrow\mathbb{C}^*$ such that

$$\Phi'(v)=f\big(\pi(x)\big)\Phi(v) \qquad \forall\, v\in L.$$

If $\Phi|_M$ and $\Phi'|_{M'}$ are homotopic, then $f|_{M'}$ is homotopic to the constant map. By the injectivity of (3.33) with $Y=M'$, $f|_{M'}$ then corresponds to the trivial element of $H^1(M';\mathbb{Z})$. By the H^1 case of (5.9), f corresponds to the trivial element of $H^1(M;\mathbb{Z})$ and so is homotopic to the identity. Thus, Φ and Φ' are homotopic as trivializations of L over M.

(2) Suppose $\mathfrak{c}\ge 3$ and $\Phi: L|_{M'}\longrightarrow\mathbb{C}$ is a trivialization of $L|_{M'}$. By the H^2 case of (5.9) and $c_1(L)|_{M'}=0$, $c_1(L)=0$. Therefore, there exists a trivialization $\Phi'\colon L\longrightarrow\mathbb{C}$ of L over M. Since Φ and $\Phi'|_{M'}$ are trivializations of L over M', there exists a smooth map $f\colon M'\longrightarrow\mathbb{C}^*$ such that

$$\Phi'(v)\equiv f\big(\pi(v)\big)\Phi(v) \qquad \forall\, v\in L|_{M'}. \qquad (5.10)$$

By the bijectivity of (3.33) and the $i=1$ case of (5.9), there exists a smooth map $g\colon M\longrightarrow\mathbb{C}^*$ such that $g|_{M'}$ is homotopic to f. Define a trivialization of L by

$$\Phi''\colon L\longrightarrow\mathbb{C},\qquad \Phi''(v)=g\big(\pi(v)\big)^{-1}\Phi'(v).$$

By (5.10), $\Phi''|_{M'}$ is homotopic to Φ. □

5.2 Equivalence of input and output trivializations

We now show that the complex line bundle (2.14) and its trivialization Φ used in the construction of Section 4 extend over a neighborhood of $\mathcal{Z}'_0\equiv\iota_{\mathcal{C},\varepsilon}(X_\emptyset)$ in the space $\mathcal{Z}'$ constructed in Section 4.2. Furthermore, these extensions can be chosen to lie in the homotopy class of trivializations determined as in Proposition 5.1 by the nearly regular symplectic fibration $(\mathcal{Z},\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)},\pi_{\mathcal{C},\varepsilon})$ constructed in Section 4. The proof of Propositions 5.5 below makes use of straightforward, though somewhat technical, Lemmas 5.6 and 5.7; they are deferred to the end of this section.

Proposition 5.5. *Suppose* $\mathbf{X}$ *is as in Theorem 2.8,* Φ *is a trivialization of the complex line bundle* $\mathcal{O}_{X_\partial}(X_\emptyset)$ *as in the first paragraph of Section 4,* ε *and* $\mathcal{C}$ *are* $\mathbb{R}^+$*-valued functions on* $X_\emptyset$ *satisfying the conditions of Propositions 4.4, and* $(\mathcal{Z},\widetilde{\omega}_{\mathcal{C}}^{(\varepsilon)},\pi_{\mathcal{C},\varepsilon})$ *is the corresponding one-parameter family of smoothings. Then there exists a complex line bundle* $\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)$ *with the canonical section* $s_\emptyset$ *and a trivialization* $\widetilde{\Phi}_{\mathcal{C},\varepsilon}$ *such that the associated embedding (4.33) induces an isomorphism*

$$\mathrm{d}\iota_{\mathcal{C},\varepsilon}|_{X_\partial}\colon\mathcal{O}_{X_\partial}(X_\emptyset)\longrightarrow\iota_{\mathcal{C},\varepsilon}^*\mathcal{O}_{\mathcal{Z}}(\mathcal{Z}_0)|_{X_\partial}\tag{5.11}$$

of complex line bundles over X_∂ *and*

$$\widetilde{\Phi}_{\mathcal{C},\varepsilon}\circ s_\emptyset=\pi_{\mathcal{C},\varepsilon}\colon\mathcal{Z}\longrightarrow\mathbb{C},\qquad \widetilde{\Phi}_{\mathcal{C},\varepsilon}\circ\mathrm{d}\iota_{\mathcal{C},\varepsilon}|_{X_\partial}=\mathcal{C}\varepsilon^{-1}\Phi\colon\mathcal{O}_{X_\partial}(X_\emptyset)\longrightarrow\mathbb{C}\,.\tag{5.12}$$

We continue with the notation and setup of Section 4. For $i\in[N]$, let $X_i^\circ\subset X_i$ be as in (4.24), $\mathcal{Z}'_i\subset\mathcal{Z}'$ be the image of X_i under the map $\iota_{\mathcal{C},\varepsilon;i}$ in (4.32), and

$$\pi_i=\pi_{\{i\}}\colon\mathcal{N}_{X_\emptyset}X_{\{i\}}=\mathcal{N}X_i\equiv X_i\times\mathbb{C}\longrightarrow X_i$$

be the projection to the first coordinate. Denote by $\rho_{\{i\};i}$ and $\alpha_{\{i\};i}$ the pullbacks of the function r^2 and the 1-form $\mathrm{d}\theta$, respectively, by the projection $\mathcal{N}X_i\longrightarrow\mathbb{C}$.

If in addition $i\in I\subset[N]$, let

$$X_{I;i}=\mathcal{Z}_I\cap\mathcal{N}_{I;i}\subset\mathrm{Dom}(\Psi_{I;i}^{(\varepsilon)})\big|_{X_I^\circ}\subset\mathcal{N}_{I;i}\qquad\text{and}\qquad X_{i;I}=\Psi_{I;i}^{(\varepsilon)}\big(X_{I;i}\big)\subset X_i$$

be as in (4.27). Denote by

$$\pi_{I;i}\colon\mathcal{N}_{I;i}\longrightarrow X_I\qquad\text{and}\qquad\mathrm{pr}_{I;i}\colon\mathcal{N}X_I=\pi_{I;i}^*\mathcal{N}_{X_{I-i}}X_I\longrightarrow\mathcal{N}_{I;i}$$

the projection maps (if $I=\{i\}$, $\mathrm{pr}_{I;i}=\pi_i$). Let $\mathfrak{i}_{I;i}$ be the complex structure on the oriented rank 2 vector bundle $\mathcal{N}_{X_{I-i}}X_I$ determined by $\rho_{I;i}^{\mathbb{R}}$ if $|I|\ge2$ and the standard complex structure if $|I|=1$.

For each $i\in[I]$, we will construct a complex line bundle and a smooth map,

$$\pi_{\mathcal{N}\mathcal{Z}_i}\colon\mathcal{N}\mathcal{Z}_i\longrightarrow X_i\qquad\text{and}\qquad\Psi'_i\colon\mathcal{N}'\mathcal{Z}_i\longrightarrow\mathcal{Z}',$$

respectively, so that the latter is a diffeomorphism from a neighborhood $\mathcal{N}'\mathcal{Z}_i$ of X_i in $\mathcal{N}\mathcal{Z}_i$ onto an open neighborhood of $\mathcal{Z}'_i$ in $\mathcal{Z}'$ and restricts to $\iota_{\mathcal{C},\varepsilon;i}$ over X_i. Under the identification of $\mathcal{N}\mathcal{Z}_i$ with the vertical tangent subbundle of $T\mathcal{N}\mathcal{Z}_i|_{X_i}$, the homomorphism

$$\begin{aligned}\Theta_i\colon \mathcal{N}\mathcal{Z}_i \longrightarrow \mathcal{N}\iota_{\mathcal{C},\varepsilon;i} \equiv \frac{\iota^*_{\mathcal{C},\varepsilon;i}T\mathcal{Z}'}{\mathrm{d}\iota_{\mathcal{C},\varepsilon;i}(TX_i)} = \iota^*_{\mathcal{C},\varepsilon;i}\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i \equiv \iota^*_{\mathcal{C},\varepsilon;i}\frac{T\mathcal{Z}'|_{\mathcal{Z}'_i}}{T\mathcal{Z}'_i},\\ \Theta_i(v) = \big[\mathrm{d}_{\pi_{\mathcal{N}\mathcal{Z}_i}(v)}\Psi'_i(v)\big],\end{aligned} \tag{5.13}$$

is an isomorphism of rank 2 real vector bundles over X_i such that

$$\Psi_i \equiv \Psi'_i\circ\Theta_i^{-1}\colon \mathcal{N}'_i \equiv \Theta_i\big(\mathcal{N}'\mathcal{Z}_i\big) \longrightarrow \mathcal{Z}' \tag{5.14}$$

is a regularization of $\mathcal{Z}'_i$ in $\mathcal{Z}'$ in the sense of Definition 3.1. We will show that the complex structure $\mathfrak{i}_i$ in the fibers of $\mathcal{N}\mathcal{Z}_i$ is $\Theta_i^*(\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C}}|_{\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i})$-compatible and that

$$\iota_{\mathcal{C},\varepsilon;j}(\Psi_{ij;j}(v)) = \Psi_i\big(\mathrm{d}_{\pi_{ij;j}(v)}\iota_{\mathcal{C},\varepsilon;j}(v)\big) \in \mathcal{Z}'_j \quad \forall\, v\in\mathcal{N}_{ij;j}(2\varepsilon^2),\ i,j\in[N],\ i\neq j. \tag{5.15}$$

The pairs $(\Psi'_i,\mathfrak{i}_i)$ and $(\Psi_i,\Theta_{i*}\mathfrak{i}_i)$ determine complex line bundles $\mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_i)$ and $\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)$, respectively, over a neighborhood $\mathcal{Z}^*$ of $\mathcal{Z}'_0$ in $\mathcal{Z}'$ as in (2.8). The maps (5.13) with $i\in[N]$ induce an isomorphism

$$\Theta\colon \mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_0) \equiv \bigotimes_{i=1}^N \mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_i) \longrightarrow \mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_0) \equiv \bigotimes_{i=1}^N \mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i) \tag{5.16}$$

of complex line bundles over $\mathcal{Z}^*$. By (5.15), the differentials of the maps $\iota_{\mathcal{C},\varepsilon;j}$ induce an isomorphism as in (5.11) with $\mathcal{Z}_0$ replaced by $\mathcal{Z}'_0$. We will describe the line bundle $\mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_0)$ explicitly over trivializing open sets and construct a section $s'_\emptyset$ and a trivialization $\widetilde{\Phi}'_{\mathcal{C},\varepsilon}$ of this bundle so that

$$\widetilde{\Phi}'_{\mathcal{C},\varepsilon}\circ s'_\emptyset = \pi_{\mathcal{C},\varepsilon}\colon \mathcal{Z}^* \longrightarrow \mathbb{C} \tag{5.17}$$

and $s_\emptyset \equiv \Theta\circ s'_\emptyset$ is the canonical section of $\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}_0)$. By (5.17), the trivialization

$$\widetilde{\Phi}_{\mathcal{C},\varepsilon} \equiv \widetilde{\Phi}'_{\mathcal{C},\varepsilon}\circ\Theta^{-1}\colon \mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}_0) \longrightarrow \mathbb{C} \tag{5.18}$$

satisfies the first equality in (5.12). We conclude by establishing the second equality in (5.12).

For all $i\in I\subsetneq I'\subset[N]$,

$$\pi^*_{I';i}\mathcal{N}_{X_{I'-i}}X_{I'}\big|_{\mathcal{N}_{I';i}(4^N)} = \pi^*_{I';i}\mathcal{N}_{I';I'-i}\big|_{\mathcal{N}_{I';i}(4^N)} \subset \pi^*_{I';I}\mathcal{N}_{I';I'-I}\big|_{\mathcal{N}_{I';I}(4^N)} \subset \mathcal{N}X_{I'}$$

under identifications as in the second equality in (3.13). Thus, the bundle homomorphism in (4.12) if $|I|\geq 2$ and in (4.20) if $|I|=1$ with I and I' switched restricts to a bundle homomorphism

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\colon \pi^*_{I';i}\mathcal{N}_{X_{I'-i}}X_{I'}\big|_{\mathcal{N}_{I';i}(4^N)} \longrightarrow \pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I\big|_{\mathcal{N}_{I;i}(4^N)|_{\mathrm{Im}(\Psi^{(\varepsilon)}_{I';I})}}. \tag{5.19}$$

If $|I|\geq 2$ or $I'=\{i\}$, (5.19) is an isomorphism of Hermitian line bundles. If $I=\{i\}$ and $|I'|\geq 2$, (5.19) restricts to an isomorphism of rank 2 vector bundles over the subspace (4.21) with I and I'

interchanged; this subspace contains $X_{I';i}$.

For each $i\in I$, define

$$\mathcal{NZ}_i=\bigg(\bigsqcup_{i\in I\subset[N]}\pi_{I;i}^*\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}}\bigg)\Big/\!\sim,$$

$$\pi_{I';i}^*\mathcal{N}_{X_{I'-i}}X_{I'}\big|_{\Psi_{I';i}^{(\varepsilon)\,-1}(X_{i;I}\cap X_{i;I'})}\ni v\sim\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}(v)\in\pi_{I;i}^*\mathcal{N}_{X_{I-i}}X_I\big|_{\Psi_{I;i}^{(\varepsilon)\,-1}(X_{i;I}\cap X_{i;I'})}\quad\forall\, i\in I\subset I'\subset[N].$$

By the first statement in (4.25) and (4.23), $\sim$ is an equivalence relation. By the first statement in (4.14), the map

$$\pi_{\mathcal{NZ}_i}\colon\mathcal{NZ}_i\longrightarrow X_i=\bigcup_{i\in I\subset[N]}\Psi_{I;i}^{(\varepsilon)}(X_{I;i}),\tag{5.20}$$
$$\pi_{\mathcal{NZ}_i}\big([v]\big)=\Psi_{I;i}^{(\varepsilon)}\big(\mathrm{pr}_{I;i}(v)\big)\quad\forall\, v\in\pi_{I;i}^*\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}},\ i\in I\subset[N],$$

is well-defined and determines a smooth rank 2 vector bundle. By Lemma 5.6 below, the complex structures $\mathfrak{i}_{I;i}$ with $i\in I$ induce a complex structure $\mathfrak{i}_i$ on the vector bundle (5.20).

For $i\in I\subset[N]$, define

$$\delta_I\colon X_I^\circ\longrightarrow\mathbb{R}^+,\qquad\delta_I(x)=\begin{cases}2, & \text{if } |I|\ge 2;\\ 2\mathcal{C}(x)^2\widetilde{\varepsilon}_i(x)^2, & \text{if } I=\{i\};\end{cases}$$
$$\mathcal{N}'\mathcal{Z}_{i;I}=\big\{v\in\pi_{I;i}^*\mathcal{N}_{X_{I-i}}X_I|_{X_{I;i}}\colon\rho_{I;i}(v)<\delta_I(\pi_I(v))\big\}.$$

By (4.29) and Proposition 4.1, the map

$$\Psi_i'\colon\mathcal{N}'\mathcal{Z}_i\equiv\bigg(\bigsqcup_{i\in I\subset[N]}\mathcal{N}'\mathcal{Z}_{i;I}\bigg)\Big/\!\sim\longrightarrow\mathcal{Z}'=\bigg(\bigsqcup_{I\in\mathcal{P}^*(N)}\mathcal{Z}_I\bigg)\Big/\!\sim,$$
$$\Psi_i'\big([v]\big)=q(v)\in q(\mathcal{Z}_{I;i})\quad\forall\, v\in\mathcal{N}'\mathcal{Z}_{i;I},\ i\in I\subset[N],\tag{5.21}$$

is well-defined and smooth. By (4.32) and the first statement in (4.14),

$$\Psi_i'\big([v]\big)=\iota_{\mathcal{C},\varepsilon;j}\big(\Psi_{ij;j}^{(\varepsilon)}\big(\mathfrak{D}\Psi_{I;ij}^{(\varepsilon)}(v)\big)\big)\quad\forall\, v\in\mathcal{N}'\mathcal{Z}_{i;I}\big|_{X_{I;i}\cap X_{I;j}},\ i,j\in I\subset[N],\ i\ne j.\tag{5.22}$$

For $I_0\subset I\subset[N]$ with $I_0\ne\emptyset$, let

$$\mathcal{Z}_{I;I_0}^*=\big\{v\in\mathcal{Z}_I\colon\ \rho_{I;i}(v)<\delta_I(\pi_I(v))\ \forall\, i\in I_0,\ \rho_{I;i}(v)\ne 0\ \forall\, i\in I-I_0\big\},$$
$$\pi_{I;I_0}^{\mathcal{O}}\colon\mathcal{O}_{I;I_0}'(\mathcal{Z}_0')=\bigotimes_{i\in I_0}\pi_I^*\mathcal{N}_{X_{I-i}}X_I\big|_{\mathcal{Z}_{I;I_0}^*}\longrightarrow\mathcal{Z}_{I;I_0}^*.$$

If $|I|=1$, then $I_0=I$ and $\mathcal{Z}_{I;I_0}^*=\mathcal{Z}_I$. If $I_0',I\subset I'\subset[N]$ with $I_0'\ne\emptyset$, then

$$\mathcal{Z}_{I';I}\cap\mathcal{Z}_{I';I_0'}^*=\emptyset\ \text{ if } I_0'\not\subset I,\quad\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}\big(\mathcal{Z}_{I';I}\cap\mathcal{Z}_{I';I_0'}^*\cap\mathcal{Z}_{I';I_0}^*\big)=\mathcal{Z}_{I;I'}\cap\mathcal{Z}_{I;I_0'}^*\cap\mathcal{Z}_{I;I_0}^*\ \text{ if } I_0'\subset I.$$

In the second case, the diffeomorphism

$$\mathfrak{D}\Psi_{I';I}^{(\varepsilon)}\colon\mathcal{Z}_{I';I}\cap\mathcal{Z}_{I';I_0'}^*\cap\mathcal{Z}_{I';I_0}^*\longrightarrow\mathcal{Z}_{I;I'}\cap\mathcal{Z}_{I;I_0'}^*\cap\mathcal{Z}_{I;I_0}^*$$

lifts to a bundle isomorphism

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\colon \mathcal{O}'_{I';I'_0}(\mathcal{Z}'_0)\big|_{\mathcal{Z}_{I';I}\cap\mathcal{Z}^*_{I';I'_0}\cap\mathcal{Z}^*_{I';I_0}} \longrightarrow \mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\big|_{\mathcal{Z}_{I;I'}\cap\mathcal{Z}^*_{I;I'_0}\cap\mathcal{Z}^*_{I;I_0}} \qquad \text{s.t.}$$

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\Big((v'_i)_{i\in I'},\bigotimes_{i\in I'_0-I_0}v'_i\otimes\bigotimes_{i\in I'_0\cap I_0}w'_i\Big)=\Big((v_i)_{i\in I},\bigotimes_{i\in I_0-I'_0}v_i\otimes\bigotimes_{i\in I'_0\cap I_0}w_i\Big)\ \text{ if}$$

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big((v'_i)_{i\in I'}\big)=(v_i)_{i\in I},\ \ \mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big((v'_i)_{i\in I'-I},(w'_i)_{i\in I'_0\cap I_0}\big)=(w_i)_{i\in I'_0\cap I_0}\in\mathcal{N}X_I\,.$$

The union $\mathcal{Z}^*$ of the open subsets $q(\mathcal{Z}^*_{I;I_0})\subset\mathcal{Z}'$ is a neighborhood of $\mathcal{Z}'_0$ in $\mathcal{Z}'$. Define

$$\mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_0)=\Bigg(\bigsqcup_{\substack{I_0\subset I\subset[N]\\ I_0\neq\emptyset}}\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\Bigg)\Big/\!\sim,$$

$$\mathcal{O}'_{I';I'_0}(\mathcal{Z}'_0)\big|_{\mathcal{Z}_{I';I}\cap\mathcal{Z}^*_{I';I'_0}\cap\mathcal{Z}^*_{I';I_0}}\ni u\sim\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}(u)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\big|_{\mathcal{Z}_{I;I}\cap\mathcal{Z}^*_{I;I'_0}\cap\mathcal{Z}^*_{I;I_0}}\quad\forall\,I_0,I'_0\subset I\subset I'.$$

By the first statement in (4.25) and (4.23), $\sim$ above is an equivalence relation. Furthermore, the map

$$\pi^{\mathcal{O}}_{\mathcal{Z}^*}\colon\mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_0)\longrightarrow\mathcal{Z}^*,\qquad\pi^{\mathcal{O}}_{\mathcal{Z}^*}\big([u]\big)=q\big(\pi^{\mathcal{O}}_{I;I_0}(u)\big),\tag{5.23}$$

is well-defined and determines a smooth complex line bundle.

For $I\in\mathcal{P}^*(N)$, let $\Phi_{\mathcal{C},\varepsilon;I}$ be as in (4.16) and (4.19). Define a smooth map

$$\widetilde{\Phi}'_{\mathcal{C},\varepsilon}\colon\mathcal{O}'_{\mathcal{Z}^*}(\mathcal{Z}'_0)\longrightarrow\mathbb{C},$$

$$\widetilde{\Phi}'_{\mathcal{C},\varepsilon}\Big(\big[(v_i)_{i\in I},\bigotimes_{i\in I_0}w_i\big]\Big)=\Phi_{\mathcal{C},\varepsilon;I}\big((v_i)_{i\in I-I_0},(w_i)_{i\in I_0}\big)\ \ \forall\,\big((v_i)_{i\in I},\bigotimes_{i\in I_0}w_i\big)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\,.\tag{5.24}$$

By the first statement in (4.25) and (4.22), this map is well-defined. The restriction of $\widetilde{\Phi}'_{\mathcal{C},\varepsilon}$ to each fiber of (5.23) is an isomorphism because

$$v_i\neq0\qquad\forall\,\big((v_i)_{i\in I},\bigotimes_{i\in I_0}w_i\big)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0),\ i\in I-I_0,$$

and (4.3) is an isomorphism of complex line bundles over $X^\circ_I\subset X^\star_I$. Define a smooth section of (5.23) by

$$s'_\emptyset\big(q\big((v_i)_{i\in I}\big)\big)=\big[(v_i)_{i\in I},\bigotimes_{i\in I_0}v_i\big]\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\qquad\forall\,(v_i)_{i\in I}\in\mathcal{Z}^*_{I;I_0},\ I_0\subset I\subset[N],\ I_0\neq\emptyset.\tag{5.25}$$

By (5.24) and (4.31), the section $s'_\emptyset$ and the trivialization $\widetilde{\Phi}'_{\mathcal{C},\varepsilon}$ of (5.23) satisfy (5.17).

For $i\in I\subset[N]$, let

$$\widetilde{\Theta}_{I;i}\colon\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}}\longrightarrow\pi^*_{I;i}\mathcal{N}X_I\big|_{X_{I;i}}\longrightarrow T\mathcal{Z}_I\big|_{X_{I;i}}\tag{5.26}$$

denote the inclusion of $\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I$ as a subbundle of the restriction of the vertical tangent bundle of the total space of the fibration $\pi_I:\mathcal{N}X_I\longrightarrow X_I$. Since the maps (4.12) and (4.20) are

isomorphisms of split vector bundles,

$$\widetilde{\Theta}_{I;i}\big(\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}(v)\big)=\mathrm{d}_{\Psi^{(\varepsilon)}_{I';I}(\pi_{I';i}(v))}\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big(\widetilde{\Theta}_{I';i}(v)\big) \tag{5.27}$$
$$\forall\, v\in\pi^*_{I';i}\mathcal{N}_{X_{I'-i}}X_{I'}\big|_{\Psi^{(\varepsilon)\,-1}_{I';i}(X_{i;I}\cap X_{i;I'})},\ i\in I\subset I'\subset[N].$$

By the first statement of Lemma 5.7 below, the composition

$$\Theta_{I;i}\colon\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}}\longrightarrow T\mathcal{Z}_I\big|_{X_{I;i}}\longrightarrow\frac{T\mathcal{Z}_I|_{X_{I;i}}}{TX_{I;i}}=\mathcal{N}_{\mathcal{Z}_I}X_{I;i} \tag{5.28}$$

of $\widetilde{\Theta}_{I;i}$ with the quotient projection is an isomorphism of vector bundles over $X_{I;i}$. By (5.27), the isomorphisms (5.28) induce an isomorphism

$$\Theta_i\colon\mathcal{N}\mathcal{Z}_i\longrightarrow\mathcal{N}\iota_{\mathcal{C},\varepsilon;i}\equiv\frac{\iota^*_{\mathcal{C},\varepsilon;i}T\mathcal{Z}'}{\mathrm{d}\iota_{\mathcal{C},\varepsilon;I}(TX_i)},$$
$$\Theta_i\big([v]\big)=\big[\Psi^{(\varepsilon)}_{I;i}\big(\mathrm{pr}_{I;i}(v)\big),\mathrm{d}_{\mathrm{pr}_{I;i}(v)}q\big(\widetilde{\Theta}_{I;i}(v)\big)\big]\quad\forall\, v\in\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}},\ i\in I\subset[N], \tag{5.29}$$

of rank 2 real vector bundles over X_i. By (5.21), this isomorphism is described by the second line in (5.13) and thus the map Ψ_i in (5.14) is a regularization of $\mathcal{Z}'_i$ in $\mathcal{Z}'$. By the statement of Lemma 5.7 below, the complex structure $\mathfrak{i}_i$ on the fibers of $\mathcal{N}\mathcal{Z}_i$ is $\Theta_i^*(\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C}}|_{\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i})$-compatible.

The complex line bundles $\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)$ over $\mathcal{Z}^*$ as in (2.8) obtained from the identifications Ψ_i and the complex structures $\Theta_{i*}\mathfrak{i}_i$ are given by

$$\begin{aligned}\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)&=\big(\Psi_i^{-1\,*}\pi^*_{\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i}\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i|_{\Psi'_i(\mathcal{N}'_i)}\sqcup(\mathcal{Z}^*-\mathcal{Z}'_i)\times\mathbb{C}\big)/\!\sim\longrightarrow\mathcal{Z}^*,\\ \Psi_i^{-1\,*}\pi^*_{\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i}\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i|_{\Psi_i(\mathcal{N}'_i)}&\ni\big(\Psi_i(v),v,cv\big)\sim\big(\Psi_i(v),c\big)\in\big(\mathcal{Z}^*-\mathcal{Z}'_i\big)\times\mathbb{C}.\end{aligned} \tag{5.30}$$

The isomorphisms Θ_i with $i\in[N]$ induce an isomorphism Θ as in (5.16). By (5.25) and (5.29), $s_\emptyset\equiv\Theta\circ s'_\emptyset$ is the canonical section of $\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_0)$. We define a trivialization $\Phi_{\mathcal{C},\varepsilon}$ of $\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_0)$ by (5.18).

Let $i\in[N]$. By (4.9) and the last condition in Definition 3.1,

$$\big[\mathrm{d}_{\pi_{ij}(v)}\Psi^{(\varepsilon)}_{ij;j}(v)\big]=\varepsilon\big(\pi_{ij}(v)\big)v\in\mathcal{N}_{X_j}X_{ij}=\frac{TX_j|_{X_{ij}}}{TX_{ij}}\quad\forall\,[v]\in\mathcal{N}_{X_j}X_{ij}\,.$$

By (4.32) and the first statement in (4.14),

$$\iota_{\mathcal{C},\varepsilon;j}\big(\Psi^{(\varepsilon)}_{ij;j}\big(\mathfrak{D}\Psi^{(\varepsilon)}_{I;ij}(v)\big)\big)=q(v)\in\mathcal{Z}'_j\quad\forall\, v\in\mathcal{N}'\mathcal{Z}_{i;I}\big|_{X_{I;i}\cap X_{I;j}},\ i,j\in I\subset[N],\ i\neq j.$$

Combining the last two statements with (5.29), we obtain

$$\Theta_i\big([v]\big)=\varepsilon\big(\pi_{ij}\big(\mathfrak{D}\Psi^{(\varepsilon)}_{I;ij}(v)\big)\big)\,\mathrm{d}_{\pi_{ij}(\mathfrak{D}\Psi^{(\varepsilon)}_{I;ij}(v))}\iota_{\mathcal{C},\varepsilon;j}\big(\mathfrak{D}\Psi^{(\varepsilon)}_{I;ij}(v)\big)\in\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i \tag{5.31}$$
$$\forall\, v\in\pi^*_{I;ij}\mathcal{N}_{X_{I-i}}X_I\big|_{X_{I;i}\cap X_{I;j}},\ i,j\in I\subset[N],\ i\neq j.$$

Along with (5.14), (5.22), and (4.9), this implies (5.15).

Let $j\in[N]-\{i\}$ and $\mathcal{Z}'_{ij}=\iota_{\mathcal{C},\varepsilon}(X_{ij})$. By (5.30) and (5.15),

$$\iota^*_{\mathcal{C},\varepsilon}\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)\big|_{X_j}=\big(\Psi_{ij;j}^{-1\,*}\{\mathrm{d}_{\pi_{ij;j}(v)}\iota_{\mathcal{C},\varepsilon;j}\}^*\pi^*_{\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i}\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i|_{\Psi_{ij;j}(\mathcal{N}_{ij;j}(2\varepsilon^2))}\sqcup(X_j-X_{ij})\times\mathbb{C}\big)/\!\sim,$$
$$\big(\Psi_{ij;j}(v),v,\mathrm{d}\iota_{\mathcal{C},\varepsilon;j}(v),c\,\mathrm{d}\iota_{\mathcal{C},\varepsilon;j}(v)\big)\sim\big(\Psi_{ij;j}(v),c\big).$$

The line bundle $\mathcal{O}_{X_j}(X_{ij})$ constructed using the identification $\Psi_{ij;j}|_{\mathcal{N}_{ij;j}(2\varepsilon^2)}$ is given by

$$\mathcal{O}_{X_j}(X_{ij})=\big(\Psi_{ij;j}^{-1\,*}\pi^*_{ij;j}\mathcal{N}_{ij;j}|_{\Psi_{ij;j}(\mathcal{N}_{ij;j}(2\varepsilon^2))}\sqcup(X_j-X_{ij})\times\mathbb{C}\big)/\!\sim,$$
$$\big(\Psi_{ij;j}(v),v,cv\big)\sim\big(\Psi_{ij;j}(v),c\big).$$

Thus, the isomorphism

$$\mathrm{d}\iota_{\mathcal{C},\varepsilon;j}\colon\mathcal{N}_{ij;j}=\mathcal{N}_{X_j}X_{ij}\longrightarrow\frac{T\mathcal{Z}'_j|_{\mathcal{Z}'_{ij}}}{T\mathcal{Z}'_{ij}}=\frac{T\mathcal{Z}'|_{\mathcal{Z}'_{ij}}}{T\mathcal{Z}'_i|_{\mathcal{Z}'_{ij}}}=\mathcal{N}_{\mathcal{Z}'}\mathcal{Z}'_i$$

induces an isomorphism from $\mathcal{O}_{X_j}(X_{ij})$ to $\iota^*_{\mathcal{C},\varepsilon}\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)|_{X_j}$. If $k\in[N]-\{i\}$, then

$$\mathrm{d}\iota_{\mathcal{C},\varepsilon;j}|_{X_{jk}}=\mathrm{d}\iota_{\mathcal{C},\varepsilon;k}|_{X_{jk}}\colon\mathcal{O}_{X_{jk}}(X_{ijk})=\mathcal{O}_{X_j}(X_{ij})|_{X_{jk}}=\mathcal{O}_{X_k}(X_{ik})|_{X_{jk}}\longrightarrow\iota^*_{\mathcal{C},\varepsilon}\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)|_{X_{ij}}\,.$$

Thus, the differentials of the maps $\iota_{\mathcal{C},\varepsilon;j}$ induce an isomorphism as in (5.11) with $\mathcal{Z}_0$ replaced by $\mathcal{Z}'_0$.

Let $i,j\in[N]$ be distinct. An element of $\mathcal{O}_{X_\partial}(X_\emptyset)|_x$ with $x\in X_{ij}$ is represented by a tensor product

$$w'_i\otimes w'_j\otimes\bigotimes_{k\in I'_0}(x,v'_k,w'_k)\qquad\text{with}$$
$$I'_0\subset[N]-\{i,j\},\ v'_k\in\mathcal{N}_{ijk;ij}(2\varepsilon^2),\ x=\Psi_{ijk;ij}(v'_k),\ (v'_k,w'_k)\in\pi^*_{ijk;ij}\mathcal{N}_{ijk;ij}\,.$$

By (4.8) and the first statement in (3.19), such an element can be written as

$$\big(x=\Psi_{I;ij}(v)=\Psi^{(\varepsilon)}_{I;ij}(\varepsilon(v)^{-1}v),v,\bigotimes_{k\in I_0}w_k\big)\qquad\text{with}$$
$$I_0=\{i,j\}\cup I'_0\subset I,\ v\equiv(v_l)_{l\in I}\in m_{\varepsilon;I}(\mathcal{Z}^*_{I;I_0})\cap\mathcal{N}_{I;ij},\ (v,w_k)\in\pi_I^*\mathcal{N}_{X_{I-k}}X_I,$$
$$w'_k=\mathfrak{D}\Psi_{I;ijk}\big((v_l)_{l\in I-\{i,j,k\}},w_k\big)=\mathfrak{D}\Psi^{(\varepsilon)}_{I;ijk}\big((\varepsilon(v)^{-1}v_l)_{l\in I-\{i,j,k\}},w_k\big)\ \forall\,k\in I_0.$$

With these identifications, the condition (3.27) becomes

$$\Phi\Big(\Psi_{I;ij}(v),v,\bigotimes_{k\in I_0}w_k\Big)=\Phi\big(\Pi_{\mathfrak{R};I}\big((v_k)_{k\in I-I_0},(w_k)_{k\in I_0}\big)\big)\tag{5.32}$$
$$\forall\ \big(v,\bigotimes_{k\in I_0}w_k\big)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\big|_{m_{\varepsilon;I}(\mathcal{Z}^*_{I;I_0})\cap\mathcal{N}_{I;ij}},\ i,j\in I_0\subset I\subset[N].$$

By (5.31), (4.9), and the second condition in (4.7), the isomorphism (5.11) satisfies

$$\mathrm{d}\iota_{\mathcal{C},\varepsilon}|_{X_{ij}}\Big(\big[\Psi_{I;ij}(v),v,\bigotimes_{k\in I_0}w_k\big]\Big)=\Theta\Big(\big[\varepsilon(v)^{-1}v,\bigotimes_{k\in I_0}\big(\varepsilon(v)^{-1}w_k\big)\big]\Big)\in\mathcal{O}_{\mathcal{Z}^*}(\mathcal{Z}'_i)$$
$$\forall\ \big(v,\bigotimes_{k\in I_0}w_k\big)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\big|_{m_{\varepsilon;I}(\mathcal{Z}^*_{I;I_0})\cap\mathcal{N}_{I;ij}},\ i,j\in I_0\subset I\subset[N].$$

Along with (5.18), (5.24), and (4.16), this implies that

$$\widetilde{\Phi}_{\mathcal{C},\varepsilon}\Big(\mathrm{d}\iota_{\mathcal{C},\varepsilon}|_{X_{ij}}\Big(\big[\Phi_{I;ij}(v),v,\bigotimes_{k\in I_0}w_k\big]\Big)\Big)=\mathcal{C}\big(\pi_I(v)\big)\varepsilon\big(\pi_I(v)\big)^{-1}\Phi\big(\Pi_{\mathfrak{R};I}\big((v_k)_{k\in I-I_0},(w_k)_{k\in I_0}\big)\big)$$
$$\forall\ \big(v,\bigotimes_{k\in I_0}w_k\big)\equiv\big((v_k)_{k\in I},\bigotimes_{k\in I_0}w_k\big)\in\mathcal{O}'_{I;I_0}(\mathcal{Z}'_0)\big|_{m_{\varepsilon;I}(\mathcal{Z}^*_{I;I_0})\cap\mathcal{N}_{I;ij}},\ i,j\in I_0\subset I\subset[N].$$

Combining this with (5.32), we obtain the second equality in (5.12).

It remains to establish the two lemmas used above.

Lemma 5.6. *For all $i\in I\subsetneq I'\subset[N]$, the bundle homomorphism (5.19) is $\mathbb{C}$-linear with respect to the complex structures $\pi^*_{I';i}\mathfrak{i}_{I';i}$ and $\pi^*_{I;i}\mathfrak{i}_{I;i}$.*

Proof. If $|I|\ge 2$, then the homomorphism (5.19) is an isomorphism intertwining the Hermitian structures $\pi^*_{I';i}(\rho_{I';i},\alpha_{I';i})$ and $\pi^*_{I;i}(\rho_{I;i},\alpha_{I;i})$. If $I'=\{i\}$, (5.19) is the identity. If $I=\{i\}$ and $|I'|\ge 2$, the homomorphism (5.19) is given by

$$\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}\big((v_j)_{j\in I'-i},v_i\big)=\mathcal{C}\big(\pi_I\big((v_j)_{j\in I'}\big)\big)\varepsilon\big(\pi_I\big((v_j)_{j\in I'-i}\big)\big)^{|I|-1}\Phi\big(\Pi_{\mathfrak{R};I}((v_j)_{j\in I'}\big).$$

Since Φ is a $\mathbb{C}$-linear isomorphism and $\Pi_{\mathfrak{R};I}$ is a $\mathbb{C}$-linear homomorphism in the v_i-input, $\mathfrak{D}\Psi^{(\varepsilon)}_{I';I}$ is a $\mathbb{C}$-linear homomorphism in this case as well. □

Lemma 5.7. *For all $i\in I\subset[N]$, the homomorphism (5.28) is an isomorphism of vector bundles over $X_{I;i}$. Furthermore, the restriction of $\pi^*_{I;i}\mathfrak{i}_{I;i}$ to $\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I|_{X_{I;i}}$ is a $\Theta^*_{I;i}(\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{\mathcal{N}_{\mathcal{Z}_I}X_{I;i}})$-compatible complex structure.*

Proof. The first claim follows from the canonical decomposition

$$T\big(\mathcal{N}X_I\big)\big|_{\mathcal{N}_{I;i}}=T\mathcal{N}_{I;i}\oplus\pi^*_{I;i}\mathcal{N}_{X_{I-i}}X_I \tag{5.33}$$

of vector bundles over $\mathcal{N}_{I;i}$. By (4.36) and (4.37),

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{T\mathcal{Z}_I|_{X_{I;i}}}=\bigg(\pi^*_{I;i}\omega_I+\frac{1}{2}\mathrm{d}\sum_{j\in I-\{i\}}\big(\varepsilon^2\rho_{I;j}\alpha_{I;j}\big)$$
$$+\frac{1}{2}\mathrm{d}\bigg(\bigg(1-\prod_{j\in I-\{i\}}\eta_{I;j}\bigg)\varepsilon^2\rho_{I;i}\alpha_{I;i}+\bigg(\prod_{j\in I-\{i\}}\eta_{I;j}\bigg)\varepsilon^2\Phi^*_{\mathcal{C},\varepsilon;I}\big(r^2\mathrm{d}\theta\big)\bigg)\bigg)\bigg|_{T\mathcal{Z}_I|_{X_{I;i}}}.$$

Since $\rho_{I;i}$ vanishes on $\mathcal{N}_{I;i}$ and $\mathrm{d}\rho_{I;i}$ vanishes on $T(\mathcal{N}X_I)|_{\mathcal{N}_{I;i}}$, Lemma 4.3 implies that there is a smooth $\mathbb{R}^+$-valued function $f_{\mathcal{C},\varepsilon;I}$ on $X^\star_I$ such that

$$\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{T\mathcal{Z}_I|_{X_{I;i}}}=\bigg(\mathrm{pr}^*_{I;I-i}\big(\widehat{\omega}^{(\varepsilon)}_I|_{\mathcal{N}_{I;i}}\big)$$
$$+\frac{\varepsilon^2}{2}\bigg(\bigg(1-\prod_{j\in I-\{i\}}\eta_{I;j}\bigg)+(f_{\mathcal{C},\varepsilon;I}\circ\pi_I)\prod_{j\in I-\{i\}}\eta_{I;j}\rho_{I;j}\bigg)\mathrm{d}\big(\rho_{I;i}\alpha_{I;i}\big)\bigg)\bigg|_{T\mathcal{Z}_I|_{X_{I;i}}} \tag{5.34}$$

if $|I|=2$. The same conclusion holds if $I=\{i\}$.

By (5.34), the image of the inclusion (5.26) is the $\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}$-orthogonal complement of $TX_{I;i}$ in $T\mathcal{Z}_I|_{X_{I;i}}$. Thus,

$$\begin{aligned}\Theta^*_{I;i}(\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}|_{\mathcal{N}_{\mathcal{Z}_I}X_{I;i}}) &= \widetilde{\Theta}^*_{I;i}\widetilde{\omega}^{(\varepsilon)}_{\mathcal{C};I}\\ &= \frac{\varepsilon^2}{2}\bigg(\bigg(1-\prod_{j\in I-\{i\}}\eta_{I;j}\bigg)+(f_{\mathcal{C},\varepsilon;I}\circ\pi_I)\prod_{j\in I-\{i\}}\eta_{I;j}\rho_{I;j}\bigg)\pi^*_{I;i}\mathrm{d}\big(\rho_{I;i}\alpha_{I;i}\big)\end{aligned} \tag{5.35}$$

under the canonical identification of $\mathcal{N}_{X_{I-i}}X_I$ with the vertical tangent bundle of $\mathcal{N}_{X_{I-i}}X_I\longrightarrow X_I$ along X_I. By Definition 3.2(1) and (3.1), the complex structure $\mathfrak{i}_{I;i}$ is compatible with

$$\omega_i\big|_{\mathcal{N}_{X_{I-i}X_I}} = \frac{1}{2}\mathrm{d}\big(\rho_{I;i}\alpha_{I;i}\big)\big|_{\mathcal{N}_{X_{I-i}X_I}} \tag{5.36}$$

under the identification of $\mathcal{N}_{X_{I-i}}X_I$ with the vertical tangent bundle of $T\mathcal{N}_{X_{I-i}}X_I$ along X_I if $|I|\ge 2$. If $|I|=1$, the complex structure $\mathfrak{i}_{I;i}$ is also compatible with (5.36). Along with (5.35), this implies the second claim. □

A Connections in vector bundles

This appendix contains a number of explicit computations involving connections in complex line bundles. All of these computations are straightforward; we include them for the sake of completeness.

Let $\pi\colon L\longrightarrow X$ be a complex line bundle and ζ_L be the radial vector field on the total space of L, i.e.

$$\zeta_L(v)=(v,v)\in\pi^*L=TL^{\mathrm{ver}}\equiv\ker\mathrm{d}\pi\longhookrightarrow TL\qquad\forall\, v\in L.$$

A connection ∇ in L and determines a horizontal tangent subbundle $TL^{\mathrm{hor}}\subset TL$ and a splitting of the exact sequence

$$0\longrightarrow\pi^*L\longrightarrow TL\xrightarrow{\mathrm{d}\pi}\pi^*TX\longrightarrow 0$$

of vector bundles over L, i.e. a splitting

$$TL\approx TL^{\mathrm{ver}}\oplus TL^{\mathrm{hor}}\approx\pi^*L\oplus\pi^*TX \tag{A.1}$$

such that the second isomorphism above is $(\mathrm{id},\mathrm{d}\pi)$; see [27, Lemma 1.1].

If $U\subset X$ is an open subset, a non-vanishing section $\xi\in\Gamma(U;L)$ induces a trivialization $L|_U\approx U\times\mathbb{C}$ and

$$\nabla\xi=\kappa_\xi\xi \tag{A.2}$$

for some $\mathbb{C}$-valued 1-form $\kappa_\xi\in\Gamma(U;T^*X\otimes_{\mathbb{R}}\mathbb{C})$. If $\zeta\in\Gamma(U;L)$ is another non-vanishing section, then $\zeta=f\xi$ for some $f\in C^\infty(U;\mathbb{C}^*)$ and

$$\kappa_\zeta=\kappa_\xi+f^{-1}\mathrm{d}f\,. \tag{A.3}$$

With respect to the trivialization of $L|_U$ determined by ξ,

$$T_{(x,v)}L^{\mathrm{hor}}=\big\{(\dot{x},-\kappa_\xi(\dot{x})v)\colon\ \dot{x}\in T_xX\big\}\subset T_xX\oplus\mathbb{C}\qquad\forall\,(x,v)\in U\times\mathbb{C}\,.$$

This is consistent with (A.3) and shows that the $\mathbb{C}^*$-action on L preserves the splitting (A.1).

If $h\colon Y\longrightarrow X$ is a smooth map, a connection ∇ in L induces a connection ∇^h in the line bundle $h^*L\longrightarrow Y$. If $U\subset X$ is an open subset, ξ is a non-vanishing section of $L|_U$, and κ_ξ is as in (A.2), then $h^*\xi=\xi\circ h$ is a non-vanishing section of h^*L over the open subset $h^{-1}(U)\subset Y$ and

$$\nabla^h(h^*\xi)=\kappa_{h^*\xi}(h^*\xi),\qquad\text{where}\quad \kappa_{h^*\xi}=h^*\kappa_\xi=\kappa_\xi\circ\mathrm{d}h\,.$$

Let $\xi\in\Gamma(U;L)$ be as in (A.2). Denote by $\mathcal{K}$ the $\mathbb{C}$-valued 1-form on $L\text{–}X$ given in the corresponding trivialization of L by

$$\mathcal{K}_{(x,v)}\big(\dot{x},\dot{v}\big)=-\mathfrak{i}\big(\kappa_\xi(\dot{x})+v^{-1}\dot{v}\big)\qquad\forall\,(\dot{x},\dot{v})\in T_{(x,v)}(U\times\mathbb{C}),\ (x,v)\in U\times\mathbb{C}^*. \tag{A.4}$$

By (A.3), $\mathcal{K}$ is well-defined (i.e. independent of the choice of ξ). It is preserved by the $\mathbb{C}^*$-action and satisfies

$$\mathcal{K}\big|_{TL^{\mathrm{hor}}|_{L-X}}=0,\quad \mathcal{K}\big(\zeta_L(v)\big)=-\mathfrak{i},\quad \mathcal{K}\bigg(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathfrak{i}\theta}v\Big|_{\theta=0}\bigg)=\mathcal{K}\big(\mathfrak{i}\zeta_L(v)\big)=1\quad\forall\,v\in L-X. \tag{A.5}$$

The choice of the normalization for $\mathcal{K}$ is motivated by the last property above and by (A.7) below.

Given a Hermitian metric ρ on L, let $SL\subset L$ denote the unit circle bundle. If ∇ is ρ-compatible, ξ is a trivialization of L over an open subset $U\subset X$, $|\xi|=1$, and κ_ξ is as in (A.2), then

$$\begin{aligned}T_{(x,v)}(SL)=\big\{(\dot{x},cv)\colon \dot{x}\in T_xX,\,c\in\mathfrak{i}\mathbb{R}\big\}\subset T_xX\oplus\mathbb{C}\qquad\forall\,(x,v)\in U\times S^1,\\ \kappa_\xi(\dot{x})+\overline{\kappa_\xi(\dot{x})}=0\ \ \forall\,\dot{x}\in TU\,.\end{aligned} \tag{A.6}$$

Thus, κ_ξ take values in $\mathfrak{i}\mathbb{R}$ and the splitting (A.1) restricts to a splitting

$$\begin{aligned}T(SL)\approx T(SL)^{\mathrm{ver}}\oplus T(SL)^{\mathrm{hor}},\qquad\text{where}\\ T(SL)^{\mathrm{ver}}=\ker\mathrm{d}\big\{\pi|_{SL}\big\}=TL^{\mathrm{ver}}\cap T(SL)\,,\qquad T(SL)^{\mathrm{hor}}=TL^{\mathrm{hor}}\big|_{SL}\,.\end{aligned}$$

There is a unique $\mathbb{R}$-valued connection 1-form α on SL such that

$$\ker\alpha=T(SL)^{\mathrm{hor}}\,,\qquad\alpha\bigg(\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{e}^{\mathfrak{i}\theta}v\Big|_{\theta=0}\bigg)=1\quad\forall\,v\in SL.$$

By the first and last statements in (A.5), it is given by the restriction of $\mathcal{K}$ to the tangent bundle of SL. Via the retraction

$$L^\circ\equiv L-V\longrightarrow SL,\qquad v\longrightarrow\frac{v}{|v|}\,,$$

the 1-form α extends to a 1-form on L°; we denote the resulting extension by α as well. By (A.5),

$$\alpha=\mathrm{Re}\,\mathcal{K}\,. \tag{A.7}$$

As in the main part of the paper, we will use the same notation ρ to denote the square of the norm function on L and the Hermitian form on L.

A connection ∇ in a vector bundle $\pi\colon L\longrightarrow X$ determines an extension Ω_∇ of a fiberwise 2-form Ω to a 2-form on the total space of L. If Ω is a fiberwise symplectic form on an oriented vector bundle $\pi\colon L\longrightarrow X$ of rank 2 and (ρ,∇) is an Ω-compatible Hermitian structure on L, then

$$\big\{\iota_{\zeta_L}\Omega_\nabla\big\}(\mathfrak{i}\zeta_L)=\Omega(\zeta_L,\mathfrak{i}\zeta_L)=\rho^{\mathbb{R}}(\zeta_L,\zeta_L)=\rho(\zeta_L,\zeta_L). \tag{A.8}$$

Since $\iota_{\zeta_L}\Omega_\nabla$ vanishes on TL^{hor} and on ζ_L, (A.7), (A.5), and (A.8) give

$$\rho\alpha=\iota_{\zeta_L}\Omega_\nabla\,. \tag{A.9}$$

Lemma A.1. *Suppose $\pi\colon L\longrightarrow X$ is a complex line bundle, (ρ,∇) is a Hermitian structure in L, and α is the connection 1-form on L° determined by (ρ,∇).*

(1) If (ρ',∇') is another Hermitian structure in L, there exist $f\in C^\infty(X;\mathbb{R}^+)$ and an $\mathbb{R}$-valued 1-form $\mu_{\mathfrak{i}\mathbb{R}}$ on X such that

$$\rho'=f^{-2}\rho,\qquad \nabla'=\nabla-f^{-1}\mathrm{d}f+\mathfrak{i}\mu_{\mathfrak{i}\mathbb{R}}\,. \tag{A.10}$$

(2) If ρ' and ∇' are given by (A.10), then (ρ',∇') is a Hermitian structure in L and the connection 1-form on L° determined by (ρ',∇') is given by

$$\alpha'=\alpha+\pi^*\mu_{\mathfrak{i}\mathbb{R}}\,. \tag{A.11}$$

Proof. Since L is a complex line bundle, the first half of (1) is clear. If ∇' is another connection in L, then

$$\nabla'=\nabla+\mu$$

for some $\mathbb{C}$-valued 1-form on X. Along with the compatibility condition on (ρ,∇) and (ρ',∇'), this implies the second half of (1) and the first half of (2). If ρ' and ∇' are given by (A.10) and $\xi\in\Gamma(U;L)$ is a local section with $\rho(\xi)=1$, then $\xi'\equiv f\xi$ is a local section with $\rho'(\xi')=1$ and

$$\kappa'_{\xi'}=\big(\kappa_\xi-f^{-1}\mathrm{d}f+\mathfrak{i}\mu_{\mathfrak{i}\mathbb{R}}\big)+f^{-1}\mathrm{d}f=\kappa_\xi+\mathfrak{i}\mu_{\mathfrak{i}\mathbb{R}}\,.$$

Along with (A.7) and (A.4), this implies the second half of (2). □

If $N\in\mathbb{Z}^+$ and $L_i\longrightarrow X$ is a complex line bundle for each $i=1,\ldots,N$, define

$$\Pi\colon\bigoplus_{i=1}^N L_i\longrightarrow\bigotimes_{i=1}^N L_i\qquad\text{by}\quad(v_1,\ldots,v_N)\longrightarrow v_1\otimes\ldots\otimes v_N\,.$$

Lemma A.2. *Let $N\in\mathbb{Z}^+$ and X be a manifold. For each $i\in[N]$, let $(L_i,\rho_i,\nabla^{(i)})$ be a Hermitian line bundle over X with induced connection 1-form α_i on L_i°. Suppose*

$$(\pi,\Phi)\colon\bigotimes_{i=1}^N(L_i,\rho_i,\nabla^{(i)})\longrightarrow(X\times\mathbb{C},\rho_{\mathbb{C}},\nabla^{\mathbb{C}})$$

is an isomorphism with the trivial Hermitian line bundle,

$$\pi_i\colon L_1^\circ\times_X\ldots\times_X L_N^\circ\longrightarrow L_i^\circ$$

is the projection onto the i-th factor, and $\widetilde\Phi=\Phi\circ\Pi$. Then,

$$\sum_{i=1}^N\pi_i^*\alpha_i=\widetilde\Phi^*\mathrm{d}\theta\in\Gamma\big(L_1^\circ\times_X\ldots\times_X L_N^\circ;T^*(L_1^\circ\times_X\ldots\times_X L_N^\circ)\big)\,.$$

Proof. For each $i \in [N]$, let $\xi_i \in \Gamma(U; L_i)$ be such that $|\xi_i| = 1$. By choosing ξ_N suitably, we can assume that

$$\Phi(\xi_1 \otimes \ldots \otimes \xi_N) = 1\,.$$

This implies that

$$0 = \nabla\big(\xi_1 \otimes \ldots \otimes \xi_N\big) = \Big(\sum_{i=1}^N \kappa_{\xi_i}\Big)\xi_1 \otimes \ldots \otimes \xi_N\,,$$
$$\widetilde{\Phi}(x, v_1, \ldots, v_N) = v_1 \ldots v_N \quad \forall\ (x, v_1, \ldots, v_N) \in U \times \mathbb{C}^N.$$

Combining this with (A.7) and (A.4), we find that

$$\begin{aligned}\sum_{i=1}^N \pi_i^* \alpha_i\big|_{(x,v_1,\ldots,v_N)}\big(\dot{x}, \dot{v}_1, \ldots, \dot{v}_N\big) &= \mathrm{Im} \sum_{i=1}^N \big(\kappa_{\xi_i}(\dot{x}) + v_i^{-1}\dot{v}_i\big) = \mathrm{Im} \sum_{i=1}^N v_i^{-1}\dot{v}_i \\ &= \mathrm{Im}\big(\big\{\mathrm{d}_{(x,v_1,\ldots,v_N)} \ln \widetilde{\Phi}\big\}\big(\dot{x}, \dot{v}_1, \ldots, \dot{v}_N\big)\big) \\ &= \big\{\widetilde{\Phi}^* \mathrm{d}\theta\big\}\big|_{(x,v_1,\ldots,v_N)}\big(\dot{x}, \dot{v}_1, \ldots, \dot{v}_N\big),\end{aligned}$$

as claimed. □

Corollary A.3. *Let N, X, $(L_i, \rho_i, \nabla^{(i)})$, and α_i be as in Lemma A.2. If*

$$(\pi, \Phi)\colon \bigotimes_{i=1}^N L_i \longrightarrow X \times \mathbb{C} \tag{A.12}$$

is an isomorphism of complex line bundles, there exist $f \in C^\infty(X; \mathbb{R}^+)$ and an $\mathbb{R}$-valued 1-form $\mu_{\mathrm{i}\mathbb{R}}$ on X such that

$$(\pi, \Phi)\colon \bigotimes_{i=1}^N (L_i, \rho_i, \nabla^{(i)}) \longrightarrow \big(X \times \mathbb{C}, f^2\rho_{\mathbb{C}}, \nabla^{\mathbb{C}} + f^{-1}\mathrm{d}f - \mathrm{i}\mu_{\mathrm{i}\mathbb{R}}\big) \tag{A.13}$$

is an isomorphism of Hermitian line bundles. If (A.13) is an isomorphism of Hermitian line bundles, then

$$\pi^*\mu_{\mathrm{i}\mathbb{R}} + \sum_{i=1}^N \pi_i^*\alpha_i = \widetilde{\Phi}^*\mathrm{d}\theta \in \Gamma\big(L_1^\circ \times_X \ldots \times_X L_N^\circ; T^*(L_1^\circ \times_X \ldots \times_X L_N^\circ)\big)\,.$$

Proof. The first claim of this corollary follows from Lemma A.1(1). The second claim is obtained by applying Lemma A.2 with

$$(L_1, \rho_1, \nabla^{(1)}) \qquad \text{replaced by} \qquad (L_1, f^{-2}\rho_1, \nabla^{(1)} - f^{-1}\mathrm{d}f + \mathrm{i}\mu_{\mathrm{i}\mathbb{R}})$$

and then using (A.11); see Lemma A.1. □

With N, X, $(L_i, \rho_i, \nabla^{(i)})$, α_i, and π_i as in Lemma A.2, let

$$\pi\colon \mathcal{N} = \bigoplus_{i=1}^N L_i \longrightarrow X\,.$$

A splitting of the exact sequence

$$0 \longrightarrow \pi^*\mathcal{N} \longrightarrow T\mathcal{N} \xrightarrow{\mathrm{d}\pi} \pi^*TX \longrightarrow 0 \tag{A.14}$$

over $L_1^\circ \times_X \ldots \times_X L_N^\circ$ is obtained by taking

$$T\mathcal{N}^{\mathrm{hor}} = \bigcap_{i=1}^N \big(\ker \pi_i^*\alpha_i \cap \ker \pi_i^* \mathrm{d}\rho_i\big) \subset T\mathcal{N}.$$

By (A.5), (A.4), and (A.6),

$$\begin{aligned} T_{(x,v_1,\ldots,v_N)}\mathcal{N}^{\mathrm{hor}} = \big\{(\dot{x}, -\kappa_{\xi_1}^{(1)}(\dot{x})v_1, \ldots, -\kappa_{\xi_N}^{(N)}(\dot{x})v_N) \colon\ \dot{x} \in T_xX\big\} \subset T_xX \times \mathbb{C}^N \\ \forall\ (x, v_1, \ldots, v_N) \in U \times (\mathbb{C}^*)^N \end{aligned} \tag{A.15}$$

in the trivialization induced by local sections $\xi_i \in \Gamma(U; L_i)$ with $|\xi_i| = 1$. The above splitting thus extends to a splitting

$$T\mathcal{N} = T\mathcal{N}^{\mathrm{ver}} \oplus T\mathcal{N}^{\mathrm{hor}} \longrightarrow \mathcal{N}$$

of (A.14) over the entire total space $\mathcal{N}$; the latter restricts to the canonical splitting over $X \subset \mathcal{N}$. Via this splitting, the complex structure $\mathfrak{i}$ on the fibers of π and an almost complex structure J on X induce an almost complex structure $\widetilde{J}$ on the total space of $\mathcal{N}$.

Lemma A.4. *If N, X, $(L_i, \rho_i, \nabla^{(i)})$, α_i, π_i, and $\widetilde{J}$ are as above and Φ is as in (A.12), then there exists a continuous function $\mathcal{C}_\Phi : X \longrightarrow \mathbb{R}^+$ with the following property. For every $\lambda \in \mathbb{C}^*$, $v \in \widetilde{\Phi}^{-1}(\lambda)$, and $\dot{v} \in T_v\widetilde{\Phi}^{-1}(\lambda)$, there exists $\dot{w} \in T_v\mathcal{N}^{\mathrm{ver}}$ such that*

$$\widetilde{J}\dot{v} + \dot{w} \in T_v\widetilde{\Phi}^{-1}(\lambda), \qquad |\dot{w}| \le \mathcal{C}_\Phi\big(\pi(v)\big)|\lambda|^{1/N}\big|\mathrm{d}\pi_v(\dot{v})\big|. \tag{A.16}$$

Proof. Let $v \in \widetilde{\Phi}^{-1}(\lambda)$, $\dot{v} \in T_v\mathcal{N}$, and $x = \pi(v)$. In a trivialization as in (A.15),

$$v = (x, v_1, \ldots, v_N), \qquad \dot{v} = \big(\dot{x}, \dot{v}_1 - \kappa_{\xi_1}^{(1)}(\dot{x})v_1, \ldots, \dot{v}_N - \kappa_{\xi_N}^{(N)}(\dot{x})v_N\big)$$

for some $v_i, \dot{v}_i \in \mathbb{C}$. Furthermore,

$$\widetilde{\Phi}(v) = f(x)v_1 \ldots v_N, \qquad \widetilde{J}\dot{v} = \big(J\dot{x}, \mathfrak{i}\dot{v}_1 - \kappa_{\xi_1}^{(1)}(J\dot{x})v_1, \ldots, \mathfrak{i}\dot{v}_N - \kappa_{\xi_N}^{(N)}(J\dot{x})v_N\big)$$

for some $f \in C^\infty(U; \mathbb{C}^*)$ determined by the trivialization. Thus,

$$\big|\widetilde{\Phi}(v)\big| = \big|f(x)\big| \cdot |v_1| \ldots |v_N|\,. \tag{A.17}$$

If $\lambda \in \mathbb{C}^*$ and $(x, v) \in \widetilde{\Phi}^{-1}(\lambda)$, then $v_i \neq 0$ for all i. By symmetry and (A.17), we can assume that

$$|v_1| \le \big|\widetilde{\Phi}(v)/f(x)\big|^{1/N}. \tag{A.18}$$

Define

$$\dot{w}_1 = \mathfrak{i}\bigg(\frac{\mathrm{d}_x f(\dot{x}) + \mathfrak{i}\,\mathrm{d}_x f(J\dot{x})}{f(x)} - \sum_{i=1}^N \big(\kappa_{\xi_i}^{(i)}(\dot{x}) + \mathfrak{i}\kappa_{\xi_i}^{(i)}(J\dot{x})\big)\bigg)v_1 \in \mathbb{C},$$
$$\dot{w} = (0, \dot{w}_1, 0, \ldots, 0) \in T_x\mathcal{N}.$$

By (A.18), $\dot{w}$ satisfies the second condition in (A.16). If $\dot{v}\in T_v\widetilde{\Phi}^{-1}(\lambda)$, then

$$\frac{\mathrm{d}_x f(\dot{x})}{f(x)}+\sum_{i=1}^N\frac{\dot{v}_i-\kappa_{\xi_i}^{(i)}(\dot{x})v_i}{v_i}=\mathrm{d}_{(x,v)}\widetilde{\Phi}(\dot{v})=0,$$

$$\mathrm{d}_{(x,v)}\widetilde{\Phi}\big(\widetilde{J}\dot{v}+\dot{w}\big)=\frac{\mathrm{d}_x f(J\dot{x})}{f(x)}+\frac{\mathfrak{i}\dot{v}_1-\kappa_{\xi_1}^{(1)}(J\dot{x})v_1+\dot{w}_1}{v_1}+\sum_{i=2}^N\frac{\mathfrak{i}\dot{v}_i-\kappa_{\xi_i}^{(i)}(J\dot{x})v_i}{v_i}=0.$$

Thus, $\dot{w}$ also satisfies the first condition in (A.16). □

Corollary A.5. *Suppose N, X, $(L_i,\rho_i,\nabla^{(i)})$, α_i, π_i, $\widetilde{J}$, and Φ are as in Lemma A.4 and $\widetilde{\omega}$ is a nondegenerate 2-form on a neighborhood $\mathcal{N}'$ of X in $\mathcal{N}$ taming $\widetilde{J}$. For every compact subset K of $\mathcal{N}'$, there exists $r_K\in\mathbb{R}^+$ such that the restriction of $\widetilde{\omega}$ to $T_v\widetilde{\Phi}^{-1}(\lambda)$ is nondegenerate for all $v\in\widetilde{\Phi}^{-1}(\lambda)\cap K$ and $\lambda\in\mathbb{C}^*$ with $|\lambda|<r_K$.*

Proof. Given $\dot{v}\in T_v\widetilde{\Phi}^{-1}(\lambda)$, let $\dot{w}\in T_v\mathcal{N}^{\mathrm{ver}}$ be as in Lemma A.5. Thus,

$$\big|\widetilde{\omega}\big(\dot{v},\widetilde{J}\dot{v}+\dot{w}\big)-\widetilde{\omega}\big(\dot{v},\widetilde{J}\dot{v}\big)\big|\le\mathcal{C}_\Phi\big(\pi(v)\big)|\lambda|^{1/N}\big|\mathrm{d}\pi_v(\dot{v})\big||\dot{v}|\le\mathcal{C}'_\Phi\big(\pi(v)\big)r_K^{1/N}|\dot{v}|^2.$$

Since $\widetilde{J}$ tames $\widetilde{\omega}$ over the compact set K, it follows that $\widetilde{\omega}(\dot{v},\widetilde{J}\dot{v}+\dot{w})$ is nonzero if r_K is sufficiently small. □

B The smoothability criterion revisited

The smoothability condition (2.17) is equivalent to the condition

$$\sum_{i=1}^N c_1\big(\mathcal{O}_{X_i^c}(X_i)\big)\big|_{X_\partial}=0\in H^2\big(X_\partial;\mathbb{Z}\big).$$

Proposition B.1 below provides a different description of the cohomology classes

$$\mathrm{PD}_{X_i^c}\big(X_i\big)\equiv c_1\big(\mathcal{O}_{X_i^c}(X_i)\big)\in H^2(X_i^c;\mathbb{Z}),\qquad i\in[N].$$

It is more conceptual and less useful, but is more intrinsic. It is also more indicative of being an obstruction to the existence of the smoothing, since the singular fiber $X_\emptyset$ of $\pi:\mathcal{Z}\longrightarrow\Delta$ is homologous to a smooth one and the normal bundle to the latter is trivial.

Proposition B.1. *Let $(X_\emptyset,(\omega_i)_{i\in[N]})$ be as SC symplectic variety as in Definition 2.5.*

(1) For each $i\in[N]$ and each connected component $X'_{\partial;i}$ of $X_\partial\cap X_i$, there exists a unique element $\mathrm{PD}_{X_i^c}(X'_{\partial;i})\in H^2(X_i^c;\mathbb{Z})$ such that

$$\mathrm{PD}_{X_i^c}\big(X'_{\partial;i}\big)\big|_{X_j}=\mathrm{PD}_{X_j}\big(X'_{\partial;i}\cap X_j\big)\in H^2(X_j;\mathbb{Z})\qquad\forall\,j\in[N]-\{i\},\tag{B.1}$$

$$\mathrm{PD}_{X_i^c}\big(X'_{\partial;i}\big)|_{X_i^c-X'_{\partial;i}}=0\in H^2(X_i^c-X'_{\partial;i};\mathbb{Z}).\tag{B.2}$$

(2) For each $i\in[N]$,

$$c_1\big(\mathcal{O}_{X_i^c}(X_i)\big)=\sum_{X'_{\partial;i}}\mathrm{PD}_{X_i^c}\big(X'_{\partial;i}\big)\in H^2(X_i^c;\mathbb{Z}),$$

where the sum is taken over the connected components of $X_\partial\cap X_i$.

Proof. For all $i,j\in[N]$ distinct, let $\Psi_{ij;j}$ and $\mathfrak{i}_{ij;j}$ be as in (2.10) and (2.11), respectively. Restricting the construction of (2.13) to each connected component $X'_{\partial;i}$ of $X_\partial\cap X_i$, we obtain a complex line bundle $\mathcal{O}_{X_i^c}(X'_{\partial;i})$ over X_i^c such that

$$c_1\big(\mathcal{O}_{X_i^c}(X'_{\partial;i})\big)\big|_{X_j}=c_1\big(\mathcal{O}_{X_j}(X'_{\partial;i}\cap X_j)\big)=\mathrm{PD}_{X_j}\big(X'_{\partial;i}\cap X_j\big)\qquad\forall\,j\in[N]-\{i\}.$$

Thus, the cohomology class

$$\mathrm{PD}_{X_i^c}\big(X'_{\partial;i}\big)\equiv c_1\big(\mathcal{O}_{X_i^c}(X'_{\partial;i})\big)\in H^2(X_i^c;\mathbb{Z})$$

satisfies (B.1). Since the restriction of $\mathcal{O}_{X_i^c}(X'_{\partial;i})$ to $X_i^c-X'_{\partial;i}$ is a trivial line bundle, it also satisfies (B.2). Along with Lemma B.2 below, this completes the proof of the first claim. It is immediate that

$$\mathcal{O}_{X_i^c}(X_i)\equiv\bigotimes_{X'_{\partial;i}}\mathcal{O}_{X_i^c}(X'_{\partial;i})\longrightarrow X_i^c\,,$$

where the tensor product is taken over the connected components of $X_\partial\cap X_i$. This implies the second claim. □

Lemma B.2. *Let $(X_\emptyset,(\omega_i)_{i\in[N]})$ be as as Proposition B.1 and $i,j\in[N]$ be distinct. If $X'_{\partial;i}\subset X_\partial\cap X_i$ is a connected component such that $X'_{\partial;i}\cap X_j\neq\emptyset$, then the homomorphism*

$$H^2(X_i^c;\mathbb{Z})\longrightarrow H^2(X_i^c-X'_{\partial;i};\mathbb{Z})\oplus H^2(X_j;\mathbb{Z}),\tag{B.3}$$

induced by the restriction homomorphisms, is injective.

Proof. The kernel of the first homomorphism in (B.3) is the image of the restriction homomorphism

$$H^2\big(X_i^c,X_i^c-X'_{\partial;i};\mathbb{Z}\big)\longrightarrow H^2(X_i^c;\mathbb{Z})\,.$$

Thus, it is sufficient to show that the composition

$$H^2\big(X_i^c,X_i^c-X'_{\partial;i};\mathbb{Z}\big)\longrightarrow H^2(X_i^c;\mathbb{Z})\longrightarrow H^2(X_j;\mathbb{Z})\tag{B.4}$$

of the two restriction homomorphisms is injective.

With notation as in (2.10), let

$$D_{X_k}(X'_{\partial;i})=\mathcal{N}'_{ik;i}\big|_{X'_{\partial;i}\cap X_k}\quad\forall\,k\in[N]-i,\qquad D_{X_i^c}(X'_{\partial;i})=\bigcup_{k\in[N]-i}D_{X_k}(X'_{\partial;i})\,.$$

We use the maps $\Psi_{ik;k}$ to identify these disk bundles with neighborhoods of $X'_{\partial;i}\cap X_k$ in X_k and of $X'_{\partial;i}$ in X_i^c. Since these disk bundles are oriented, there is a commutative diagram

$$\begin{array}{ccccc}
H^0(X'_{\partial;i}) & \xrightarrow{\approx} & H^2\big(D_{X_i^c}(X'_{\partial;i}),D_{X_i^c}(X'_{\partial;i})-X'_{\partial;i}\big) & \xleftarrow{\approx} & H^2\big(X_i^c,X_i^c-X'_{\partial;i}\big)\\
\downarrow & & \downarrow & & \downarrow\\
H^0\big(X'_{\partial;i}\cap X_j\big) & \xrightarrow{\approx} & H^2\big(D_{X_j}(X'_{\partial;i}),D_{X_j}(X'_{\partial;i})-X'_{\partial;i}\cap X_j\big) & \xleftarrow{\approx} & H^2\big(X_j,X_j-X'_{\partial;i}\cap X_j\big)
\end{array}$$

where the vertical arrows are the restriction homomorphisms, the right horizontal arrows are the excision isomorphisms [24, Corollary 4.6.5], and the left horizontal arrows are the Thom isomorphisms [17, Theorem 10.4] for the disk bundle $D_{X_i^c}(X'_{\partial;i}) \longrightarrow X'_{\partial;i}$ and its restriction to $X'_{\partial;i} \cap X_j$. They send the unit 1 in $H^0(X'_{\partial;i};\mathbb{Z})$ and $H^0\big(X'_{\partial;i} \cap X_j;\mathbb{Z}\big)$ to the Thom class u_i for $D_{X_i^c}(X'_{\partial;i})$ and its restriction $u_i|_{X'_{\partial;i}\cap X_j}$, respectively. Let

$$u_i'|_{X'_{\partial;i}\cap X_j} \in H^2\big(X_j, X_j - X'_{\partial;i}\cap X_j;\mathbb{Z}\big)$$

denote the element corresponding to $u_i|_{X'_{\partial;i}\cap X_j}$ under the excision isomorphism. By [17, Exercise 11-C], the restriction of $u_i'|_{X'_{\partial;i}\cap X_j}$ to X_j is $\mathrm{PD}_{X_j}(X'_{\partial;i}\cap X_j)$. Since $X'_{\partial;i}\cap X_j$ is a symplectic submanifold of X_j,

$$\big\langle \omega^{n-1}\mathrm{PD}_{X_j}(X'_{\partial;i}\cap X_j), X_j\big\rangle = \big\langle \omega^{n-1}, X'_{\partial;i}\cap X_j\big\rangle \neq 0$$

if $2n = \dim_{\mathbb{R}} X_j$. Thus, $\mathrm{PD}_{X_j}(X'_{\partial;i}\cap X_j) \neq 0$ and the composition

$$H^2\big(X_i^c, X_i^c - X'_{\partial;i};\mathbb{Z}\big) \longrightarrow H^2\big(X_j, X_j - X'_{\partial;i}\cap X_j;\mathbb{Z}\big) \longrightarrow H^2(X_j;\mathbb{Z})$$

of the two restriction homomorphisms is nonzero even after tensoring with $\mathbb{Q}$. Since $H^0(X'_{\partial;i};\mathbb{Z}) = \mathbb{Z}$, it follows that this composition is injective. Therefore, the composition (B.4) is also injective, as needed. □

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References

[1] D. Abramovich and Q. Chen, *Stable logarithmic maps to Deligne-Faltings pairs II*, Asian J. Math. 18 (2014), no. 3, 465-488

[2] T. Bröcker and K. Jänich, *Introduction to Differential Topology*, Cambridge University Press, 1982

[3] Q. Chen, *Stable logarithmic maps to Deligne-Faltings pairs I*, Ann. of Math. 180 (2014), no. 2, 455-521

[4] M. Farajzadeh Tehrani, M. McLean, and A. Zinger, *Normal crossings singularities for symplectic topology*, math/1410.0609v3

[5] M. Farajzadeh Tehrani, M. McLean, and A. Zinger, *Towards a theory of singular symplectic varieties*, in preparation

[6] M. Farajzadeh Tehrani and A. Zinger, *Normal crossings degenerations of symplectic manifolds*, math/1603.07661

[7] M. Farajzadeh Tehrani and A. Zinger, *On the multifold symplectic sum and cut constructions*, in preparation

[8] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. Math. 118 (1983), no. 1, 75–114

[9] R. Gompf, *A new construction of symplectic manifolds*, Ann. of Math. 142 (1995), no. 3, 527–595

[10] M. Gromov, *Partial Differential Relations*, Results in Mathematics and Related Areas (3), Springer-Verlag, 1986

[11] M. Gross and B. Siebert, *Affine manifolds, log structures, and mirror symmetry*, Turkish J. Math. 27 (2003), no. 1, 33-60

[12] M. Gross and B. Siebert, *Logarithmic Gromov-Witten invariants*, J. Amer. Math. Soc. 26 (2013), no. 2, 451–510

[13] E. Lerman, *Symplectic cuts*, Math. Res. Lett. 2 (1995), no. 3, 247–258

[14] J. McCarthy and J. Wolfson, *Symplectic normal connect sum*, Topology 33 (1994), no. 4, 729–764

[15] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, 2nd Ed., Oxford University Press, 1998

[16] J. Milnor, *Singular Points of Complex Hypersurfaces*, Princeton University Press, 1968

[17] J. Milnor and J. Stasheff, *Characteristic Classes*, Annals of Mathematics Studies 76, Princeton University Press 1974

[18] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, 1984

[19] B. Parker, *Exploded fibrations*, Proceedings of Gokova Geometry-Topology Conference 2006 (2007), 52-90

[20] B. Parker, *Gromov-Witten invariants of exploded manifolds*, math/1102.0158

[21] U. Persson, *On degenerations of algebraic surfaces*, Mem. AMS 11 (1977), no. 189

[22] U. Persson and H. Pinkham, *Some examples of nonsmoothable varieties with normal crossings*, Duke Math. J. 50 (1983), no. 2, 477–486

[23] P. Seidel, *Vanishing cycles and mutation*, Proc. 3rd European Congress of Mathematics, Vol. II, Progr. Math. 202 (2001), 65–85

[24] E. Spanier, *Algebraic Topology*, Springer-Verlag, 1981

[25] M. Symington, *New constructions of symplectic four-manifolds*, Ph.D. Thesis, Stanford University, 1996

[26] M. Symington, *A new symplectic surgery: the 3-fold sum*, Topology and its Applications, 88 (1998), no. 1, 27–53

[27] A. Zinger, *Basic Riemannian geometry and Sobolev estimates used in symplectic topology*, math/1012.3980