# Math53: Ordinary Differential Equations
# Winter 2004

## Unit 4 Summary
*Systems of Linear ODEs*

### Linear Algebra

Throughout this section $A$ denotes an $n \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

(1) Matrix $A$ is *nonsingular* if for every $\mathbf{v} \in \mathbb{R}^n$, there exists $\mathbf{x} \in \mathbb{R}^n$ such that

$$A\mathbf{x} = \mathbf{v} \quad \text{or} \quad \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad \text{if} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Matrix $A$ is *invertible* if it has an *inverse*, i.e. there exists a matrix $B$ such that $AB = I = BA$, where $I = I_n$ is the *identity matrix.* If $AB = I$, then $BA = I$, provided that $A$ and $B$ are square matrices. If $AB = I$ and $AC = I$, then $B = C$. Thus, if $A$ has an inverse, it is unique, and denoted by $A^{-1}$. Furthermore,

$$\boxed{A \text{ is nonsingular} \iff A \text{ is invertible} \iff \det A \neq 0}$$

If $\det A \neq 0$, in the $n=2$ case $A^{-1}$ is given by

$$\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \det A = ad - bc}$$

In general, there is a three-step procedure for computing $A^{-1}$. The last step of this procedure involves division by $\det A$. If $A$ and $B$ are square matrices,

$$\boxed{\det(AB) = (\det A) \cdot (\det B) = \det(BA), \qquad \text{but} \qquad \det(A+B) \neq (\det A) + (\det B)}$$

(2) The set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in $\mathbb{R}^n$, or in any *vector space*, is *linearly independent* if

$$c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0}, \quad c_1, \dots, c_k \in \mathbb{R} \ (\text{or } \mathbb{C}) \qquad \implies \qquad c_1, \dots, c_k = 0.$$

In other words, no nontrivial *linear combination* of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the zero vector $\mathbf{0}$. The set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$, or in any *vector space* $V$, is a *basis* for $\mathbb{R}^n$, or for $V$, if for every $\mathbf{v}$ in $\mathbb{R}^n$, or in $V$, there exists a *unique* tuple $(c_1, \dots, c_n)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n.$$

Equivalently, the set of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a *basis* for $V$ if the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent and *span* $V$, i.e. for every $\mathbf{v}$ in $V$, there exists a tuple $(c_1, \ldots, c_n)$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n.$$

*Can you show that these two definitions are equivalent?* In the case of $\mathbb{R}^n$:
(i) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ if and only if
(ii) $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$ if and only if
(iii) $\det \begin{pmatrix} | & \ldots & | \\ \mathbf{v}_1 & \ldots & \mathbf{v}_n \\ | & \ldots & | \end{pmatrix} \neq 0.$

(3) An *eigenvector* $\mathbf{v}$ for $A$ with *eigenvalue* $\lambda \in \mathbb{R}$ is a nonzero column $n$-vector such that

$$\boxed{A\mathbf{v} = \lambda\mathbf{v}} \quad \text{or} \quad \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{pmatrix} \quad \text{if} \quad \mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

If $\mathbf{v}$ is an eigenvector for $A$ with eigenvalue $\lambda$, so is $c\mathbf{v}$ for any number $c$. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for $A$ with the *same* eigenvalue $\lambda$, so is $\mathbf{v}_1 + \mathbf{v}_2$. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors for $A$ with *distinct eigenvalues* $\lambda_1, \ldots, \lambda_k$, i.e. $\lambda_i \neq \lambda_j$ whenever $i \neq j$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. If some of these eigenvalues are the same, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ may or may not be linearly independent.

(4) The eigenvalues of $A$ are the roots of the *characteristic polynomial* for $A$:

$$\det(A - \lambda I) = \det \begin{pmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1,n-1} & a_{1n} \\ a_{21} & a_{22} - \lambda & & & a_{2n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{n,n-1} & a_{nn} - \lambda \end{pmatrix}$$

However, repeated roots of the characteristic polynomial may or may not correspond to different linearly independent eigenvectors. If the multiplicity of a root $\lambda$ of the characteristic polynomial is $q$, there exist $q$ linearly independent *generalized eigenvectors* $\mathbf{v}_1, \ldots, \mathbf{v}_q$ for $A$ with eigenvalue $\lambda$, i.e.

$$\boxed{(A - \lambda)^r \mathbf{v}_i = \mathbf{0} \quad \text{for some} \quad \mathrm{r}}$$

In fact, $r = q$ works in the given case. If $\mathbf{v}_i$ is an actual eigenvector, $r = 1$ suffices, by definition. Furthermore, $\mathbf{v}_1, \ldots, \mathbf{v}_q$ can be chosen in such a way that

$$A\mathbf{v}_1 = \lambda\mathbf{v}_1 \qquad \text{and} \qquad A\mathbf{v}_{i+1} = \mathbf{v}_i + \lambda\mathbf{v}_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, q-1.$$

Thus, it is always possible to find a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of generalized eigenvectors for $A$ such that

$$A\mathbf{v}_i = \lambda_i\mathbf{v}_i + a_i\mathbf{v}_{i-1}, \qquad \text{where} \qquad a_i = 0 \text{ or } a_i = 1, \quad a_i = 0 \;\; \text{if } i = 1 \text{ or } \lambda_{i-1} \neq \lambda_i,$$

where $\lambda_i$ is the eigenvalue corresponding to the generalized eigenvector $\mathbf{v}_i$. Then,

$$A = B^{-1}DB, \qquad \text{where} \qquad D = \begin{pmatrix} \lambda_1 & a_2 & 0 & \dots \\ 0 & \lambda_2 & a_3 & \dots \\ \vdots & \dots & \ddots & \ddots \\ 0 & \dots & 0 & \lambda_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} | & \dots & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & \dots & | \end{pmatrix}. \tag{1}$$

*Can you check this?* The above basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ and matrix $B$, however, may be complex. In such a case, $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a $\mathbb{C}$-basis for $\mathbb{C}^n$, not an $\mathbb{R}$-basis for $\mathbb{R}^n$.

(5) If $A$ is an $n\times n$ matrix, the *exponential* of $A$ is the $n\times n$ matrix given by

$$\boxed{e^A = I_n + \frac{1}{1!}A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots = \sum_{k=0}^{k=\infty} \frac{1}{k!}A^k} \qquad \text{where } A^0 = I_n,\ A^2 = AA,\ A^3 = AAA, \dots$$

Note that this is the same power series as for $e^a$, if $a$ is a real or complex number. By definition, if $A$ is the zero matrix, $e^A = I_n$. Another property of the matrix exponential is

$$\boxed{\text{If } AB = BA, \quad \text{then} \quad e^{A+B} = e^A e^B = e^B e^A} \tag{2}$$

Using this property, we can conclude that
 (i) $e^A$ is an invertible matrix and $(e^A)^{-1} = e^{-A}$;
 (ii) if $H(t) = e^{tA}$, then $H'(t) = AH(t) = H(t)A$.
If $A$ is a diagonal matrix, then $e^A$ is also a diagonal matrix, and the diagonal entires of $e^A$ are the exponentials of the corresponding diagonal entries of $A$. For example,

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \Longrightarrow \qquad e^A = \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix}$$

However, if $A$ is not a diagonal matrix, the entries of $e^A$ are *not* usually the exponentials of the entries of $A$, and it may be very hard to determine them directly from the power series definition of the exponential. On the other hand, it may be possible to find a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for $\mathbb{R}^n$, or $\mathbb{C}^n$, such that $e^A\mathbf{v}_i$ is easy to compute for each $i$. Since $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis, an arbitrary vector $\mathbf{v}$ has the form

$$\mathbf{v} = C_1\mathbf{v}_1 + \dots + C_n\mathbf{v}_n, \quad C_1, \dots, C_n \in \mathbb{C} \qquad \Longrightarrow \qquad e^A\mathbf{v} = C_1 e^A\mathbf{v}_1 + \dots + C_n e^A\mathbf{v}_n.$$

This is usually sufficient for solving systems of linear ODEs with constant coefficients. The product $e^A\mathbf{v}_i$ can be computed for generalized eigenvectors of $A$. For example,

$$\boxed{A\mathbf{v}_1 = \lambda\mathbf{v}_1, \quad A\mathbf{v}_2 = a\mathbf{v}_1 + \lambda\mathbf{v}_2 \quad \Longrightarrow \quad e^A\mathbf{v}_1 = e^\lambda\mathbf{v}_1, \quad e^A\mathbf{v}_2 = ae^\lambda\mathbf{v}_1 + e^\lambda\mathbf{v}_2} \tag{3}$$

These two relations are sufficient for the $n=2$ case.

(6) In order to compute $e^A$ for an arbitrary square matrix, one makes use of the relation

$$e^{B^{-1}DB} = B^{-1}e^D B$$

and (eq1). The exponential of the matrix $D$ as in (eq1) can be computed directly from the definition. This approach is analogous to the one described in Section 9.8: if $\{\mathbf{v}_1(t),\ldots,\mathbf{v}_n(t)\}$ is a fundamental set of solutions for the ODE, then

$$\boxed{Y(t)=\begin{pmatrix} | & \ldots & | \\ \mathbf{v}_1(t) & \ldots & \mathbf{v}_n(t) \\ | & \ldots & | \end{pmatrix} \qquad\Longrightarrow\qquad e^{tA}=Y(t)Y(0)^{-1}} \tag{4}$$

On the other hand, if $A$ has only one eigenvalue $\lambda$, $(A-\lambda I)^n$ is the zero matrix, and the power series for the exponential of $A-\lambda I$ quickly truncates. Since $\lambda I$ commutes with all matrices, one can compute $e^A$ by using (eq2) with $A=\lambda I$ and $B=A-\lambda I$.

## Systems of Linear ODEs with Constant Coefficients

(1) A system of first-order linear ODEs with constant coefficients can be written as

$$\mathbf{y}'=A\mathbf{y}+\mathbf{f},\quad \mathbf{y}=\mathbf{y}(t)=\begin{pmatrix} y_1(t)\\ \vdots \\ y_n(t)\end{pmatrix}, \quad \text{where} \quad A=\begin{pmatrix} a_{11} & \ldots & a_{1n}\\ \vdots & & \vdots\\ a_{n1} & \ldots & a_{nn}\end{pmatrix},\quad \mathbf{f}=\mathbf{f}(t)=\begin{pmatrix} f_1(t)\\ \vdots \\ f_n(t)\end{pmatrix}.$$

This system is called *homogeneous* if $\mathbf{f}=0$. A system of first-order linear ODEs with constant coefficients can be solved by the integrating factor method for first-order linear ODEs:

$$\boxed{\mathbf{y}'=A\mathbf{y}+\mathbf{f} \qquad\Longrightarrow\qquad \mathbf{y}(t)=e^{tA}\mathbf{v}+e^{tA}\int_{t_0}^t e^{-sA}\mathbf{f}(s)\,ds,\quad \mathbf{v}\in\mathbb{R}^n} \tag{5}$$

Note that the function $\mathbf{y}_h=\mathbf{y}_h(t)$ defined by (eq5) with $\mathbf{f}=\mathbf{0}$, i.e. the first term on the right-hand side, is the general solution of the corresponding homogeneous system of ODEs. Thus, the general solution to an inhomogeneous system of ODEs is given by

$$\boxed{\mathbf{y}'=A\mathbf{y}+\mathbf{f} \qquad\Longrightarrow\qquad \mathbf{y}=\mathbf{y}_p+\mathbf{y}_h} \tag{6}$$

where $\mathbf{y}_p$ is a solution to the inhomogeneous system, e.g. the function $\mathbf{y}$ corresponding to $\mathbf{v}=\mathbf{0}$ to (eq5). The relation (eq6) is valid for any system of linear ODEs, with constant or non-constant coefficients.

(2) The main difficulty in solving a system of linear ODEs with constant coefficients is dealing with the terms in (eq5) involving $e^{tA}$. This is not difficult to do if there is a basis for $\mathbb{R}^n$, or $\mathbb{C}^n$, of eigenvectors for $A$:

$$\boxed{\begin{array}{l} \mathbf{y}'=A\mathbf{y} \qquad\Longrightarrow\qquad \mathbf{y}(t)=C_1e^{\lambda_1 t}\mathbf{v}_1+\ldots+C_ne^{\lambda_n t}\mathbf{v}_n,\quad C_1,\ldots,C_n\in\mathbb{R}\ (\text{or }\mathbb{C})\\ \quad\text{if}\quad \{\mathbf{v}_1,\ldots,\mathbf{v}_n\}\text{ is a basis for }\mathbb{R}\ (\text{or }\mathbb{C})\quad\text{and}\quad A\mathbf{v}_1=\lambda_1\mathbf{v}_1,\ldots,A\mathbf{v}_n=\lambda_n\mathbf{v}_n\end{array}} \tag{7}$$

(3) If we are looking for real solutions, we will need to rewrite (eq7) in a different way if some of the eigenvalues $\lambda_i$ are complex, and not real. If $\mathbf{v}_i$ is an eigenvector for $A$ with eigenvalue $\lambda_i$ and

$\lambda_i$ is complex, $\bar{\mathbf{v}}_i$ is an eigenvector for $A$ with eigenvalue $\bar{\lambda}_i$ and the vectors $\mathbf{v}_i$ and $\bar{\mathbf{v}}_i$ are linearly independent. Thus, if $n = 2$ and $A$ has an eigenvector $\mathbf{v}_1$ with a complex eigenvalue $\lambda_1$, then the two eigenvalues of $A$ are complex conjugates, $\lambda_1, \lambda_2 = a \pm ib$, and $\mathbb{C}^2$ has a basis of conjugate eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2 = \mathbf{w}_1 \pm i\mathbf{w}_2\}$. The general solution in this case can be written as

$$\mathbf{y}' = A\mathbf{y} \implies \begin{aligned} \mathbf{y}(t) &= \big(A_1 \cos bt + A_2 \sin bt\big)e^{at}\mathbf{w}_1 + \big(A_2 \cos bt - A_1 \sin bt\big)e^{at}\mathbf{w}_2, \\ &= e^{at}(\mathbf{w}_1\ \mathbf{w}_2)\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \end{aligned} \qquad A_1, A_2 \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}$$
$$\text{if} \quad \lambda_1 = a + ib, \quad b \neq 0, \quad \mathbf{v}_1 = \mathbf{w}_1 + i\mathbf{w}_2 \neq 0, \text{ and } A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$$

This expression is obtained by setting $C_1, C_2 = (A_1 \mp iA_2)/2$ in (eq7). Note that if $A_1$ and $A_2$ are arbitrary complex constants, so are $C_1$ and $C_2$. On the other hand, the solution corresponding to $A_1$ and $A_2$ is real if and only if $A_1$ and $A_2$ are real.

(3) Another potential problem with (eq7) is that $\mathbb{R}^n$, or $\mathbb{C}^n$, may not have a basis of eigenvectors for $A$. If so, we can use a basis of generalized eigenvectors. If $n = 2$ and $A$ has only one eigenvalue $\lambda$, by (eq3),

$$\mathbf{y}' = A\mathbf{y} \implies \mathbf{y}(t) = \big(C_1 e^{\lambda t} + C_2 ate^{\lambda t}\big)\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2, \quad C_1, C_2 \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}$$
$$\text{if} \quad \mathbf{v}_1, \mathbf{v}_2 \text{ are lin. indep.}, \ A\mathbf{v}_1 = \lambda\mathbf{v}_1, \text{ and } A\mathbf{v}_2 = a\mathbf{v}_1 + \lambda\mathbf{v}_2$$

Once an eigenvector $\mathbf{v}_1$ for the eigenvalue $\lambda$ is found, $\mathbf{v}_2$ can be taken to be any vector in $\mathbb{R}^2$ which is not a multiple of $\mathbf{v}_1$, and the number $a$ is determined by computing $A\mathbf{v}_2$.

(4) The general solution to an inhomogeneous system of linear first-order ODEs with constant-coefficients is given by (eq5), or more generally by

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f} \implies \mathbf{y}(t) = (e^{tA}B)\mathbf{v} + (e^{tA}B)\int_{t_0}^{t}(e^{sA}B)^{-1}\mathbf{f}(s)\,ds, \quad \mathbf{v} \in \mathbb{R}^n \tag{8}$$

for any invertible $n \times n$-matrix $B$. For a good choice of $B$, the product $e^{tA}B$ may be easier to compute than $e^{tA}$. For example,

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f} \implies \mathbf{y}(t) = Y(t)\mathbf{v} + Y(t)\int_{t_0}^{t} Y(s)^{-1}\mathbf{f}(s)\,ds, \quad \mathbf{v} \in \mathbb{R}^n$$
$$\text{if} \quad Y = Y(t) \text{ is a fundamental matrix for } \mathbf{y}' = A\mathbf{y} \text{ as in (eq4)}$$

(5) A solution to an initial value problem can be obtained directly by

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \ \ \mathbf{y}(t_0) = \mathbf{y}_0 \implies \mathbf{y}(t) = e^{tA}\big(e^{-t_0 A}\mathbf{y}_0 + \int_{t_0}^{t} e^{-sA}\mathbf{f}(s)\,ds\big)$$

More generally, if $Y = Y(t)$ is any fundamental matrix for $\mathbf{y}' = A\mathbf{y}$,

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \ \ \mathbf{y}(t_0) = \mathbf{y}_0 \implies \mathbf{y}(t) = Y(t)\big(Y(t_0)^{-1}\mathbf{y}_0 + \int_{t_0}^{t} Y(s)^{-1}\mathbf{f}(s)\,ds\big)$$

## Qualitative Descriptions

(1) As is the case for linear ODEs, every initial-value problem

$$\mathbf{y}' = A\mathbf{y} + \mathbf{f}, \quad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad A = A(t), \quad \mathbf{f} = \mathbf{f}(t), \tag{9}$$

has a unique solution, provided the functions $A$ and $\mathbf{f}$ are continuous near $t_0$. Furthermore, the interval of the existence of the solution to (eq9) is the largest interval on which $A$ and $\mathbf{f}$ are defined. If $A$ is a constant matrix, it follows that the phase-space solution curves for the system $\mathbf{y}' = A\mathbf{y}$ do not intersect. *Can you explain why?*

(2) Every homogeneous system of linear ODEs $\mathbf{y}' = A\mathbf{y}$ has an equilibrium solution, $\mathbf{y}(t) = \mathbf{0}$. This solution can be *asymptotically stable*, *stable*, or *unstable*. If $A$ is a constant matrix and the real part of every eigenvalue of $A$ is negative, all solutions $\mathbf{y} = \mathbf{y}(t)$ approach $\mathbf{0}$ at $t \longrightarrow \infty$, and thus $\mathbf{0}$ is an *asymptotically stable* equilibrium point of the system. If the real parts of some eigenvalues of $A$ are negative and of some are zero, some solutions $\mathbf{y} = \mathbf{y}(t)$ approach $\mathbf{0}$ at $t \longrightarrow \infty$, while others approach closed orbits. In this case, $\mathbf{0}$ is a *stable* equilibrium point of the system, as every solution starting near $\mathbf{0}$ stays near $\mathbf{0}$. Finally, if the real part of at least one eigenvalue of $A$ is positive, some solutions $\mathbf{y} = \mathbf{y}(t)$ move away from $\mathbf{0}$ and approach $\infty$ at $t \longrightarrow 0$, and thus $\mathbf{0}$ is an *unstable* equilibrium point of the system.

(3) If $A$ is a constant matrix, the system $\mathbf{y}' = A\mathbf{y}$ is *autonomous*, i.e. it does not involve $t$ explicitly. Thus, if $\mathbf{y} = \mathbf{y}(t)$ is a solution to this system, so is $\tilde{\mathbf{y}}(t) = \mathbf{y}(t-c)$. The latter solution traces the same curve $\mathbf{y}(t)$ in $\mathbb{R}^n$, but is delayed by time $c$. For this reason, the qualitative behavior of solutions of $\mathbf{y}' = A\mathbf{y}$ is well represented by the non-intersecting curves $\mathbf{y}(t)$ traced out in the *phase space*, i.e. $\mathbb{R}^n$. For some sketches in the $n = 2$ case, see Figures 2-4 of the solutions to PS4.

(4) While systems of first-order ODEs arise in applications by themselves, they can also be used to replace high-order ODEs. For example, the initial value problem

$$y''' + y'y'' + ty = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1, \quad y''(t_0) = y_2,$$

is equivalent to the initial value problem

$$\begin{pmatrix} y \\ u \\ v \end{pmatrix}' = \begin{pmatrix} u \\ v \\ -uv - ty \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} y(0) \\ u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}.$$

*Can you explain why?* Such replacements are often useful, because many numerical methods and methods of qualitative analysis apply only to first-order ODEs and systems of first-order ODEs.