Math53: Ordinary Differential Equations
Winter 2004

Unit 6 Summary

Qualitative Analysis of Autonomous Systems of ODEs

Introduction

(1) An *autonomous system of ODEs* is a system of the form

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \qquad \mathbf{y} = \mathbf{y}(t), \tag{1}$$

where $\mathbf{y}=\mathbf{y}(t)$ is an n-vector of smooth functions of t and $\mathbf{f}(\mathbf{y})$ is an n-vector for each n-vector $\mathbf{y}$. For example,

$$\begin{cases} x' = f(x,y) = -x(1-y) \\ y' = g(x,y) = -4y(1+x) \end{cases} \qquad (x,y) = \big(x(t), y(t)\big) \tag{2}$$

is a two-dimensional autonomous system. The corresponding vector-valued functions are

$$\mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad \mathbf{f}(\mathbf{y}) = \mathbf{f}(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} -x(1-y) \\ -4y(1+x) \end{pmatrix}.$$

The systems (eq1) and (eq2) are called autonomous because they are not explicitly dependent on t.

(2) We are interested in knowing what happens with the vector $\mathbf{y}(t)$, where $\mathbf{y}=\mathbf{y}(t)$ is a solution of (eq1), as t increases. One special property of autonomous equations is that if $\mathbf{y} = \mathbf{y}(t)$ is a solution of such an ODE, e.g. of (eq1), then so is the function $\mathbf{z} = \mathbf{z}(t) = \mathbf{y}(t-c)$. As the time parameter t increases, the points $\mathbf{z}(t)$ and $\mathbf{y}(t)$ trace the same path in $\mathbb{R}^n$, but $\mathbf{z}=\mathbf{z}(t)$ is delayed by c. Thus, the behavior of a solution $\mathbf{y}=\mathbf{y}(t)$ of (eq1) is well-represented by the solution curve in $\mathbb{R}^n$ corresponding to $\mathbf{y}=\mathbf{y}(t)$, i.e. the directed curve in $\mathbb{R}^n$ traced by $\mathbf{y}(t)$ as t increases. Such a solution curve in $\mathbb{R}^n$, the *phase space* for the system (eq1), shows every point in $\mathbb{R}^n$ the path $\mathbf{y}(t)$ passes through as t increases, including what happens to $\mathbf{y}(t)$ as $t \longrightarrow \infty$, even though the curve does not specify at what value of t the solution $\mathbf{y}=\mathbf{y}(t)$ arrives at each given point. The *phase-space portrait* for (eq1) is the space $\mathbb{R}^n$ with all solution curves of (eq1) shown. By the uniqueness theorem for system of ODEs, such solution curves do not intersect, *provided* $\mathbf{f}=\mathbf{f}(\mathbf{y})$ is a smooth function.

Local Descriptions

(1) The first step in analyzing the system (eq1) is to find the *equilibrium points* of (eq1). These are the points $\mathbf{y}_i$ of $\mathbb{R}^n$ such that each constant function $\mathbf{y}(t)=\mathbf{y}_i$ is a solution of (eq1). The physical interpretation of this is that if the system starts at an equilibrium point, it stays there forever. In mathematical terms, this means that if the initial value $\mathbf{y}_0$ of a solution $\mathbf{y}=\mathbf{y}(t)$ to the ODE (eq1) is an equilibrium point, $\mathbf{y}(t)=\mathbf{y}_0$ for all $t\in\mathbb{R}$. Since the derivative of a constant function is zero, the constant function $\mathbf{y}(t)=\mathbf{y}_i$ is a solution of (eq1) if and only if $\mathbf{f}(\mathbf{y}_i)=\mathbf{0}$. Thus,

$$\boxed{\mathbf{y}_i \in \mathbb{R}^n \text{ is an equilibrium point for } \; \mathbf{y}' = \mathbf{f}(\mathbf{y}),\; \mathbf{y} = \mathbf{y}(t) \qquad \Longleftrightarrow \qquad \mathbf{f}(\mathbf{y}_i) = \mathbf{0}}$$

In order to find the equilibrium points, we only need to solve the system $\mathbf{f}(\mathbf{y}) = \mathbf{0}$ for $\mathbf{y}$. For example, we find the equilibrium points for (eq2) by setting $x'=0$ and $y'=0$:

$$\begin{cases} x'=0 \\ y'=0 \end{cases} \iff \begin{cases} -x(1-y) = 0 \\ -4y(1+x) = 0 \end{cases} \iff \begin{cases} x=0 \text{ or } y=1 \\ y=0 \text{ or } x=-1 \end{cases} \tag{3}$$

Thus, the equilibrium points of the system (eq2) are $(0,0)$ and $(-1,1)$; they are indicated by large dots on the first sketch in Figure 1. In contrast, the set of equilibrium points of the system $\mathbf{y}'=A\mathbf{y}$, where A is a constant $n\times n$ matrix, is always a *linear* subspace of $\mathbb{R}^n$ passing through the origin, such as $\mathbf{0}$ by itself, or a line through the origin, or a plane through the origin, etc. *What is the set of equilibrium points for the system* $\mathbf{y}'=A\mathbf{y}$*, in terms the constant* $n\times n$ *matrix* A*?*

Note: While it is usually not hard to find the equilibrium points of an autonomous system of ODEs, some care is often needed. For example, after the last step in (eq3), we need to determine all pairs (x,y) such that one of the two conditions on the top line is satisfied, so that $x'=0$, *and* one of the two conditions on the bottom line is satisfied, so that $y'=0$. This is different from finding (x,y) such that any two of the four conditions in (eq3) are satisfied. Thus, it is essential to keep the conditions for $x'=0$ and the condition for $y'=0$ separately, e.g. on separate lines.

(2) The next step is to determine the *stability type* of every equilibrium point $\mathbf{y}_i$ of (eq1). An equilibrium point $\mathbf{y}_i$ for (eq1) is

asymptotically stable if every solution curve for (eq1) that passes sufficiently close to $\mathbf{y}_i$ approaches $\mathbf{y}_i$ as $t\longrightarrow\infty$;

stable if every solution curve for (eq1) that passes very close stays very close to $\mathbf{y}_i$ as $t\longrightarrow\infty$;

unstable if there exist solution curves that pass very close to $\mathbf{y}_i$ and then move away from $\mathbf{y}_i$.

More formal, and perhaps more confusing, definitions of these notions can be found on p512 and in the solutions to PS6-Problem 1. The equilibrium point $\mathbf{y}_i$ is unstable if and only if it is not stable. Since the stable-equilibrium condition requires that *every* solution curve that comes close to $\mathbf{y}_i$ stay near $\mathbf{y}_i$ after that, $\mathbf{y}_i$ is unstable if there is *at least one* solution curve that comes very close to $\mathbf{y}_i$ and then moves away. In particular, there may even be solution curves that approach $\mathbf{y}_i$ as $t\longrightarrow\infty$, as is the case for saddle points.

Note: The three stability types only describe what happens very close to an equilibrium point $\mathbf{y}_i$ of (eq1). They have *nothing* to do with what the phase-space portrait may look like away from $\mathbf{y}_i$. In particular, moving "away" does not need to mean very far. For example, $\mathbf{0}$ is an unstable equilibrium for the system (eq10) below, even though any solution curve passing close to $\mathbf{0}$ does not move past the circle of radius one centered at the origin.

(3) If $\mathbf{y}_i$ is an equilibrium point for the system (eq1), we often, but not always, can determine whether $\mathbf{y}_i$ is asymptotically stable or unstable from the Jacobian of $\mathbf{f}$ at $\mathbf{y}=\mathbf{y}_i$:

$$J\mathbf{f}(\mathbf{y}_i) = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\bigg|_{\mathbf{y}_i}.$$

This is the $n\times n$ matrix consisting of the partial derivatives of the n component functions of $\mathbf{f}$ with respect to the n components of $\mathbf{y}$, evaluated at $\mathbf{y}=\mathbf{y}_i$. For example, in the case of (eq2),

$$J\mathbf{f}(x,y) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} = \begin{pmatrix} y-1 & x \\ -4y & -4-4x \end{pmatrix}. \tag{4}$$

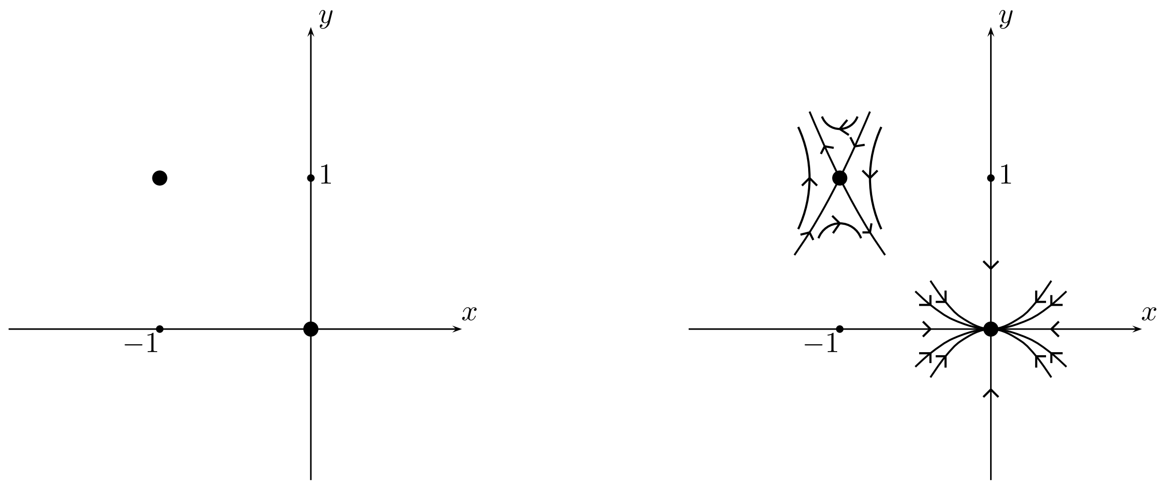

Figure 1: Sketching the Phase-Plane Portrait for (eq2): Steps 1 and 2

The Jacobian stability test for an *equilibrium* point $\mathbf{y}_i$ of (eq1) is

$$\boxed{\begin{array}{ll} \text{If the real part } \mathrm{Re}\,\lambda \text{ of} & \textit{every} \text{ eigenvalue } \lambda \text{ of } J\mathbf{f}(\mathbf{y}_i) \text{ is negative,} \\ & \text{then } \;\; \mathbf{y}_i \text{ is an asymptotically stable equilibrium for (eq1).} \\ \text{If the real part } \mathrm{Re}\,\lambda \text{ of} & \textit{some} \text{ eigenvalue } \lambda \text{ of } J\mathbf{f}(\mathbf{y}_i) \text{ is positive,} \\ & \text{then } \;\; \mathbf{y}_i \text{ is an unstable equilibrium for (eq1).} \end{array}} \tag{5}$$

For example, by (1) above, the equilibrium points of (eq2) are $(0,0)$ and $(-1,1)$, while by (eq4),

$$\begin{aligned} J\mathbf{f}(0,0) &= \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \implies \lambda_1 = -1,\; \lambda_2 = -4; \\ J\mathbf{f}(-1,1) &= \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \implies \lambda^2 + 0\lambda - 4 = 0 \implies \lambda_1 = 2,\; \lambda_2 = -2. \end{aligned} \tag{6}$$

Since *both* eigenvalues of $J\mathbf{f}(0,0)$ are negative, $(0,0)$ is an asymptotically stable equilibrium point of (eq2), by the first statement of the Jacobian test (eq5). On the other hand, since one of the eigenvalues of $J\mathbf{f}(-1,1)$ is positive, $(-1,1)$ is an unstable equilibrium point of (eq2), by the second statement of the Jacobian test.

The derivative test for autonomous ODEs of Section 1.9 is the one-dimensional version of the Jacobian test (eq5). *Do you see why?* The derivative test for autonomous ODEs cannot be used in some cases. Similarly, the Jacobian test cannot be applied in the cases not covered by (eq5), i.e. if at least one of the eigenvalue is zero and none is positive. In such cases, a more complicated test can be used, as described in Section 10.7 and in PS6-Problem 1.

The reason for why the Jacobian stability test (eq5) works is roughly the following. Near the equilibrium point $\mathbf{y}_i$, the system (eq1) can be written as

$$\mathbf{w}' = J\mathbf{f}(\mathbf{y}_i)\mathbf{w} + \mathbf{R}_i(\mathbf{w}), \tag{7}$$

where $\mathbf{w}=\mathbf{y}-\mathbf{y}_i$ measures the shift from $\mathbf{y}_i$ and $\mathbf{R}_i(\mathbf{w})$ is the remainder term for the first-order Taylor expansion of $\mathbf{f}$ at $\mathbf{y}_i$. In particular, $|\mathbf{R}_i(\mathbf{w})| \le C|\mathbf{w}|^2$ for some constant C and all small $\mathbf{w}$. Near $\mathbf{w}=0$, the remainder term $\mathbf{R}_i(\mathbf{w})$ can be thought of as a tiny correction to the constant $n\times n$ matrix $J\mathbf{f}(\mathbf{y}_i)$, which in turn results in tiny changes in the eigenvalues of $J\mathbf{f}(\mathbf{y}_i)$. If all eigenvalues of $J\mathbf{f}(\mathbf{y}_i)$ are negative, they'll still be negative after tiny changes in the matrix $J\mathbf{f}(\mathbf{y}_i)$, and thus all solution curves of (eq7) that start with small $\mathbf{w}$ will approach $\mathbf{w}=\mathbf{0}$, i.e. $\mathbf{y}=\mathbf{y}_i$.

(4) In some cases, the Jacobian $J\mathbf{f}(\mathbf{y}_i)$ can be used to determine the behavior of solution curves even more specifically than in (eq5). In the two-dimensional case of (eq1), if $\mathbf{y}_i$ is an equilibrium:

$$\boxed{\begin{array}{r}\text{If } \mathbf{w}=\mathbf{0} \text{ is a nodal/spiral sink/source or a saddle point for } \mathbf{w}'=J\mathbf{f}(\mathbf{y}_i)\mathbf{w},\\ \text{then}\quad \mathbf{y}=\mathbf{y}_i \text{ is the same type of equilibrium point for (eq1).}\end{array}} \tag{8}$$

Furthermore, if $J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$ has real distinct nonzero eigenvalues, i.e. $\mathbf{w}=\mathbf{0}$ is a nodal source, a nodal sink, or a saddle point for $\mathbf{w}'=J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of $J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$ with eigenvalues λ_1 and λ_2, then there exist solution curves for (eq1) that approach $\mathbf{y}_i$ as $t\longrightarrow-\infty$ or $t\longrightarrow\infty$, depending on the signs of λ_1 and λ_2, such that their slope approaches that of $\mathbf{v}_1$ or $\mathbf{v}_2$ as they approach $\mathbf{y}_i$. These solution curves correspond to the four half-line solutions, but need not be half-lines themselves. Instead, each of them approximates one of the four half-lines near $w=\mathbf{0}$, i.e. $\mathbf{y}=\mathbf{y}_i$. For example, continuing from (eq6) with the example of (eq2), we find

$$\begin{aligned} J\mathbf{f}(0,0) &= \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix} \Longrightarrow \lambda_1=-1,\ \lambda_2=-4,\ \mathbf{v}_1=\begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}0\\1\end{pmatrix};\\ J\mathbf{f}(-1,1) &= \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix} \Longrightarrow \lambda_1=2,\ \lambda_2=-2,\ \mathbf{v}_1=\begin{pmatrix}1\\-2\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\2\end{pmatrix}. \end{aligned} \tag{9}$$

This means that $(0,0)$ is a nodal sink and nearly all solutions approach $(0,0)$ tangent to the x-axis, since $\lambda_1>\lambda_2$, but there is also a pair of solution curves approaching the origin tangent to the y-axis corresponding to the two $\mathbf{v}_2$ half-lines. Similarly, $(-1,1)$ is a sink, and there is a pair of solution curves approaching $(-1,1)$ tangent to the two half-lines of slope 2 through $(-1,1)$ and a pair of solution curves leaving $(-1,1)$ tangent to the two half-lines of slope -2. These conclusions are shown in the second sketch of Figure 1. The solution curves corresponding to the four half-lines at each equilibrium point need not be the half-lines themselves. However, we will see in the next section that in the case of $(0,0)$, they are in fact the expected half-lines, and we sketch them as such.

The reasoning behind the second Jacobian test, i.e. (eq8), is the same as behind the first one, i.e. (eq5). The five types of equilibrium that appear in (eq8) are characterized by various conditions on the two eigenvalues. These conditions remain satisfied if the matrix and the eigenvalues are altered just tiny bit. While $J\mathbf{f}(\mathbf{y}_i)$ does not specify what type of equilibrium $\mathbf{y}_i$ is in other cases, this reasoning tells us what types are possible. For example, if $\mathbf{w}=\mathbf{0}$ is a center for $\mathbf{w}'=J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$, i.e. the two eigenvalues λ_1 and λ_2 of $J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$ are complex and $\operatorname{Re}\lambda_1=\operatorname{Re}\lambda_2=0$, then $\mathbf{y}_i$ is either a center, a spiral source, or spiral sink for (eq1). Furthermore, the direction of rotation is as predicted by $J\mathbf{f}(\mathbf{y}_i)$. *If $\mathbf{w}=\mathbf{0}$ is a degenerate nodal sink for $\mathbf{w}'=J\mathbf{f}(\mathbf{y}_i)\mathbf{w}$, what can $\mathbf{y}_i$ be, as an equilibrium point for (eq1)?*

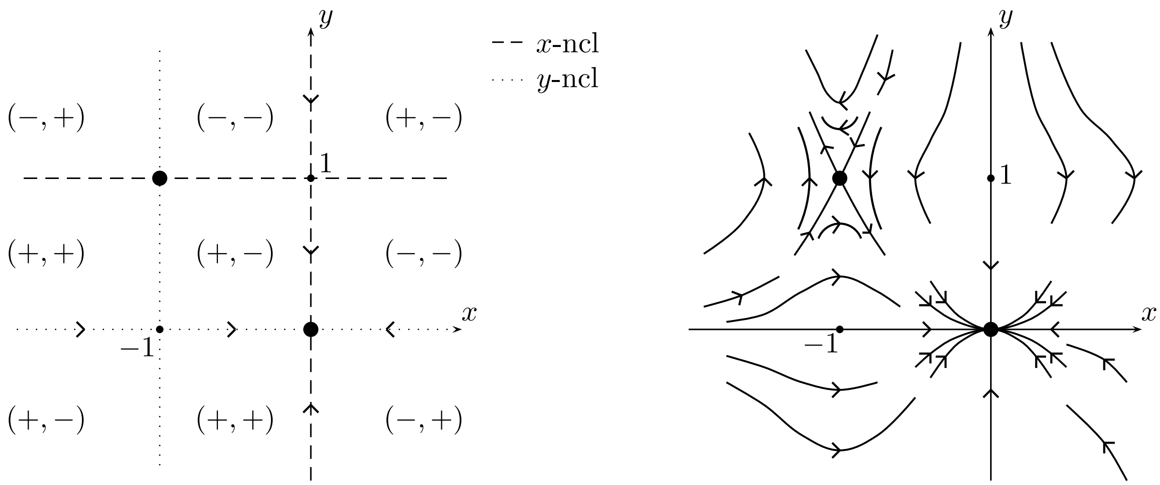

Figure 2: Sketching the Phase-Plane Portrait for (eq2): Steps 3 and 4

Global Descriptions

(1) From now on, we will assume that the system (eq1) is two-dimensional. Once we have found all equilibrium points for (eq1) and described the behavior of solution curves for (eq1) near each equilibrium point, we need to see what happens to the solution curves away from the equilibrium points. For this purpose, it is useful to do draw the two *nullclines*, i.e. the curves along which the x-component and the y-component of the two-vector $\mathbf{f} = \mathbf{f}(\mathbf{y})$ vanish separately. The nullclines will typically be a collection of curves, and not necessarily of lines. The x-nullclines and y-nullclines will divide the xy-plane into several regions. Within each region, the signs of the x-component of $\mathbf{f}$ and of the y-component of $\mathbf{f}$ will not change. Since $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, in each such region *all* solution curves must move in the same general direction. This direction can be indicated by a pair like $(+,-)$, showing the signs of the two components of $\mathbf{f}$. We do not need to compute the signs of both components in every region cut out by the nullclines, since the first sign can change only after crossing the x-nullcline and the second only after crossing the y-nullcline. Furthermore, along every x-nullcline, the vector field $\mathbf{f}$ has to be vertical. Similarly, along every y-nullcline, the vector field $\mathbf{f}$ has to be horizontal.

In the case of (eq2), the x-nullcline is described by

$$x' = f(x,y) = 0 \quad \Longleftrightarrow \quad -x(1-y) = 0 \quad \Longleftrightarrow \quad x = 0 \text{ or } y = 1.$$

Thus, the x-nullcline consists of the lines $x=0$ and $y=1$, which are shown as dashed lines in the first plot of Figure 2. Similarly, the y-nullcline is described by

$$y' = g(x,y) = 0 \quad \Longleftrightarrow \quad -4y(1+x) = 0 \quad \Longleftrightarrow \quad y = 0 \text{ or } x = -1.$$

Thus, the y-nullcline consists of the lines $y=0$ and $x=-1$, which are shown as dotted lines in the first plot of Figure 2. Note the equilibrium points are the intersections of the dashed lines with the dotted lines. We indicate the flow direction on the vertical segments of the x-nullclines and on the horizontal segments of the y-nullcline, which in this case are the y-axis and the x-axis, respectively.

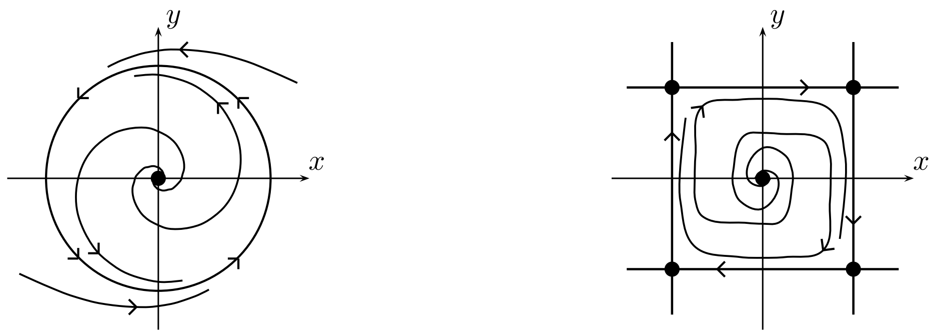

Figure 3: Examples of a Limit Cycle and of a Limit Polygon

We then see that if $\mathbf{y}=\mathbf{y}(t)$ is a solution of (eq2) and $\mathbf{y}(t_0)$ lies on the x-axis (y-axis) for some t_0, $\mathbf{y}(t)$ lies on the x-axis (y-axis) for all t. Since the only equilibrium point lying on the x-axis (y-axis) is $(0,0)$, it follows that $\mathbf{y}(t)$ approaches $(0,0)$, along the x-axis (y-axis), as $t \longrightarrow \infty$. Thus, the positive and negative x-axis and y-axis are solution curves for (eq2). We next label each region cut out by the nullclines by $(\pm,\pm)$, indicating the flow direction in each of the nine rectangular regions. We can determine the signs by looking at the two equations in (eq2), but for the seven regions touching one of the equilibrium points the signs can in fact be read off from the second sketch in Figure 1. For example, solution curves move up and left in the bottom right region; thus, the sign pair is $(-,+)$. This leaves only the bottom left and the top right regions, where the signs can be determined from the signs in nearby regions and the separating nullclines. We can then sketch solution curves away from the equilibrium points; see the second sketch in Figure 2. Note that solution curves must be horizontal when crossing the vertical y-nullcline $x=-1$ and vertical when crossing the horizontal x-nullcline $y=1$.

Note: It is essential not to mix the curves that constitute the x-nullcline with the curves that constitute the y-nullcline.

(2) We also need to determine what happens to every solution $\mathbf{y} = \mathbf{y}(t)$ of (eq1) as $t \longrightarrow \pm\infty$. A solution curve can of course approach an equilibrium point for (eq1) or go off to infinity, leaving every bounded region of the plane. However, a solution curve can also approach a *limit cycle* for (eq1) or an *oriented (or directed) polygon of solution curves* for (eq1). A *limit cycle* for (eq1) is a simple closed solution curve. Such a curve describes a solution $\mathbf{x}=\mathbf{x}(t)$, which is periodic in t. For example, the system

$$\begin{cases} x' = -y + x\big(1-(x^2+y^2)\big) \\ y' = x + y\big(1-(x^2+y^2)\big) \end{cases} \qquad x = x(t),\ y = y(t), \tag{10}$$

can be written in the standard polar coordinates (r,θ) as

$$\begin{cases} r' = r(1-r^2) \\ \theta' = 1 \end{cases} \qquad r = r(t),\ \theta = \theta(t). \tag{11}$$

Can you check this? We can find the general solution of (eq11) and thus of (eq10). One solution is $(r(t),\theta(t))=(1,t)$. The corresponding solution curve is $r=1$, i.e. the circle of radius one centered

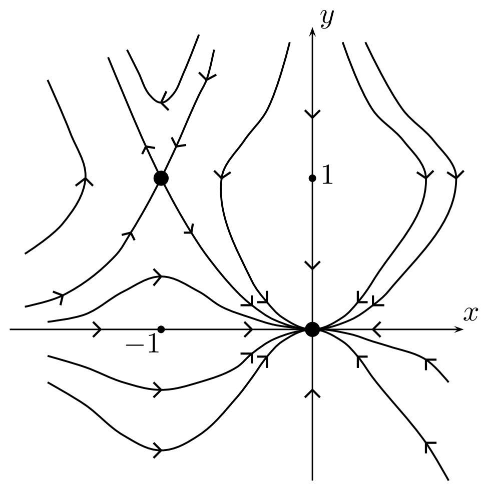

Figure 4: Phase-Plane Portrait for (eq2)

at the origin. It is positively oriented, since $\theta' = 1 > 0$. The first equation in (eq11) implies that all other curves approach the unit circle as $t \longrightarrow \infty$. They all spiral counterclockwise, since $\theta' > 0$. The unit circle is an *attracting* cycle, since solution curves approach it from both sides. The phase-plane sketch for (eq10) is the first plot in Figure 3.

An *oriented (or directed) polygon of solution curves* for (eq1) is a polygon in the plane, whose vertices are equilibrium points for (eq1) and whose edges are solution curves for (eq1). Furthermore, the edges are oriented in the same direction. For example, the system

$$\begin{cases} x' = 2(x+y)(1-x^2) \\ y' = -(2x+y)(1-y^2) \end{cases} \qquad x = x(t),\ y = y(t), \tag{12}$$

has seven equilibrium points: $(0,0)$, $(\pm 1, \pm 1)$, $(-1,2)$, and $(1,-2)$. The origin is a nodal source, by the Jacobian test. The vertical lines $x = \pm 1$ are components of the x-nullcline; thus, they are made up of solution curves. Similarly, the horizontal lines $y = \pm 1$ are also made up of solution curves. After determining the flow directions on various segments of the nullclines, we see that the square with vertices at $(\pm 1, \pm 1)$ forms an oriented polygon of solutions. There are no equilibrium points, other than the origin, inside of the square. Thus, any solution curve spiraling out from the origin approaches either the square or a cycle inside of it. If the unit square contains no cycles, then this oriented polygon of solution curves is the limit set for all solution curves spiraling out from the origin, as shown in the second sketch in Figure 3.

We now finish sketching the phase-plane portrait for (eq2). The first sketch in Figure 2 shows that the system (eq2) has no cycles or oriented polygons. Thus, all solution curves must approach one of the two equilibrium points or move off to infinity. This allows us to finish up the second sketch in Figure 2. Figure 4 represents the phase-plane portrait for (eq2). In particular, it shows that all solutions that start either in the first, third, or fourth quadrants and some solutions that start in the second quadrant end up at the origin. Some solutions come very close to the other equilibrium point, $(-1,1)$, and then move off to infinity.

In the case of (eq2), we are able to rule out the existence of any cycles and directed polygons of solution curves just by looking at its sign diagram, i.e. the first sketch in Figure 4. In other cases, we may be able to rule out the existence of any cycles and directed polygons for the system

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad x = x(t),\ y = y(t), \tag{13}$$

in a simply connected region A, i.e. one that has no holes, by using the following theorem:

$$\boxed{\text{If } A \text{ is sc and } f_x(x,y)+g_y(x,y)>0 \text{ for all } (x,y)\in A \text{ or } f_x(x,y)+g_y(x,y)<0 \text{ for all } (x,y)\in A, \text{ then } A \text{ contains no cycles or oriented polygons for (eq13)}}$$

This theorem follows easily from Green's Theorem, covered in Math52.

(3) Finally, one can also try to describe the behavior of solution curves (eq1) relative to the level sets of some function $V = V(\mathbf{y})$. For example, every solution for the system

$$\begin{cases} x' = y^2 \\ y' = 2x + x^2 \end{cases} \qquad x = x(t),\ y = y(t), \tag{14}$$

satisfies $3x^2 + x^3 - y^3 = C$, for some constant C. *Can you verify this without solving the system?* Thus, all solution curves for (eq14) lie on the level sets of the function $H(x,y) = 3x^2 + x^3 - y^3$. In general, if the system has the form

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases} \qquad x = x(t),\ y = y(t), \tag{15}$$

such a function H can be found by solving the equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}, \qquad y = y(x),$$

implicitly as $H(x,y) = 0$. In many cases, it may be difficult to solve this ODE. We may instead be able to find a function $V = V(x,y)$ such that

$$\vec{\nabla} V|_{(x,y)} \cdot (f,g) \geq 0 \qquad \text{for all} \qquad (x,y).$$

In such a case, solution curves for (eq15) can move only to higher level sets of V.