On the Topology of Real Bundle Pairs over Nodal Symmetric Surfaces

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December 18, 2015

Abstract

We give an alternative argument for the classification of real bundle pairs over smooth symmetric surfaces and extend this classification to nodal symmetric surfaces. We also classify the homotopy classes of automorphisms of real bundle pairs over symmetric surfaces. The two statements together describe the isomorphisms between real bundle pairs over symmetric surfaces up to deformation.

1 Introduction

The study of symmetric surfaces goes back to at least [10]. They have since played important roles in different areas of mathematics, as indicated by [1] and its citations. Real bundle pairs, or Real vector bundles in the sense of [2], over smooth symmetric surfaces are classified in [3]. In this paper, we give an alternative proof of this core result of [3], obtain its analogue for nodal symmetric surfaces, and classify the automorphisms of real bundle pairs over symmetric surfaces. Special cases of the main results of this paper, Theorems 1.1 and 1.2 below, are one of the ingredients in the construction of positive-genus real Gromov-Witten invariants in [8] and in the study of their properties in [9].

An involution on a topological space X is a homeomorphism $\phi\colon X\longrightarrow X$ such that $\phi\circ\phi=\mathrm{id}_X$. A symmetric surface (Σ,σ) is a closed oriented (possibly nodal) surface Σ with an orientation-reversing involution σ. If Σ is smooth, the fixed locus Σ^σ of σ is a disjoint union of circles. In general, Σ^σ consists of isolated points (called E nodes in [11, Section 3.2]) and circles identified at pairs of points (called H nodes in [11, Section 3.2]).

Let (X,ϕ) be a topological space with an involution. A conjugation on a complex vector bundle $V\longrightarrow X$ lifting ϕ is a vector bundle homomorphism $\varphi\colon V\longrightarrow V$ covering ϕ (or equivalently a vector bundle homomorphism $\varphi\colon V\longrightarrow \phi^*V$ covering id_X) such that the restriction of φ to each fiber is anti-complex linear and $\varphi\circ\varphi=\mathrm{id}_V$. A real bundle pair $(V,\varphi)\longrightarrow(X,\phi)$ consists of a complex vector bundle $V\longrightarrow X$ and a conjugation φ on V lifting ϕ. For example,

$$(X\times\mathbb{C}^n,\phi\times\mathfrak{c})\longrightarrow(X,\phi),$$

*Supported by ERC grant STEIN-259118

†Partially supported by NSF grant DMS 1500875 and MPIM

where $\mathfrak{c}\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is the standard conjugation on $\mathbb{C}^n$, is a real bundle pair; we call it the trivial rank n real bundle pair over (X,ϕ). For any real bundle pair (V,φ) over (X,ϕ), the fixed locus

$$V^{\varphi} \equiv \big\{v\in V:\ \varphi(v)=v\big\}$$

of φ is a real vector bundle over the fixed locus X^{ϕ} of ϕ with $\mathrm{rk}_{\mathbb{R}}V^{\varphi}=\mathrm{rk}_{\mathbb{C}}V$.

If (V_1,φ_1) and (V_2,φ_2) are real vector bundle pairs over (X,ϕ), an isomorphism

$$\Phi\colon (V_1,\varphi_1) \longrightarrow (V_2,\varphi_2) \tag{1.1}$$

of real bundle pairs over (X,ϕ) is a $\mathbb{C}$-linear isomorphism $\Phi: V_1 \longrightarrow V_2$ covering the identity id_X such that $\Phi\circ\varphi_1=\varphi_2\circ\Phi$. We call two real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (X,ϕ) isomorphic if there exists an isomorphism of real bundle pairs as in (1.1). Our first theorem classifies real bundle pairs over symmetric surfaces up to isomorphism.

Theorem 1.1. *Suppose (Σ,σ) is a (possibly nodal) symmetric surface. Two real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (Σ,σ) are isomorphic if and only if*

$$\mathrm{rk}_{\mathbb{C}}V_1 = \mathrm{rk}_{\mathbb{C}}V_2, \qquad w_1\big(V_1^{\varphi_1}\big) = w_1\big(V_2^{\varphi_2}\big) \in H^1(\Sigma^{\sigma};\mathbb{Z}_2),$$

and $\deg(V_1|_{\Sigma'})=\deg(V_2|_{\Sigma'})$ for each irreducible component $\Sigma'\subset\Sigma$.

Let X be a topological space. We denote by $\mathcal{C}(X;\mathbb{R}^*)$ and $\mathcal{C}(X;\mathbb{C}^*)$ the topological groups of $\mathbb{R}^*$-valued and $\mathbb{C}^*$-valued, respectively, continuous functions on X. For a real vector bundle V over X, let $\mathrm{GL}(V)$ be the topological group of vector bundle isomorphisms of V with itself covering id_X and $\mathrm{SL}(V)\subset\mathrm{GL}(V)$ be the subgroup of isomorphisms ψ so that the induced isomorphism

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\psi\colon \Lambda_{\mathbb{R}}^{\mathrm{top}}V \longrightarrow \Lambda_{\mathbb{R}}^{\mathrm{top}}V$$

is the identity. If V is a line bundle, then $\mathrm{GL}(V)$ is naturally identified with $\mathcal{C}(X;\mathbb{R}^*)$ and $\mathrm{SL}(V)\subset\mathrm{GL}(V)$ is the one-point set consisting of the constant function 1. For an arbitrary real vector bundle V over X and $\psi\in\mathrm{GL}(V)$, we denote by $\det_{\mathbb{R}}\psi$ the continuous function on X corresponding to the isomorphism $\Lambda_{\mathbb{R}}^{\mathrm{top}}\psi$ of $\Lambda_{\mathbb{R}}^{\mathrm{top}}V$.

Let (X,ϕ) be a topological space with an involution. Denote by

$$\mathcal{C}\big(X,\phi;\mathbb{C}^*\big) \subset \mathcal{C}\big(X;\mathbb{C}^*\big)$$

the subgroup of continuous maps f such that $f(\phi(z))=\overline{f(z)}$ for all $z\in X$. The restriction of such a function to the fixed locus $X^{\phi}\subset\Sigma$ takes values in $\mathbb{R}^*$, i.e. gives rise to a homomorphism

$$\mathcal{C}\big(X,\phi;\mathbb{C}^*\big) \longrightarrow \mathcal{C}\big(X;\mathbb{R}^*\big), \qquad f \longrightarrow f\big|_{X^{\phi}}.$$

For a real bundle pair (V,φ) over (X,ϕ), let $\mathrm{GL}(V,\varphi)$ be the topological group of real bundle isomorphisms of (V,φ) with itself over (X,ϕ) and $\mathrm{SL}(V,\varphi)\subset\mathrm{GL}(V,\varphi)$ be the subgroup of isomorphisms Ψ so that the induced isomorphism

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi\colon \Lambda_{\mathbb{C}}^{\mathrm{top}}(V,\varphi) \longrightarrow \Lambda_{\mathbb{C}}^{\mathrm{top}}(V,\varphi)$$

is the identity. If (V,φ) is a rank 1 real bundle pair, $\mathrm{GL}(V,\varphi)$ is naturally identified with $\mathcal{C}(X,\phi;\mathbb{C}^*)$ and $\mathrm{SL}(V,\varphi)$ is the one-point set consisting of the constant function 1. For an arbitrary real vector bundle pair (V,φ) and $\Psi\in\mathrm{GL}(V,\varphi)$, we denote by $\det_{\mathbb{C}}\Psi$ the element of $\mathcal{C}(X,\phi;\mathbb{C}^*)$ corresponding to the isomorphism $\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi$ of $\Lambda_{\mathbb{C}}^{\mathrm{top}}(V,\varphi)$. Let

$$\mathrm{GL}'(V,\varphi) = \big\{(f,\psi)\in\mathcal{C}(X,\phi;\mathbb{C}^*)\times\mathrm{GL}(V^\varphi)\colon\ f|_{X^\phi}=\det_{\mathbb{R}}\psi\big\}.$$

Our second theorem describes the topological components of $\mathrm{GL}(V,\varphi)$ and $\mathrm{SL}(V,\varphi)$ for real bundle pairs over symmetric surfaces.

Theorem 1.2. *Let (Σ,σ) be a (possibly nodal) symmetric surface and (V,φ) be a real bundle pair over (Σ,σ). Then the homomorphisms*

$$\begin{gathered}\mathrm{GL}(V,\varphi)\longrightarrow\mathrm{GL}'(V,\varphi)\longrightarrow\mathcal{C}(\Sigma,\sigma;\mathbb{C}^*),\mathrm{GL}(V^\varphi),\\ \Psi\longrightarrow\big(\det\Psi,\Psi|_{V^\varphi}\big),\qquad (f,\psi)\longrightarrow f,\psi,\end{gathered}\tag{1.2}$$

are surjective. Two automorphisms of (V,φ) lie in the same path component of $\mathrm{GL}(V,\varphi)$ if and only if their images in $\mathcal{C}(\Sigma,\sigma;\mathbb{C}^*)$ and in $\mathrm{GL}(V^\varphi)$ lie in the same path components of the two spaces. Furthermore, every path (f_t,ψ_t) in $\mathrm{GL}'(V,\varphi)$ passing through the images of some $\Psi,\Phi\in\mathrm{GL}(V,\varphi)$ lifts to a path in $\mathrm{GL}(V,\varphi)$ passing through Ψ and Φ. The analogous statements hold for the homomorphism

$$\mathrm{SL}(V,\varphi)\longrightarrow\mathrm{SL}(V^\varphi),\qquad \Psi\longrightarrow\Psi|_{V^\varphi},\tag{1.3}$$

in place of (1.2).

For a smooth symmetric surface (Σ,σ), Theorem 1.1 reduces to [3, Propositions 4.1,4.2]. We give a completely different proof of this result in Section 2; see Proposition 2.1 and its proof. The portion of Theorem 1.2 concerning the surjectivity of the homomorphism (1.3) and its analogue for $\mathrm{GL}(V,\varphi)$ is established in Section 3; see Proposition 3.1. We use this proposition to complete the proof of Theorem 1.1 by induction from the base case of Proposition 2.1 in Section 4. Proposition 5.1 is the crucial step needed for the lifting of homotopies in Theorem 1.2; it is obtained in Section 5. This theorem is then proved in Section 6. Section 7 describes connections of Theorems 1.1 and 1.2 with recent advances in real Gromov-Witten theory made in [7, 8, 9].

The second author would like to thank Max-Planck-Institut für Mathematik for the hospitality during the preparation of this paper.

2 The smooth case of Theorem 1.1

We begin by establishing the smooth case of Theorem 1.1.

Proposition 2.1 ([3, Propositions 4.1,4.2])**.** *Theorem 1.1 holds if (Σ,σ) is a smooth symmetric surface.*

Let (Σ,σ) be a smooth genus g symmetric surface. We denote by $|\sigma|_0\in\mathbb{Z}^{\geqslant 0}$ the number of connected components of Σ^σ; each of them is a circle. Let $\langle\sigma\rangle=0$ if the quotient Σ/σ is orientable, i.e. $\Sigma-\Sigma^\sigma$ is disconnected, and $\langle\sigma\rangle=1$ otherwise. There are $\left\lfloor\frac{3g+4}{2}\right\rfloor$ different topological types of

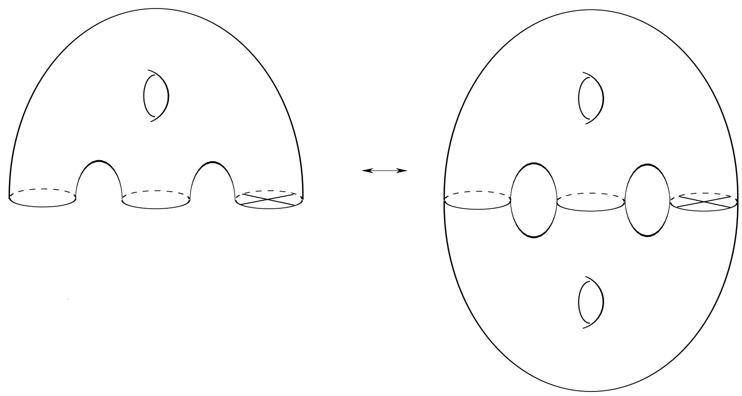

Figure 1: Doubling an oriented sh-surface

orientation-reversing involutions σ on Σ classified by the triples $(g,|\sigma|_0,\langle\sigma\rangle)$; see [12, Corollary 1.1].

An oriented symmetric half-surface (or simply oriented sh-surface) is a pair (Σ^b,c) consisting of an oriented bordered smooth surface Σ^b and an involution $c\colon \partial\Sigma^b \longrightarrow \partial\Sigma^b$ preserving each component and the orientation of $\partial\Sigma^b$. The restriction of c to a boundary component is either the identity or the antipodal map

$$\mathfrak{a}\colon S^1 \longrightarrow S^1, \qquad z \longrightarrow -z,$$

for a suitable identification of $(\partial\Sigma^b)_i$ with $S^1\subset\mathbb{C}$; the latter type of boundary structure is called crosscap in the string theory literature. We define

$$c_i = c|_{(\partial\Sigma^b)_i}, \qquad |c_i| = \begin{cases} 0, & \text{if } c_i = \mathrm{id};\\ 1, & \text{otherwise};\end{cases} \qquad |c|_k = \big|\{(\partial\Sigma^b)_i\subset\Sigma^b\colon\, |c_i|=k\}\big| \quad k=0,1.$$

Thus, $|c|_0$ is the number of standard boundary components of $(\Sigma^b,\partial\Sigma^b)$ and $|c|_1$ is the number of crosscaps. Up to isomorphism, each oriented sh-surface (Σ^b,c) is determined by the genus g of Σ^b, the number $|c|_0$ of ordinary boundary components, and the number $|c|_1$ of crosscaps.

An oriented sh-surface (Σ^b,c) of type (g,m_0,m_1) doubles to a symmetric surface (Σ,σ) of type

$$(g(\Sigma),|\sigma|_0,\langle\sigma\rangle) = \begin{cases} (2g+m_0+m_1-1,m_0,0), & \text{if } m_1=0;\\ (2g+m_0+m_1-1,m_0,1), & \text{if } m_1\neq 0;\end{cases}$$

so that σ restricts to c on the cutting circles (the boundary of Σ^b); see [6, (1.6)] and Figure 1. Since this doubling construction covers all topological types of orientation-reversing involutions σ on Σ, for every symmetric surface (Σ,σ) there is an oriented sh-surface (Σ^b,c) which doubles to (Σ,σ). In general, the topological type of such an sh-surface is not unique.

Let (Σ,σ) be a smooth symmetric surface and (Σ^b,c) be an oriented sh-surface doubling to (Σ,σ). For each i, let

$$(\partial\Sigma^b)_i\times(-2,2) \longrightarrow \Sigma$$

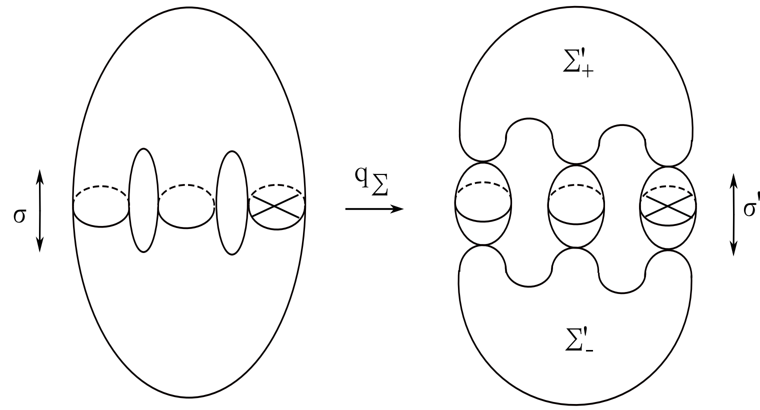

Figure 2: A smooth symmetric surface (Σ,σ) and its associated pinched surface (Σ',σ')

be a parametrization of a neighborhood U_i of $(\partial\Sigma^b)_i$ such that $(\partial\Sigma^b)_i\times 0$ corresponds to $(\partial\Sigma^b)_i$ and

$$\sigma(x,\tau)=(x,-\tau) \qquad \forall\ (x,\tau)\in(\partial\Sigma^b)_i\times(-2,2).$$

We assume that these neighborhoods are disjoint. Let (Σ',σ') be the nodal symmetric surface obtained from (Σ,σ) by collapsing the circles $\tau=\pm1$ in each U_i. Since $(\partial\Sigma^b)_i$ is a separating collection, the surface Σ' consists of two closed surfaces, Σ'_+ and Σ'_-, interchanged by σ' and attached to a collection $\{S^2_i\}$ of σ'-invariant spheres with $(\partial\Sigma^b)_i\subset S^2_i$; see Figure 2. We will call the latter the central components of Σ'. Let

$$q_\Sigma\colon \Sigma\longrightarrow\Sigma'$$

be the quotient map. In particular, $q_\Sigma\circ\sigma=\sigma'\circ q_\Sigma$.

For each cutting circle $(\partial\Sigma^b)_i$ with $|c_i|=0$, let

$$D_i^+=q_\Sigma\big((\partial\Sigma^b)_i\times[0,1]\big), \qquad D_i^-=q_\Sigma\big((\partial\Sigma^b)_i\times[-1,0]\big).$$

Choose a homeomorphism $f_i\colon(\partial\Sigma^b)_i\longrightarrow S^1$ and define a rank 1 real bundle pair $(\gamma_i,\widetilde\sigma'_i)$ over (S^2_i,σ') by

$$\gamma_i\equiv\big(D_i^+\times\mathbb{C}\sqcup D_i^-\times\mathbb{C}\big)/\sim, \qquad D_i^+\times\mathbb{C}\ni(x,0,v)\sim\big(x,0,f_i(x)v\big)\in D_i^-\times\mathbb{C},$$
$$\widetilde\sigma'_i\big([q_\Sigma(x,\tau),v]\big)=\big[q_\Sigma(x,-\tau),\bar v\big].$$

We will call the restriction of a rank n real bundle pair (V',φ') over (Σ',σ') to a central component S^2_i standard if it equals either

$$\big(S^2_i\times\mathbb{C}^n,\sigma'\times\mathfrak{c}\big) \qquad\text{or}\qquad \big(\gamma_i,\widetilde\sigma'_i\big)\oplus\big(S^2_i\times\mathbb{C}^{n-1},\sigma'\times\mathfrak{c}\big);$$

the latter is a possibility only if $|c_i|=0$.

Lemma 2.2. *Let (Σ,σ) be a smooth symmetric surface. For every real bundle pair (V,φ) over (Σ,σ), there exists a real bundle pair (V',φ') over (Σ',σ') such that (V,φ) is isomorphic to $q_\Sigma^*(V',\varphi')$ and the restriction of (V',φ') to each central component (S^2_i,σ_i) is a standard real bundle pair.*

Proof. Let $n = \mathrm{rk}_{\mathbb{C}} V$. If $|c_i| = 0$, define a rank 1 real bundle pair $(\widetilde{\gamma}_i, \widetilde{\sigma}_i)$ over (U_i, σ) by

$$\widetilde{\gamma}_i \equiv \big((\partial\Sigma^b)_i \times [0,2) \times \mathbb{C} \sqcup (\partial\Sigma^b)_i \times (-2,0] \times \mathbb{C}\big)/\sim,$$
$$(\partial\Sigma^b)_i \times [0,2) \times \mathbb{C} \ni (x,0,v) \sim \big(x, 0, f_i(x)v\big) \in (\partial\Sigma^b)_i \times (-2,0] \times \mathbb{C},$$
$$\widetilde{\sigma}_i(x,\tau,v) = \big(x, -\tau, \bar{v}\big).$$

If in addition the restriction of V^φ to $(\partial\Sigma^b)_i$ is not orientable, then

$$V^\varphi \approx \big\{(x,v) \in (\partial\Sigma^b)_i \times \mathbb{C} \colon\ f_i(x)v = \bar{v}\big\} \oplus (\partial\Sigma^b)_i \times \mathbb{R}^{n-1}$$

and thus

$$(V,\varphi)\big|_{U_i} \approx (\widetilde{\gamma}_i, \widetilde{\sigma}_i) \oplus \big(U_i \times \mathbb{C}^{n-1}, \mathrm{id} \times \mathfrak{c}\big)$$

as real bundle pairs over (U_i, σ). If the restriction of V^φ to $(\partial\Sigma^b)_i$ is instead orientable, then $V^\varphi|_{(\partial\Sigma^b)_i} \approx (\partial\Sigma^b)_i \times \mathbb{R}^n$ and thus

$$(V,\varphi)\big|_{(\partial\Sigma^b)_i} \approx \big((\partial\Sigma^b)_i \times \mathbb{C}^n, \mathrm{id} \times \mathfrak{c}\big) = \big((\partial\Sigma^b)_i \times \mathbb{C}^n, \sigma' \times \mathfrak{c}\big). \tag{2.1}$$

as real bundle pairs over $((\partial\Sigma^b)_i, \sigma)$. If $|c_i| = 1$, then

$$(V,\varphi)\big|_{(\partial\Sigma^b)_i} \approx \big((\partial\Sigma^b)_i \times \mathbb{C}^n, \sigma \times \mathfrak{c}\big) \tag{2.2}$$

as real bundle pairs over $((\partial\Sigma^b)_i, \sigma) = ((\partial\Sigma^b)_i, \mathfrak{a})$; see [4, Lemma 2.4]. The isomorphisms (2.1) and (2.2) extend to isomorphisms

$$(V,\varphi)\big|_{U_i} \approx \big(U_i \times \mathbb{C}^n, \sigma \times \mathfrak{c}\big)$$

of real bundle pairs over (U_i, σ). In all three cases, the restriction (V,φ) to the union of small neighborhoods of the pinching circles $\tau = \pm 1$ in each U_i is trivialized as a real bundle pair. Therefore, (V,φ) descends to a real bundle pair (V',φ') over (Σ',σ') such that (V,φ) is isomorphic to $q_\Sigma^*(V',\varphi')$. By construction, the restriction of (V',φ') to each central component (S_i^2, σ_i) is a standard real bundle pair. □

Proof of Proposition 2.1. The necessity of the conditions is clear. By Lemma 2.2, it thus remains to show that real bundle pairs (V_1', φ_1') and (V_2', φ_2') over (Σ', σ') that restrict to the same standard real bundle pair on each central component S_i^2 and to bundles of the same degree over Σ'_+ are isomorphic as real bundle pairs over (Σ', σ'). The identifications of (V_1', φ_1') and (V_2', φ_2') over the central components determine identifications of the restrictions of $V_1'|_{\Sigma'_\pm}$ and $V_2'|_{\Sigma'_\pm}$ at the nodes carried by $\Sigma'_\pm$ that commute with φ_1' and φ_2'. Since $V_1'|_{\Sigma'_+}$ and $V_2'|_{\Sigma'_+}$ are complex bundles of the same degree and rank and $\mathrm{GL}_n\mathbb{C}$ is connected, we can choose an isomorphism Ψ'_+ between them that respects the identifications at the nodal points. Let

$$\Psi'_- = \varphi_2 \circ \Psi'_+ \circ \varphi_1 \colon V_1'|_{\Sigma'_-} \longrightarrow V_2'|_{\Sigma'_-}\,.$$

This isomorphism again respects the identifications at the nodal points. We take

$$\Psi' \colon (V_1', \varphi_1') \longrightarrow (V_2', \varphi_2')$$

to be the identity on the central components of Σ' and $\Psi'_\pm$ on Σ'_+. This is a well-defined isomorphism of real bundle pairs. □

3 Construction of automorphisms

In this section, we establish the surjectivity of the homomorphism (1.3) and its $\mathrm{GL}(V,\varphi)$ version and show that there is no obstruction to lifting paths with basepoints.

Proposition 3.1. *Let (Σ,σ) be a symmetric surface and (V,φ) be a real bundle pair over (Σ,σ). Then the homomorphism (1.3) is surjective. Furthermore, every path ψ_t in $\mathrm{SL}(V^\varphi)$ passing through $\Psi|_{V^\varphi}$ for some $\Psi\in\mathrm{SL}(V,\varphi)$ lifts to a path in $\mathrm{SL}(V,\varphi)$ passing through Ψ. The same is the case with $\mathrm{SL}(V^\varphi)$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V^\varphi)$ and $\mathrm{GL}(V,\varphi)$, respectively. In all cases, the lifts can be chosen to restrict to the identity outside of an arbitrary small neighborhood of Σ^σ.*

Lemma 3.2. *Let (Σ,σ) be a smooth symmetric surface with fixed components $\Sigma_1^\sigma,\ldots,\Sigma_m^\sigma$ and (V,φ) be a real bundle pair over (Σ,σ). For every $i=1,\ldots,m$ and a path ψ_t in $\mathrm{SL}(V^\varphi|_{\Sigma_i^\sigma})$, there exists a path Ψ_t in $\mathrm{SL}(V,\varphi)$ such that each Ψ_t is the identity outside of an arbitrarily small neighborhood of Σ_i^σ and restricts to ψ_t on $V^\varphi|_{\Sigma_i^\sigma}$. The same is the case with $\mathrm{SL}(V^\varphi|_{\Sigma_i^\sigma})$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V^\varphi|_{\Sigma_i^\sigma})$ and $\mathrm{GL}(V,\varphi)$, respectively.*

Proof. Let $n=\mathrm{rk}_{\mathbb{C}}V$ and $\mathbb{I}=[0,1]$. Since every complex vector bundle over Σ_i^σ is trivial,

$$\psi_t\in\mathrm{SL}\big(V^\varphi|_{\Sigma_i^\sigma}\big)\subset\mathrm{SL}\big(V|_{\Sigma_i^\sigma}\big)$$

determines a path of loops in $\mathrm{SL}_n\mathbb{C}$. Since $\pi_1(\mathrm{SL}_n\mathbb{C})$ is trivial, there exists a continuous map

$$H\colon\mathbb{I}^2\longrightarrow\mathrm{SL}\big(V|_{\Sigma_i^\sigma}\big),\quad (t,\tau)\longrightarrow H_{t,\tau},\qquad \text{s.t.}\quad H_{t,0}=\psi_t,\quad H_{t,1}=\mathrm{Id}_{V|_{\Sigma_i^\sigma}}\quad\forall\,t\in\mathbb{I}.$$

Let $\Sigma_i^\sigma\times(-2,2)\longrightarrow\Sigma$ be a parametrization of a neighborhood U of Σ_i^σ such that $\Sigma_i^\sigma\times 0$ corresponds to Σ_i^σ and

$$\sigma(x,\tau)=(x,-\tau)\qquad\forall\,(x,\tau)\in\Sigma_i^\sigma\times(-2,2).$$

Identifying $(V,\varphi)|_U$ with $V|_{\Sigma_i^\sigma}\times(-2,2)$, we define Ψ_t on U by

$$\Psi_t|_{(x,\tau)}=\begin{cases}H_{t,\tau}|_x, & \text{if } \tau\in[0,1];\\ \mathrm{Id}_{V|_{\Sigma_i^\sigma}}, & \text{if } \tau\in[1,2);\\ \varphi\circ H_{t,-\tau}|_x\circ\varphi, & \text{if } \tau\in(-2,0];\end{cases}$$

and extend it as the identity over $\Sigma-U$.

A similar argument applies with $\mathrm{SL}(V^\varphi|_{\Sigma_i^\sigma})$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V^\varphi|_{\Sigma_i^\sigma})$ and $\mathrm{GL}(V,\varphi)$, respectively. A path ψ_t in $\mathrm{GL}(V^\varphi|_{\Sigma_i^\sigma})$ determines a path of loops in

$$\big\{\psi\in\mathrm{GL}_n\mathbb{C}\colon\ \det_{\mathbb{C}}\psi\in\mathbb{R}^*\big\}.$$

These loops are again contractible, and the remainder of the above reasoning still applies. □

Corollary 3.3. *The first statement Proposition 3.1, its analogue as in the third statement, and its sharpening as in the fourth statement hold if (Σ,σ) is a smooth symmetric surface.*

Proof. Let $\Sigma_1^\sigma,\ldots,\Sigma_m^\sigma$ be the connected components of the fixed locus $\Sigma^\sigma\subset\Sigma$. If $\psi\in\mathrm{SL}(V^\varphi)$, let $\Psi_i\in\mathrm{SL}(V,\varphi)$ be an automorphism as in Lemma 3.2 corresponding to the restriction of ψ to $V^\varphi|_{\Sigma_i^\sigma}$ and define

$$\Psi=\Psi_1\circ\ldots\circ\Psi_m\in\mathrm{SL}(V,\varphi).$$

Since each Ψ_i is the identity outside of a small neighborhood of Σ_i^σ, $\Psi|_{V^\varphi}=\psi$.

The same arguments apply with $\mathrm{SL}(V^\varphi)$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V^\varphi)$ and $\mathrm{GL}(V,\varphi)$, respectively. □

A nodal oriented surface Σ is obtained from a smooth oriented surface $\widetilde\Sigma$ by identifying the two points in each of finitely many disjoint pairs of points of $\widetilde\Sigma$; the images of these pairs in Σ are the nodes of Σ. The surface $\widetilde\Sigma$ is called the normalization of Σ; it is unique up to a diffeomorphism respecting the distinguished pairs of points. An orientation-reversing involution σ on Σ lifts to an orientation-reversing involution $\widetilde\sigma$ on $\widetilde\Sigma$. There are three distinct types of nodes a nodal symmetric surface (Σ,σ) may have

(H) a non-isolated real node x_{ij}, i.e. x_{ij} is not an isolated point of Σ^σ and is obtained by identifying distinct points $\widetilde x_i,\widetilde x_j\in\widetilde\Sigma^{\widetilde\sigma}$;

(E) an isolated real node x_i, i.e. x_i is an isolated point of Σ^σ and is obtained by identifying a point $\widetilde x_i^+\in\widetilde\Sigma-\widetilde\Sigma^{\widetilde\sigma}$ with $\widetilde x_i^-=\widetilde\sigma(\widetilde x_i^+)$;

(C) a pair $\{x_{ij}^+,x_{ij}^-\}$ of conjugate nodes, i.e. $x_{ij}^\pm\notin\Sigma^\sigma$ and $x_{ij}^-=\sigma(x_{ij}^+)$, with $x_{ij}^\pm$ obtained by identifying distinct points $\widetilde x_i^\pm,\widetilde x_j^\pm\in\widetilde\Sigma-\widetilde\Sigma^{\widetilde\sigma}$ such that $\widetilde x_i^-=\widetilde\sigma(\widetilde x_i^+)$ and $\widetilde x_j^-=\widetilde\sigma(\widetilde x_j^+)$.

Proof of Proposition 3.1. Let $\widetilde\Sigma\subset\Sigma$ be the normalization of Σ and $S_H,S_E,S_C\subset\widetilde\Sigma$ be the preimages of the H, E, and C nodes of Σ, respectively. Choose disjoint subsets $S_1,S_2\subset\widetilde\Sigma$ consisting of one point from the preimage of each node of Σ. Let $\widetilde V\longrightarrow\widetilde\Sigma$ be a complex vector bundle and

$$\vartheta\colon\widetilde V\big|_{S_1}\longrightarrow\widetilde V\big|_{S_2}$$

be an isomorphism of complex vector bundles such that

$$V=\widetilde V/\!\sim,\quad v\sim\vartheta(v)\ \forall\,v\in\widetilde V\big|_{S_1}.$$

Denote by $\widetilde\varphi$ the lift of φ to $\widetilde V$. Thus, $(\widetilde V,\widetilde\varphi)$ is a real bundle pair over $(\widetilde\Sigma,\widetilde\sigma)$ that descends to the real bundle pair (V,φ). An automorphism $\widetilde\Psi$ of $(\widetilde V,\widetilde\varphi)$ descends to an automorphism of (V,φ) if and only if

$$\widetilde\Psi\circ\vartheta=\vartheta\circ\widetilde\Psi|_{\widetilde V|_{S_1}}. \tag{3.1}$$

An automorphism $\psi\in\mathrm{SL}(V^\varphi)$ induces automorphisms

$$\widetilde\psi_{\mathbb R}\in\mathrm{SL}\big(\widetilde V^{\widetilde\varphi}\big)\qquad\text{s.t.}\quad\widetilde\psi_{\mathbb R}\circ\vartheta=\vartheta\circ\widetilde\psi_{\mathbb R}\qquad\text{on}\ \ \widetilde V\big|_{S_1\cap S_H}, \tag{3.2}$$

$$\widetilde\psi_E\in\mathrm{SL}\big(\widetilde V|_{S_E}\big)\quad\text{s.t.}\quad\widetilde\psi_E\circ\vartheta=\vartheta\circ\widetilde\psi_E,\ \ \widetilde\psi_E\circ\widetilde\varphi=\widetilde\varphi\circ\widetilde\psi_E\quad\text{on}\ \ \widetilde V\big|_{S_1\cap S_E}. \tag{3.3}$$

By Corollary 3.3, there exists $\widetilde{\Psi}'\in\mathrm{SL}(\widetilde{V},\widetilde{\varphi})$ such that $\widetilde{\Psi}'|_{\widetilde{V}^\varphi}=\widetilde{\psi}_{\mathbb{R}}$ and $\widetilde{\Psi}'$ restricts to the identity outside of an arbitrarily small neighborhood of $\widetilde{\Sigma}^{\widetilde{\sigma}}$. By the second condition on $\widetilde{\Psi}'$, we can assume that $\widetilde{\Psi}'$ restricts to the identity over S_C and over a collection of small disjoint neighborhoods $U_{\widetilde{x}}$ of the elements $\widetilde{x}\in S_E$ not intersecting S_C. By (3.2) and the first condition on $\widetilde{\Psi}'$,

$$\widetilde{\Psi}'\circ\vartheta=\vartheta\circ\widetilde{\Psi}'|_{\widetilde{V}|_{S_1}}\,. \tag{3.4}$$

By Lemma 3.4 below, for each $\widetilde{x}\in S_1\cap S_E$ there exists $\widetilde{\Psi}_{\widetilde{x}}\in\mathrm{SL}(\widetilde{V},\widetilde{\varphi})$ such that $\widetilde{\Psi}_{\widetilde{x}}|_{\widetilde{x}}=\widetilde{\psi}_E$ and $\widetilde{\Psi}_{\widetilde{x}}=\mathrm{Id}_{\widetilde{V}}$ outside of $U_{\widetilde{x}}\cup U_{\widetilde{\sigma}(\widetilde{x})}$. By (3.3) and the two conditions on $\widetilde{\Psi}_{\widetilde{x}}$,

$$\widetilde{\Psi}_{\widetilde{x}}\circ\vartheta=\vartheta\circ\widetilde{\Psi}_{\widetilde{x}}|_{\widetilde{V}|_{S_1}}\,. \tag{3.5}$$

Let $\widetilde{\Psi}$ be the composition of the automorphisms $\widetilde{\Psi}'$ and $\widetilde{\Psi}_{\widetilde{x}}$ with $\widetilde{x}\in S_1\cap S_E$. Since the subsets of $\widetilde{\Sigma}$ where these automorphisms differ from the identity are disjoint, $\widetilde{\Psi}$ does not depend on their ordering in this composition and satisfies

$$\widetilde{\Psi}|_{\widetilde{V}^\varphi}=\widetilde{\psi}_{\mathbb{R}}\,,\qquad \widetilde{\Psi}|_{\widetilde{V}|_{S_1\cap S_E}}=\widetilde{\psi}_E\,. \tag{3.6}$$

By (3.1), (3.4), and (3.5), $\widetilde{\Psi}$ descends to an element $\Psi\in\mathrm{SL}(V,\varphi)$. By (3.6), the latter satisfies $\Psi|_{V^\varphi}=\psi$.

Suppose $\Psi\in\mathrm{SL}(V,\varphi)$ and ψ_t is a path in $\mathrm{SL}(V^\varphi)$ such that $\Psi|_{V^\varphi}=\psi_0$. Let $\Phi_t\in\mathrm{SL}(V,\varphi)$ be a path of automorphisms with $\Phi_t|_{V^\varphi}=\psi_t$ constructed as above and define

$$\Psi_t=\Psi\circ\Phi_0^{-1}\circ\Phi_t.$$

Since $\Psi|_{V^\varphi}=\psi_0$ and $\Phi_t|_{V^\varphi}=\psi_t$, $\Psi_t|_{V^\varphi}=\psi_t$.

The same arguments apply with $\mathrm{SL}(V^\varphi)$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V^\varphi)$ and $\mathrm{GL}(V,\varphi)$, respectively. □

Lemma 3.4. *Let (Σ,σ) be a symmetric surface and (V,φ) be a real bundle pair over (Σ,σ). For every $x\in\Sigma-\Sigma^\sigma$, an open neighborhood $U\subset\Sigma$ of x, and a path $\psi_{t;x}\in\mathrm{SL}(V_x)$, there exists a path $\Psi_t\in\mathrm{SL}(V,\varphi)$ such that $\Psi_t|_x=\psi_{t;x}$ and $\Psi_t=\mathrm{Id}$ on $\Sigma-U\cup\sigma(U)$. The same is the case with $\mathrm{SL}(V_x)$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V_x)$ and $\mathrm{GL}(V,\varphi)$, respectively.*

Proof. By shrinking U, we can assume that $U\cap\sigma(U)=\varnothing$ and that $V|_U=U\times V_x$. Let $\rho\colon\Sigma\longrightarrow[0,1]$ be a smooth σ-invariant function such that $\rho(x)=0$ and $\rho=1$ on $\Sigma-U\cup\sigma(U)$. Since $\mathrm{SL}(V_x)$ is connected, there exists a continuous map

$$H\colon\mathbb{I}^2\longrightarrow\mathrm{SL}(V_x),\quad (t,\tau)\longrightarrow H_{t,\tau},\quad \text{s.t.}\quad H_{t,0}=\psi_{t;x},\quad H_{t,1}=\mathrm{Id}_{V_x}\quad\forall\ t\in\mathbb{I}.$$

The path $\Psi_t\in\mathrm{SL}(V,\varphi)$ given by

$$\Psi_t(z)=\begin{cases}H_{t,\rho(z)}, & \text{if } z\in U;\\ \varphi\circ H_{t,\rho(z)}\circ\varphi, & \text{if } z\in\sigma(U);\\ \mathrm{Id}_{V_x}, & \text{if } z\notin U\cup\sigma(U);\end{cases}$$

has the desired properties. The same arguments apply with $\mathrm{SL}(V_x)$ and $\mathrm{SL}(V,\varphi)$ replaced by $\mathrm{GL}(V_x)$ and $\mathrm{GL}(V,\varphi)$, respectively. □

4 Construction of isomorphisms

We will next establish Theorem 1.1 by induction on the number of nodes from the base case of Proposition 2.1. We will need the following lemma.

Lemma 4.1. *Let $(V,\mathfrak{i})$ be a finite-dimensional complex vector space and $A,B: V\longrightarrow V$ be $\mathbb{C}$-antilinear isomorphisms such that $A^2,B^2=\mathrm{Id}_V$. Then there exists a $\mathbb{C}$-linear isomorphism $\psi\colon V\longrightarrow V$ such that $\psi=A\circ\psi\circ B$. If*

$$\big\{\Lambda_{\mathbb{C}}^{\mathrm{top}}A\big\}\circ\big\{\Lambda_{\mathbb{C}}^{\mathrm{top}}B\big\}=\big\{\Lambda_{\mathbb{C}}^{\mathrm{top}}B\big\}\circ\big\{\Lambda_{\mathbb{C}}^{\mathrm{top}}A\big\}\colon \Lambda_{\mathbb{C}}^{\mathrm{top}}V\longrightarrow\Lambda_{\mathbb{C}}^{\mathrm{top}}V, \tag{4.1}$$

then ψ can be chosen so that $\Lambda_{\mathbb{C}}^{\mathrm{top}}\psi=\mathrm{Id}$.

Proof. Since $A^2,B^2=\mathrm{Id}_V$, the isomorphisms A,B are diagonalizable with all eigenvalues ±1. Since A,B are $\mathbb{C}$-antilinear, we can choose $\mathbb{C}$-bases $\{v_i\}$ and $\{w_i\}$ for V such that

$$A(v_i)=v_i,\quad A(\mathfrak{i}v_i)=-\mathfrak{i}v_i,\qquad B(w_i)=w_i,\quad B(\mathfrak{i}w_i)=-\mathfrak{i}w_i.$$

The $\mathbb{C}$-linear isomorphism $\psi\colon V\longrightarrow V$ defined by $\psi(w_i)=v_i$ then has the first desired property.

The automorphisms $\Lambda_{\mathbb{C}}^{\mathrm{top}}A$ and $\Lambda_{\mathbb{C}}^{\mathrm{top}}B$ are $\mathbb{C}$-antilinear and have one eigenvalue of $+1$ and one of -1. If (4.1) holds, the eigenspaces of $\Lambda_{\mathbb{C}}^{\mathrm{top}}A$ and $\Lambda_{\mathbb{C}}^{\mathrm{top}}B$ are the same and so

$$v_1\wedge_{\mathbb{C}}\ldots\wedge_{\mathbb{C}}v_n=r\cdot w_1\wedge_{\mathbb{C}}\ldots\wedge_{\mathbb{C}}w_n\in\Lambda_{\mathbb{C}}^{\mathrm{top}}V$$

for some $r\in\mathbb{R}^*$. Replacing w_1 by rw_1 in the previous paragraph, we obtain an isomorphism ψ that also satisfies the second property. □

Proof of Theorem 1.1. By Proposition 2.1, we can assume that (Σ,σ) is singular and that Theorem 1.1 holds for all symmetric surfaces $(\widetilde\Sigma,\widetilde\sigma)$ with fewer nodes than (Σ,σ).

If (Σ,σ) contains a real node x, i.e. an H or E node as described on page 8, let $(\widetilde\Sigma,\widetilde\sigma)$ be the (possibly nodal) symmetric surface obtained from (Σ,σ) by desingularizing a small neighborhood of x. The preimage of x under the natural projection $\widetilde\Sigma\longrightarrow\Sigma$ is then two distinct points $\widetilde x_1,\widetilde x_2$; (Σ,σ) is obtained from $(\widetilde\Sigma,\widetilde\sigma)$ by identifying $\widetilde x_1$ with $\widetilde x_2$ to form the additional node x. Let $S_1=\{\widetilde x_1\}$ and $S_2=\{\widetilde x_2\}$ in this case. If (Σ,σ) does not contain a real node, let $\{x^+,x^-\}$ be a pair of conjugate nodes and $(\widetilde\Sigma,\widetilde\sigma)$ be the (possibly nodal) symmetric surface obtained from (Σ,σ) by desingularizing small neighborhoods of x^+ and x^-. The preimage of each point $x^\pm$ under the natural projection $\widetilde\Sigma\longrightarrow\Sigma$ is then two distinct points $\widetilde x_1^\pm,\widetilde x_2^\pm$; (Σ,σ) is obtained from $(\widetilde\Sigma,\widetilde\sigma)$ by identifying $\widetilde x_1^+$ with $\widetilde x_2^+$ and $\widetilde x_1^-$ with $\widetilde x_2^-$ to form the additional nodes x^+ and x^-, respectively. Let $S_1=\{\widetilde x_1^+,\widetilde x_1^-\}$ and $S_2=\{\widetilde x_2^+,\widetilde x_2^-\}$ in this case.

Let $\widetilde V_1,\widetilde V_2\longrightarrow\widetilde\Sigma$ be complex vector bundles and

$$\vartheta_1\colon \widetilde V_1\big|_{S_1}\longrightarrow\widetilde V_1\big|_{S_2}\qquad\text{and}\qquad \vartheta_2\colon \widetilde V_2\big|_{S_1}\longrightarrow\widetilde V_2\big|_{S_2}$$

be isomorphisms of complex vector bundles such that

$$V_1=\widetilde V_1/\!\sim,\quad v\sim\vartheta_1(v)\ \forall\,v\in\widetilde V_1\big|_{S_1},\qquad\text{and}\qquad V_2=\widetilde V_2/\!\sim,\quad v\sim\vartheta_2(v)\ \forall\,v\in\widetilde V_2\big|_{S_1}.$$

Denote by $\widetilde{\varphi}_1$ and $\widetilde{\varphi}_2$ the lift of φ_1 to $\widetilde{V}_1$ and the lift of φ_2 to $\widetilde{V}_2$, respectively. Thus, $(\widetilde{V}_1,\widetilde{\varphi}_1)$ and $(\widetilde{V}_2,\widetilde{\varphi}_2)$ are real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$ that descend to the real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (Σ,σ). Furthermore,

$$\vartheta_i(v)=\begin{cases}\widetilde{\varphi}_i(\vartheta_i^{-1}(\widetilde{\varphi}_i(v))), & \text{if } |S_1|=1 \text{ and } x \text{ is } E \text{ node;}\\ \widetilde{\varphi}_i(\vartheta_i(\widetilde{\varphi}_i(v))), & \text{otherwise;}\end{cases} \tag{4.2}$$

for all $v\in\widetilde{V}_i\big|_{S_1}$.

Since $(\widetilde{\Sigma},\widetilde{\sigma})$ satisfies Theorem 1.1, there exists an isomorphism

$$\widetilde{\Phi}\colon(\widetilde{V}_1,\widetilde{\varphi}_1)\longrightarrow(\widetilde{V}_2,\widetilde{\varphi}_2)$$

of real bundle pairs over $(\widetilde{\Sigma},\widetilde{\sigma})$. We show below that there exists $\widetilde{\Psi}\in\mathrm{GL}(\widetilde{V}_1,\widetilde{\varphi}_1)$ so that

$$\widetilde{\Phi}\circ\widetilde{\Psi}\circ\vartheta_1=\vartheta_2\circ\widetilde{\Phi}\circ\widetilde{\Psi}\colon\widetilde{V}_1|_{S_1}\longrightarrow\widetilde{V}_2|_{S_2}\,. \tag{4.3}$$

This implies that $\widetilde{\Phi}\circ\widetilde{\Psi}$ descends to an isomorphism of real bundles as in (1.1).

Suppose $|S_1|=1$ and x is an E node. By the first case in (4.2), the $\mathbb{C}$-linear isomorphisms

$$\widetilde{\Phi}^{-1}\circ\vartheta_2^{-1}\circ\widetilde{\Phi}\circ\widetilde{\varphi}_1=\widetilde{\Phi}^{-1}\circ\vartheta_2^{-1}\circ\widetilde{\varphi}_2\circ\widetilde{\Phi},\,\widetilde{\varphi}_1\circ\vartheta_1\colon\widetilde{V}_1\big|_{\widetilde{x}_1}\longrightarrow\widetilde{V}_1\big|_{\widetilde{x}_1}$$

square to the identity. By Lemma 4.1, there thus exists $\psi\in\mathrm{GL}(\widetilde{V}_1|_{\widetilde{x}_1})$ such that

$$\psi=\widetilde{\Phi}^{-1}\circ\vartheta_2^{-1}\circ\widetilde{\Phi}\circ\widetilde{\varphi}_1\circ\psi\circ\widetilde{\varphi}_1\circ\vartheta_1\colon\widetilde{V}_1|_{\widetilde{x}_1}\longrightarrow\widetilde{V}_1|_{\widetilde{x}_1}\,. \tag{4.4}$$

By Lemma 3.4, there exist $\widetilde{\Psi}\in\mathrm{GL}(\widetilde{V}_1,\widetilde{\varphi}_1)$ and a neighborhood U of $\widetilde{x}_1$ in $\widetilde{\Sigma}$ such that

$$\widetilde{\Psi}|_z=\begin{cases}\psi, & \text{if } z=\widetilde{x}_1;\\ \mathrm{id}, & \text{if } z\notin U\cup\sigma(U);\end{cases}\qquad U\cap\sigma(U)=\varnothing. \tag{4.5}$$

By (4.4) and (4.5), $\widetilde{\Psi}$ satisfies (4.3).

In the two remaining cases, let

$$\psi=\widetilde{\Phi}^{-1}\circ\vartheta_2^{-1}\circ\widetilde{\Phi}\circ\vartheta_1\colon\widetilde{V}_1\big|_{S_1}\longrightarrow\widetilde{V}_1\big|_{S_1}\,.$$

This $\mathbb{C}$-linear automorphism satisfies

$$\widetilde{\Phi}\big(\vartheta_1(v)\big)=\vartheta_2\big(\widetilde{\Phi}(\psi(v))\big)\qquad\forall\ v\in\widetilde{V}_1\big|_{S_1}\,. \tag{4.6}$$

By the second case in (4.2), $\psi\circ\widetilde{\varphi}_1=\widetilde{\varphi}_1\circ\psi$.

Suppose $|S_1|=1$ and x is an H node. If $\widetilde{x}_1$ and $\widetilde{x}_2$ lie on different topological components $\widetilde{\Sigma}_1^{\widetilde{\sigma}},\widetilde{\Sigma}_2^{\widetilde{\sigma}}$ of $\widetilde{\Sigma}^{\widetilde{\sigma}}$, extend ψ to some $\widetilde{\psi}\in\mathrm{GL}(\widetilde{V}_1^{\widetilde{\varphi}_1}|_{\widetilde{\Sigma}_1^{\widetilde{\sigma}}})$. If $\widetilde{x}_1$ and $\widetilde{x}_2$ lie on the same topological component $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ of $\widetilde{\Sigma}^{\widetilde{\sigma}}$, the w_1-assumption applied to either of the two circles in the connected component of Σ^σ containing x implies that ψ is orientation-preserving. Therefore, it can be extended to some

$\widetilde{\psi}\in\mathrm{SL}(\widetilde{V}_1^{\widetilde{\varphi}_1}|_{\widetilde{\Sigma}_1^{\widetilde{\sigma}}})$ such that $\widetilde{\psi}$ is the identity over $\widetilde{x}_2$. In both cases, there exist $\widetilde{\Psi}\in\mathrm{GL}(\widetilde{V}_1,\widetilde{\varphi}_1)$ and a neighborhood U of $\widetilde{\Sigma}_1^{\widetilde{\sigma}}$ in $\widetilde{\Sigma}$ such that

$$\widetilde{\Psi}|_z = \begin{cases} \psi, & \text{if } z=\widetilde{x}_1; \\ \mathrm{id}, & \text{if } z\notin U; \end{cases} \qquad U\cap\big(\widetilde{\Sigma}^{\widetilde{\sigma}}-\widetilde{\Sigma}_1^{\widetilde{\sigma}}\big)=\varnothing; \tag{4.7}$$

see Proposition 3.1. By (4.6) and (4.7), $\widetilde{\Psi}$ satisfies (4.3).

Suppose $|S_1|=2$. Since $\psi\circ\widetilde{\varphi}_1=\widetilde{\varphi}_1\circ\psi$, there exist $\widetilde{\Psi}\in\mathrm{Aut}(\widetilde{V},\widetilde{\varphi}_1)$ and a neighborhood U of S_1 in $\widetilde{\Sigma}$ such that

$$\widetilde{\Psi}|_z = \begin{cases} \psi|_{S_1}, & \text{if } z\in S_1; \\ \mathrm{Id}_{\widetilde{V}_1}, & \text{if } z\notin U; \end{cases} \qquad U\cap S_2=\varnothing; \tag{4.8}$$

see Lemma 3.4. By (4.6) and (4.8), $\widetilde{\Psi}$ satisfies (4.3). □

5 Homotopies between automorphisms

In this section, we establish the main statement needed to lift homotopies in Theorem 1.2.

Proposition 5.1. *Let (Σ,σ) be a symmetric surface, (V,φ) be a real bundle pair over (Σ,σ), and $\Psi\in\mathrm{SL}(V,\varphi)$. If $\Psi|_{V^\varphi}=\mathrm{Id}_{V^\varphi}$, then Ψ is homotopic to Id_V through automorphisms $\Psi_t\in\mathrm{SL}(V,\varphi)$ such that $\Psi_t|_{V^\varphi}=\mathrm{Id}_{V^\varphi}$.*

Proof. Let $n=\mathrm{rk}_{\mathbb{C}}V$. By Lemma 5.2 below, we can assume that $\Psi|_x=\mathrm{Id}_{V_x}$ for every C node $x\in\Sigma$. Let

$$\big(\widetilde{\Sigma},\widetilde{\sigma}\big)\longrightarrow(\Sigma,\sigma), \qquad S_E,S_C\subset\widetilde{\Sigma}, \qquad \text{and} \quad \widetilde{\Psi}\in\mathrm{SL}(\widetilde{V},\widetilde{\varphi})$$

be as in the proof of Proposition 3.1. Let $(\widetilde{\Sigma}^b,\widetilde{c})$ be an oriented sh-surface which doubles to $(\widetilde{\Sigma},\widetilde{\sigma})$ so that the boundary $\partial\widetilde{\Sigma}^b$ of $\widetilde{\Sigma}^b$ is disjoint from $S_E\cup S_C$ and set

$$S^+=\big(S_E\cup S_C\big)\cap\widetilde{\Sigma}^b\,.$$

Since $\Psi|_x=\mathrm{Id}_{V_x}$ for every node $x\in\Sigma$, $\widetilde{\Psi}|_{\widetilde{x}}=\mathrm{Id}_{\widetilde{V}_{\widetilde{x}}}$ for every $\widetilde{x}\in S^+$.

Let $\partial_1\widetilde{\Sigma}^b$ be the union of the boundary components of $(\widetilde{\Sigma}^b,\widetilde{c})$ with $|c_i|=1$. Since $\det\widetilde{\Psi}=1$, the restriction of $\widetilde{\Psi}$ to $(\widetilde{V},\widetilde{\varphi})|_{\partial_1\widetilde{\Sigma}^b}$ is homotopic to the identity through automorphisms $\widetilde{\Phi}_t\in\mathrm{SL}((\widetilde{V},\widetilde{\varphi})|_{\partial_1\widetilde{\Sigma}^b})$; see [4, Lemma 2.4]. We extend $\widetilde{\Phi}_t$ over $\widetilde{\Sigma}^b$ as follows. Let $\partial_1\widetilde{\Sigma}^b\times\mathbb{I}\longrightarrow U$ be a parametrization of a (closed) neighborhood U of $\partial_1\widetilde{\Sigma}^b\subset\widetilde{\Sigma}^b-S^+$ with coordinates (x,s). Identifying $\widetilde{V}|_U$ with $\widetilde{V}|_{\partial_1\widetilde{\Sigma}^b}\times\mathbb{I}$, define

$$\widetilde{\Psi}_t\in\mathrm{SL}\big(\widetilde{V}|_{\widetilde{\Sigma}^b}\big) \qquad \text{by} \quad \widetilde{\Psi}_t|_z=\begin{cases} \widetilde{\Phi}_{(1-s)t}|_x\circ\widetilde{\Psi}|_x^{-1}, & \text{if } z=(x,s)\in U\approx\partial_1\widetilde{\Sigma}^b\times\mathbb{I}; \\ \mathrm{Id}_{\widetilde{V}_z}, & \text{if } z\in\widetilde{\Sigma}^b-U. \end{cases}$$

Since $\widetilde{\Psi}_t|_{(x,1)}$ is the identity for all t, this map is continuous. Moreover, $\widetilde{\Psi}_0|_z$ is the identity for all $z\in\widetilde{\Sigma}^b$ and

$$\widetilde{\Psi}_t|_{(x,0)}=\widetilde{\Phi}_t|_x\circ\widetilde{\Psi}|_x^{-1}$$

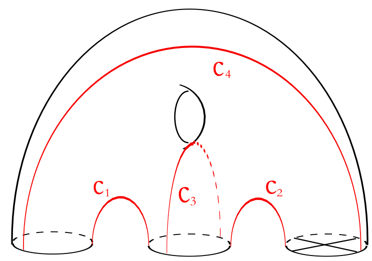

Figure 3: The paths $C_1, \ldots, C_4$ cut $\widetilde{\Sigma}^b$ to a disk.

is a homotopy between the identity and $\widetilde{\Psi}^{-1}$ on $\widetilde{V}|_{\partial_1\widetilde{\Sigma}^b}$. Thus, $\widetilde{\Psi}_t\circ\widetilde{\Psi}$ is a homotopy from $\widetilde{\Psi}$ over $\widetilde{\Sigma}^b$ extending $\widetilde{\Phi}_t$.

Choose embedded non-intersecting paths $\{C_i\}$ in $\widetilde{\Sigma}^b - S^+$ with endpoints on $\partial\widetilde{\Sigma}^b$ which cut $\widetilde{\Sigma}^b$ into a disk D_0^2; see Figure 3. Choose an embedded path $\gamma_{\widetilde{x}}$ in D_0^2 from each point $\widetilde{x}\in S^+$ to ∂D_0^2 so that these curves are pairwise disjoint and cut D_0^2 into another disk D^2. Thus, ∂D^2 is subdivided into arcs each of which was contained in $\partial\widetilde{\Sigma}^b$ before the two cuttings, or had precisely the two endpoints in common with $\partial\widetilde{\Sigma}^b$, or had one endpoint on $\partial\widetilde{\Sigma}^b$ and the other in S^+. Furthermore, every point in S^+ is an endpoint of some arc in ∂D^2.

By the assumption $\Psi|_{V^\varphi} = \mathrm{Id}_{V^\varphi}$ and the first two paragraphs, we may assume that $\widetilde{\Psi}$ is the identity over $\partial\widetilde{\Sigma}^b$ and S^+. Since the restriction of $\widetilde{V}$ to each C_i is trivial, the restriction of $\widetilde{\Psi}$ to C_i defines an element of

$$\pi_1\big(\mathrm{SL}_n\mathbb{C}, I_n\big) \approx \pi_1\big(\mathrm{SU}_n, I_n\big) = 0. \tag{5.1}$$

Thus, we can homotope $\widetilde{\Psi}$ to the identity over C_i while keeping it fixed at the endpoints. Similarly to the second paragraph, this homotopy extends over $\widetilde{\Sigma}^b$ without changing $\widetilde{\Psi}$ over $\partial\widetilde{\Sigma}^b$, over C_j for any $C_j \neq C_i$, or over S^+. In particular, this homotopy descends to $\widetilde{\Sigma}^b$ restricting to the identity over $\partial\widetilde{\Sigma}^b$ and S^+.

By the previous paragraph, we may assume that $\widetilde{\Psi}$ restricts to the identity over ∂D_0^2. Since the restriction of $\widetilde{V}$ to $\gamma_{\widetilde{x}}$ is trivial for each $\widetilde{x}\in S^+$, the restriction of $\widetilde{\Psi}$ to $\gamma_{\widetilde{x}}$ defines an element of

$$\pi_1\big(\mathrm{SL}_n\mathbb{C}, I_n\big) \approx \pi_1\big(\mathrm{SU}_n, I_n\big) = 0.$$

Thus, we can homotope $\widetilde{\Psi}$ to the identity over $\gamma_{\widetilde{x}}$ while keeping it fixed at the endpoints. Similarly to the second paragraph, this homotopy extends over D_0^2 without changing $\widetilde{\Psi}$ over ∂D_0^2 or over $\gamma_{\widetilde{x}'}$ for any $\widetilde{x}'\in S^+$ different from $\widetilde{x}$. In particular, this homotopy descends to $\widetilde{\Sigma}^b$ restricting to the identity over $\partial\widetilde{\Sigma}^b$ and S^+.

By the previous paragraph, we may assume that $\widetilde{\Psi}$ restricts to the identity over the boundary S^1 of D^2. Since every vector bundle over D^2 is trivial and

$$\pi_2\big(\mathrm{SL}_n\mathbb{C}, I_n\big) \approx \pi_2\big(\mathrm{SU}_n, I_n\big) = 0,$$

the map $\widetilde{\Psi}\colon (D^2, S^1) \longrightarrow (\mathrm{SL}_n\mathbb{C}, I_n)$ can be homotoped to the identity as a relative map. Doubling such a homotopy $\widetilde{\Psi}_t$ by the requirement that $\widetilde{\Psi}_t \circ \varphi = \varphi \circ \widetilde{\Psi}_t$, we obtain a homotopy $\widetilde{\Psi}_t$ from $\widetilde{\Psi}$ to $\mathrm{Id}_{\widetilde{V}}$ through automorphisms $\widetilde{\Psi}_t \in \mathrm{SL}(\widetilde{V}, \widetilde{\varphi})$ such that $\widetilde{\Psi}_t$ restricts to the identity over $\widetilde{\Sigma}^{\widetilde{\sigma}}$ and S^+. Thus, they descend to automorphisms $\Psi_t \in \mathrm{SL}(V, \varphi)$ with the required properties. □

Lemma 5.2. *Let (Σ, σ) be a symmetric surface and (V, φ) be a real bundle pair over (Σ, σ). For every $x \in \Sigma - \Sigma^\sigma$, an open neighborhood $U \subset \Sigma$ of x, and $\Psi \in \mathrm{SL}(V, \varphi)$, there exists a path $\Psi_t \in \mathrm{SL}(V, \varphi)$ such that $\Psi_0 = \Psi$, $\Psi_1|_x = \mathrm{Id}_{V_x}$, and $\Psi_t = \Psi$ on $\Sigma - U \cup \sigma(U)$. The same is the case with $\mathrm{SL}(V, \varphi)$ replaced by $\mathrm{GL}(V, \varphi)$.*

Proof. Since $\mathrm{SL}(V_x)$ is connected, there exists a path

$$\psi_{x;t} \in \mathrm{SL}(V_x) \qquad \text{s.t.} \quad \psi_{x;0} = \mathrm{Id}_{V_x}, \quad \psi_{x;1} = \Psi^{-1}|_{V_x}.$$

By Lemma 3.4, there exists a path $\Phi_t \in \mathrm{SL}(V, \varphi)$ such that $\Phi_t|_x = \psi_{x;t}$ and $\Phi_t = \mathrm{Id}$ on $\Sigma - U \cup \sigma(U)$. The path $\Psi_t = \Phi_t \circ \Phi_0^{-1} \circ \Psi$ then has the desired properties. □

6 Classification of automorphisms

By the next lemma, the first composite homomorphism in (1.2) is surjective. We use it to complete the proof of Theorem 1.2 in this section.

Lemma 6.1. *Let (Σ, σ) be a symmetric surface and (V, φ) be a real bundle pair over (Σ, σ). Then the homomorphism*

$$\mathrm{GL}(V, \varphi) \longrightarrow \mathcal{C}(\Sigma, \sigma; \mathbb{C}^*), \qquad \Psi \longrightarrow \det \Psi,$$

is surjective. Furthermore, every path f_t in $\mathcal{C}(\Sigma, \sigma; \mathbb{C}^*)$ passing through $\det \Psi$ for some $\Psi \in \mathrm{GL}(V, \varphi)$ lifts to a path in $\mathrm{GL}(V, \varphi)$ passing through Ψ.

Proof. Let $n = \mathrm{rk}_{\mathbb{C}} V$. By Theorem 1.1, we can assume that

$$(V, \varphi) = \Lambda^{\mathrm{top}}_{\mathbb{C}}(V, \varphi) \oplus \big(\Sigma \times \mathbb{C}^{n-1}, \sigma \times \mathfrak{c}_{\mathbb{C}^{n-1}}\big).$$

If $f \in \mathcal{C}(\Sigma, \sigma; \mathbb{C}^*)$, then

$$\Psi \equiv f\mathrm{Id}_{\Lambda^{\mathrm{top}}_{\mathbb{C}} V} \oplus \mathrm{Id}_{\Sigma \times \mathbb{C}^{n-1}} \colon (V, \varphi) \longrightarrow (V, \varphi)$$

is an element of $\mathrm{GL}(V, \varphi)$ such that $\det \Psi = f$. If

$$\Psi \equiv f\mathrm{Id}_{\Lambda^{\mathrm{top}}_{\mathbb{C}} V} \oplus \Phi$$

is an arbitrarily element of $\mathrm{GL}(V, \varphi)$ and f_t is a path in $\mathcal{C}(\Sigma, \sigma; \mathbb{C}^*)$ such that $\det \Psi = f_0$, then

$$\Psi_t \equiv \frac{f_t}{\det \Phi} \mathrm{Id}_{\Lambda^{\mathrm{top}}_{\mathbb{C}} V} \oplus \Phi$$

is a path in $\mathrm{GL}(V, \varphi)$ such that $\Psi_0 = \Psi$ and $\det \Psi_t = f_t$. □

Proof of Theorem 1.2. By Lemma 6.1 and Proposition 3.1, the compositions

$$\begin{aligned} &\mathrm{GL}(V,\varphi)\longrightarrow \mathrm{GL}'(V,\varphi)\longrightarrow \mathcal{C}(\Sigma,\sigma;\mathbb{C}^*), &&\Psi\longrightarrow \det\Psi,\\ &\mathrm{GL}(V,\varphi)\longrightarrow \mathrm{GL}'(V,\varphi)\longrightarrow \mathrm{GL}(V^{\varphi}), &&\Psi\longrightarrow \Psi|_{V^{\varphi}}, \end{aligned}$$

are surjective. This implies that the projection homomorphisms

$$\mathrm{GL}'(V,\varphi)\longrightarrow \mathcal{C}(\Sigma,\sigma;\mathbb{C}^*), \mathrm{GL}(V^{\varphi})$$

are surjective.

Suppose $\Psi,\Phi\in\mathrm{GL}(V,\varphi)$ are such that $\det\Psi$ and $\det\Phi$ lie in the same path component of $\mathcal{C}(\Sigma,\sigma;\mathbb{C}^*)$ and $\Psi|_{V^{\varphi}}$ and $\Phi|_{V^{\varphi}}$ lie in the same path component of $\mathrm{GL}(V^{\varphi})$. We will show that Ψ and Φ lie in the same path component of $\mathrm{GL}(V,\varphi)$. By the second statement of Lemma 6.1, we may assume that $\det\Psi=\det\Phi$ and thus

$$\Theta\equiv\Phi\circ\Psi^{-1}\in\mathrm{SL}(V,\varphi), \qquad \psi\equiv\Theta|_{V^{\varphi}}\in\mathrm{SL}(V^{\varphi}).$$

Since $\Psi|_{V^{\varphi}}$ and $\Phi|_{V^{\varphi}}$ lie in the same path component of $\mathrm{GL}(V^{\varphi})$, there exists a path ψ_t in $\mathrm{SL}(V^{\varphi})$ from $\psi_0\equiv\mathrm{Id}_{V^{\varphi}}$ to $\psi_1\equiv\psi$. By Proposition 3.1, there exists a path Θ_t in $\mathrm{SL}(V,\varphi)$ such that $\Theta_t|_{V^{\varphi}}=\psi_t$ and $\Theta_1=\Theta$. In particular, Φ and $\Theta_0\circ\Psi$ lie in the same path component of $\mathrm{GL}(V,\varphi)$,

$$\det(\Theta_0\circ\Psi)=\det\Psi, \qquad \text{and} \qquad (\Theta_0\circ\Psi)\big|_{V^{\varphi}}=\Psi|_{V^{\phi}}.$$

By Proposition 5.1, Ψ lies in the same path component of $\mathrm{GL}(V,\varphi)$ as $\Theta_0\circ\Psi$ and Φ.

Suppose $\Psi,\Phi\in\mathrm{GL}(V,\varphi)$ and (f_t,ψ_t) is a path in $\mathrm{GL}'(V,\varphi)$ such that

$$(f_0,\psi_0)=\big(\det\Psi,\Psi|_{V^{\varphi}}\big), \qquad (f_1,\psi_1)=\big(\det\Phi,\Phi|_{V^{\varphi}}\big).$$

By the second statement of Lemma 6.1, there exists a path $\Phi_t\in\mathrm{GL}(V,\varphi)$ such that $\Phi_0=\Psi$ and $\det\Phi_t=f_t$. Let

$$\widehat{\psi}_t=\psi_t\circ\{\Phi_t\}^{-1}|_{V^{\varphi}}\in\mathrm{GL}(V^{\varphi}).$$

Since $\det\psi_t=f_t|_{\Sigma^{\sigma}}$, $\widehat{\psi}_t\in\mathrm{SL}(V^{\varphi})$. Furthermore, $\widehat{\psi}_0=\mathrm{Id}|_{V^{\varphi}}$. By Proposition 3.1, $\widehat{\psi}_t$ thus extends to a path $\widehat{\Psi}_t$ in $\mathrm{SL}(V,\varphi)$ such that $\widehat{\Psi}_0=\mathrm{Id}_V$. Let $\widetilde{\Psi}_t=\widehat{\Psi}_t\circ\Phi_t$. In particular,

$$\widetilde{\Psi}_0=\Psi, \quad \det\widetilde{\Psi}_t=f_t, \quad \widetilde{\Psi}_t\big|_{V^{\varphi}}=\psi_t, \qquad \det\big(\Phi\circ\widetilde{\Psi}_1^{-1}\big)=\mathrm{Id}_V, \quad \big(\Phi\circ\widetilde{\Psi}_1^{-1}\big)|_{V^{\varphi}}=\mathrm{Id}_{V^{\varphi}}.$$

By Proposition 5.1, the last two properties imply that there exists a path Θ_t in $\mathrm{SL}(V,\varphi)$ from Id_V to $\Phi\circ\widetilde{\Psi}_1^{-1}$ such that $\Theta_t|_{V^{\varphi}}=\mathrm{Id}_{V^{\varphi}}$. The path $\Psi_t\equiv\Theta_t\circ\widetilde{\Psi}_t$ in $\mathrm{GL}(V,\varphi)$ runs from Ψ to Φ and lifts (f_t,ψ_t). □

7 Connections with real Gromov-Witten theory

Let X be a topological space. For real vector bundles $V_1,V_2\longrightarrow X$ of the same rank, let $\mathrm{Isom}(V_1,V_2)$ be the space of vector bundle isomorphisms

$$\psi\colon V_1\longrightarrow V_2$$

covering id_X. For any such isomorphism, let

$$\Lambda_{\mathbb{R}}^{\mathrm{top}}\psi\colon \Lambda_{\mathbb{R}}^{\mathrm{top}}V_1 \longrightarrow \Lambda_{\mathbb{R}}^{\mathrm{top}}V_2$$

be the induced element of $\mathrm{Isom}(\Lambda_{\mathbb{R}}^{\mathrm{top}}V_1, \Lambda_{\mathbb{R}}^{\mathrm{top}}V_2)$.

Let (X,ϕ) be a topological space with an involution. For real bundle pairs (V_1,φ_1) and (V_2,φ_2) over (X,ϕ) satisfying the conditions of Theorem 1.1, let $\mathrm{Isom}((V_1,\varphi_1),(V_2,\varphi_2))$ be the space of isomorphisms

$$\Psi\colon (V_1,\varphi_1) \longrightarrow (V_2,\varphi_2)$$

of real bundle pairs covering id_X. For any such isomorphism, let

$$\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi\colon \Lambda_{\mathbb{C}}^{\mathrm{top}}(V_1,\varphi_1) \longrightarrow \Lambda_{\mathbb{C}}^{\mathrm{top}}(V_2,\varphi_2)$$

be the induced element of $\mathrm{Isom}(\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_1,\varphi_1), \Lambda_{\mathbb{C}}^{\mathrm{top}}(V_2,\varphi_2))$. Define

$$\begin{split}\mathrm{Isom}'\big((V_1,\varphi_1),(V_2,\varphi_2)\big) = \big\{(f,\psi)\in\mathrm{Isom}\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_1,\varphi_1),\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_2,\varphi_2)\big)\times\mathrm{Isom}\big(V_1^{\varphi_1},V_2^{\varphi_2}\big)\colon\\ f|_{\Lambda_{\mathbb{R}}^{\mathrm{top}}V^{\varphi_1}}=\Lambda_{\mathbb{R}}^{\mathrm{top}}\psi\big\}.\end{split}$$

The next statement is an immediate consequence of Theorems 1.1 and 1.2.

Corollary 7.1 (of Theorems 1.1,1.2)**.** *Let (Σ,σ) be a (possibly nodal) symmetric surface and (V_1,φ_1) and (V_2,φ_2) over (X,ϕ) such that*

$$\mathrm{rk}_{\mathbb{C}}V_1 = \mathrm{rk}_{\mathbb{C}}V_2, \qquad w_1\big(V_1^{\varphi_1}\big) = w_1\big(V_2^{\varphi_2}\big) \in H^1(\Sigma^\sigma;\mathbb{Z}_2),$$

and $\deg(V_1|_{\Sigma'})=\deg(V_2|_{\Sigma'})$ for each irreducible component $\Sigma'\subset\Sigma$. Then the maps

$$\begin{aligned}\mathrm{Isom}\big((V_1,\varphi_1),(V_2,\varphi_2)\big) &\longrightarrow \mathrm{Isom}'\big((V_1,\varphi_1),(V_2,\varphi_2)\big), & \Psi &\longrightarrow \big(\Lambda_{\mathbb{C}}^{\mathrm{top}}\Psi, \Psi|_{V_1^{\varphi_1}}\big),\\ \mathrm{Isom}'\big((V_1,\varphi_1),(V_2,\varphi_2)\big) &\longrightarrow \mathrm{Isom}\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_1,\varphi_1),\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_2,\varphi_2)\big), & (f,\psi) &\longrightarrow f,\\ \mathrm{Isom}'\big((V_1,\varphi_1),(V_2,\varphi_2)\big) &\longrightarrow \mathrm{Isom}\big(V_1^{\varphi_1},V_2^{\varphi_2}\big), & (f,\psi) &\longrightarrow \psi,\end{aligned}$$

are surjective. Two isomorphisms from (V_1,φ_1) to (V_2,φ_2) lie in the same path component of $\mathrm{Isom}((V_1,\varphi_1),(V_2,\varphi_2))$ *if and only if their images in*

$$\mathrm{Isom}\big(\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_1,\varphi_1),\Lambda_{\mathbb{C}}^{\mathrm{top}}(V_2,\varphi_2)\big) \qquad \textit{and} \qquad \mathrm{Isom}\big(V_1^{\varphi_1},V_2^{\varphi_2}\big)$$

lie in the same path components of the two spaces. Furthermore, every path (f_t,ψ_t) in the space $\mathrm{Isom}'((V_1,\varphi_1),(V_2,\varphi_2))$ *passing through the images of some $\Psi,\Phi\in\mathrm{Isom}((V_1,\varphi_1),(V_2,\varphi_2))$ lifts to a path in* $\mathrm{Isom}((V_1,\varphi_1),(V_2,\varphi_2))$ *passing through Ψ and Φ.*

A special case of this corollary underpins the perspective on orientability in real Gromov-Witten theory and for orienting naturally twisted determinants of Fredholm operators on real bundle pairs over symmetric surfaces introduced in [7] and built upon in [8, 9]. This perspective motivated the following definition.

Definition 7.2 ([8, Definition 5.1])**.** Let (X,ϕ) be a topological space with an involution and (V,φ) be a real bundle pair over (X,ϕ). A real orientation on (V,φ) consists of

(RO1) a rank 1 real bundle pair $(L,\widetilde{\phi})$ over (X,ϕ) such that

$$w_2(V^{\varphi}) = w_1(L^{\widetilde{\phi}})^2 \qquad \text{and} \qquad \Lambda_{\mathbb{C}}^{\text{top}}(V,\varphi) \approx (L,\widetilde{\phi})^{\otimes 2}, \tag{7.1}$$

(RO2) a homotopy class of isomorphisms of real bundle pairs in (7.1), and

(RO3) a spin structure on the real vector bundle $V^{\varphi}\oplus 2(L^*)^{\widetilde{\phi}^*}$ over X^{ϕ} compatible with the orientation induced by (RO2).

We recall that a spin structure on a rank n oriented real vector bundle $W\longrightarrow X^{\phi}$ with a Riemannian metric is a Spin_n principal bundle $\widetilde{\mathrm{Fr}}(W)\longrightarrow X^{\phi}$ factoring through a double cover $\widetilde{\mathrm{Fr}}(W)\longrightarrow \mathrm{Fr}(W)$ equivariant with respect to the canonical homomorphism $\mathrm{Spin}_n\longrightarrow \mathrm{SO}_n$. Since SO_n is a deformation retract of the identity component $\mathrm{GL}_n^+\mathbb{R}$ of $\mathrm{GL}_n\mathbb{R}$, this notion is independent of the choice of the metric on W. If X^{ϕ} is a CW-complex, a spin structure on W is equivalent to a trivialization of W over the 2-skeleton of X^{ϕ}.

An isomorphism Θ in (7.1) restricts to an isomorphism

$$\Lambda_{\mathbb{R}}^{\text{top}} V^{\varphi} \approx (L^{\widetilde{\phi}})^{\otimes 2} \tag{7.2}$$

of real line bundles over X^{ϕ}. Since the vector bundles $(L^{\widetilde{\phi}})^{\otimes 2}$ and $2(L^*)^{\widetilde{\phi}^*}$ are canonically oriented, Θ determines orientations on V^{φ} and $V^{\varphi}\oplus 2(L^*)^{\widetilde{\phi}^*}$. We will call them the orientations determined by (RO2) if Θ lies in the chosen homotopy class. An isomorphism Θ in (7.1) also induces an isomorphism

$$\begin{aligned} \Lambda_{\mathbb{C}}^{\text{top}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big) &\approx \Lambda_{\mathbb{C}}^{\text{top}}(V,\varphi)\otimes (L^*,\widetilde{\phi}^*)^{\otimes 2} \\ &\approx (L,\widetilde{\phi})^{\otimes 2}\otimes (L^*,\widetilde{\phi}^*)^{\otimes 2} \approx \big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big), \end{aligned} \tag{7.3}$$

where the last isomorphism is the canonical pairing. We will call the homotopy class of isomorphisms (7.3) induced by the isomorphisms Θ in (RO2) the homotopy class determined by (RO2).

By the above, a real orientation on a rank n real bundle pair (V,φ) over a symmetric surface (Σ,σ) determines a topological component of the space

$$\begin{aligned} &\mathrm{Isom}'\big((V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*),(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c})\big) \\ &\quad\subset \mathrm{Isom}'\big(\Lambda_{\mathbb{C}}^{\text{top}}(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*),\Lambda_{\mathbb{C}}^{\text{top}}(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c})\big)\times \mathrm{Isom}'\big((V\oplus 2L^*)^{\varphi\oplus 2\widetilde{\phi}^*},(\Sigma\times\mathbb{C}^{n+2})^{\sigma\times\mathfrak{c}}\big). \end{aligned}$$

The next proposition, established for smooth and one-nodal symmetric surfaces in [8] and for symmetric surfaces with one pair of conjugate nodes in [9], is thus a special case of Corollary 7.1.

Proposition 7.3. *Let (Σ,σ) be a symmetric surface and (V,φ) be a rank n real bundle pair over (Σ,σ). A real orientation on (V,φ) determines a homotopy class of isomorphisms*

$$\Psi\colon \big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big) \approx \big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big)$$

of real bundle pairs over (Σ,σ). An isomorphism Ψ belongs to this homotopy class if and only if the restriction of Ψ to the real locus induces the chosen spin structure (RO3) and the isomorphism

$$\Lambda_{\mathbb{C}}^{\text{top}}\Psi\colon \Lambda_{\mathbb{C}}^{\text{top}}\big(V\oplus 2L^*,\varphi\oplus 2\widetilde{\phi}^*\big) \longrightarrow \Lambda_{\mathbb{C}}^{\text{top}}\big(\Sigma\times\mathbb{C}^{n+2},\sigma\times\mathfrak{c}\big) = \big(\Sigma\times\mathbb{C},\sigma\times\mathfrak{c}\big)$$

lies in the homotopy class determined by (RO2).

A symmetric Riemann surface $(\Sigma, \sigma, \mathfrak{j})$ is a symmetric surface with a complex structure $\mathfrak{j}$ on Σ such that $\sigma^*\mathfrak{j} = -\mathfrak{j}$. A real Cauchy-Riemann operator on a real bundle pair (V, φ) over such a surface is a linear map of the form

$$D = \bar{\partial}_V + A \colon \Gamma(\Sigma; V)^\varphi \equiv \{\xi \in \Gamma(\Sigma; V) \colon \xi \circ \sigma = \varphi \circ \xi\} \\ \longrightarrow \Gamma^{0,1}_{\mathfrak{j}}(\Sigma; V)^\varphi \equiv \{\zeta \in \Gamma(\Sigma; (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} V) \colon \zeta \circ \mathrm{d}\sigma = \varphi \circ \zeta\},$$

where $\bar{\partial}_V$ is the holomorphic $\bar{\partial}$-operator for some holomorphic structure in V and

$$A \in \Gamma\big(\Sigma; \mathrm{Hom}_{\mathbb{R}}(V, (T^*\Sigma, \mathfrak{j})^{0,1} \otimes_{\mathbb{C}} V)\big)^\varphi$$

is a zeroth-order deformation term. Let $\bar{\partial}_{\Sigma;\mathbb{C}}$ denote the real Cauchy-Riemann operator on the trivial rank 1 real bundle $(\Sigma \times \mathbb{C}, \sigma \times \mathfrak{c})$ with the standard holomorphic structure and $A = 0$. Any real Cauchy-Riemann operator D on a real bundle pair is Fredholm in the appropriate completions. We denote by

$$\det D \equiv \Lambda^{\mathrm{top}}_{\mathbb{R}}(\ker D) \otimes \big(\Lambda^{\mathrm{top}}_{\mathbb{R}}(\mathrm{cok}\, D)\big)^*$$

its determinant line.

If (X, ϕ) is a topological space with an involution, a real map $u \colon (\Sigma, \sigma) \longrightarrow (X, \phi)$ is a continuous map $u \colon \Sigma \longrightarrow X$ such that $u \circ \sigma = \phi \circ u$. Such a map pulls back a real bundle pair (V, φ) over (X, ϕ) to a real bundle $u^*(V, \varphi)$ over (Σ, σ) and a real orientation on the former to a real orientation on the latter. By Proposition 7.3, a real orientation on a rank n real bundle pair (V, φ) over (X, ϕ) thus determines an orientation on the relative determinant

$$\widehat{\det}\, D_u \equiv \big(\det D_u\big) \otimes \big(\det \bar{\partial}_{\Sigma;\mathbb{C}}\big)^{\otimes n} \tag{7.4}$$

for every real Cauchy-Riemann operator D_u on the real bundle pair $u^*(V, \varphi)$ over (Σ, σ) for every real map $u \colon (\Sigma, \sigma) \longrightarrow (X, \phi)$. This observation plays a central role in the construction of positive-genus real Gromov-Witten invariants in [8]. If Σ is of genus 0, $\det \bar{\partial}_{\Sigma;\mathbb{C}}$ has a canonical orientation and an orientation on $\widehat{\det}\, D_u$ is canonically equivalent to an orientation on $\det D_u$. In particular, [4, Theorem 1.3] is essentially equivalent to the case of this observation with $\Sigma = \mathbb{P}^1$ and σ being an involution without fixed points.

The analogue of the last factor in (7.4) in the case of bordered surfaces is canonically oriented. Thus, the role of the real orientations of Definition 7.2 for Cauchy-Riemann operators in real Gromov-Witten theory is analogous to that of the relative spin structures of [5, Definition 8.1.2] in open Gromov-Witten theory.

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