Math53: Ordinary Differential Equations
Winter 2004

Solutions to Problem Set 1

Problem 1 (20pts)

(a; **2pts**) *State the two Fundamental Theorems of Calculus.*
(a-i) If F is a continuously differentiable function on the interval (a,b) and $t_0 \in (a,b)$, then

$$F(t) = F(t_0) + \int_{t_0}^{t} F'(s)ds \qquad \text{for all} \quad t \in (a,b). \tag{1}$$

(a-ii) If f is a continuous function on the interval (a,b), $t_0 \in (a,b)$,

$$\text{and} \qquad F(t) \equiv \int_{t_0}^{t} f(s)ds \qquad \text{for all} \quad t \in (a,b),$$
$$\text{then} \qquad F'(t) = f(t) \qquad \text{for all} \quad t \in (a,b).$$

(b; **2pts**) *State the chain rule for the one-variable differentiation.*
If f and g are continuously differentiable functions on (a,b) and (c,d), respectively, and $a < g(t) < b$ for all $t \in (c,d)$, then the function

$$h(t) \equiv f(g(t)), \qquad t \in (c,d),$$

is defined and continuously differentiable on (c,d) and

$$h'(t) = f'(g(t)) \cdot g'(t) \qquad \text{for all} \quad t \in (c,d).$$

(c; **2pts**) *State the product rule for the one-variable differentiation.*
If f and g are continuously differentiable functions on (a,b), then the function

$$h(t) \equiv f(t) \cdot g(t), \qquad t \in (a,b),$$

is also continuously differentiable and

$$h'(t) = f'(t) \cdot g(t) + f(t) \cdot g'(t) \qquad \text{for all} \quad t \in (a,b). \tag{2}$$

(d; **1pt**) *If a is a real number and $f(x) = x^a$, what is $f'(x)$? (no proof necessary)*

$$f'(x) = a \cdot x^{a-1}.$$

(e; **1pt**) *If $f(x) = e^x$, what is $f'(x)$? (no proof necessary)*

$$f'(x) = e^x.$$

(f; **3pts**) *State the quotient rule for the one-variable differentiation. Deduce it from (b)-(d).*
If f and g are continuously differentiable functions on (a,b) and $g(t)\neq 0$ for all $t\in(a,b)$, then the function

$$h(t) \equiv f(t)/g(t), \qquad t \in (a,b),$$

is also continuously differentiable and

$$h'(t) = \frac{f'(t)}{g(t)} - \frac{f(t)\cdot g'(t)}{g(t)^2} \qquad \text{for all} \;\; t\in(a,b). \tag{3}$$

In order to prove (3), we apply (c) to the functions f and $G(t)=1/g(t)$. Since $h=f\cdot G$,

$$h'(t) = f'(t)\cdot G(t) + f(t)\cdot G'(t) = \frac{f'(t)}{g(t)} + f(t)\cdot G'(t). \tag{4}$$

In order to compute $G'(t)$, we apply (b) to the functions $y(x)=x^{-1}$ and g and use (d) with $a=-1$. Since $G(t)=y(g(t))$,

$$G'(t) = y'(g(t))\cdot g'(t) = (-1)\cdot g(t)^{-2}\cdot g'(t) = -\frac{g'(t)}{g(t)^2}. \tag{5}$$

The quotient rule, i.e. (3), is obtained by plugging (5) into (4).

(g; **3pts**) *State the change-of-variables formula for the one-variable integration. Deduce it from (a) and (b).*
If f is a continuous function on (a,b), g is a continuously differentiable function on (c,d) such that $a<g(t)<b$ for all $t\in(c,d)$, $t_0\in(c,d)$,

$$\text{and} \qquad F(x) \equiv \int_{g(t_0)}^{x} f(y)dy \qquad \text{for all} \;\; x\in(a,b), \tag{6}$$

$$\text{then} \qquad \int_{t_0}^{t} f(g(s))\cdot g'(s)ds = F(g(t)) \qquad \text{for all} \;\; t\in(c,d). \tag{7}$$

By (c) applied to F, g, and $h(t)=F(g(t))$,

$$h'(t) = F'(g(t))\cdot g'(t). \tag{8}$$

Since $h(t_0)=0$ by (6), the change-of-variables formula, i.e. (7), follows from (1) and (8).

(h; **2pts**) *State the integration-by-parts formula for the one-variable integration. Deduce it from (a) and (c).*
If f and g are continuously differentiable functions on (a,b) and $t_0\in(c,d)$,

$$\int_{t_0}^{t} f(s)\cdot g'(s)ds = \big(f(t)g(t)-f(t_0)g(t_0)\big) - \int_{t_0}^{t} f'(s)\cdot g(s)ds \qquad \text{for all} \;\; t\in(a,b). \tag{9}$$

Rearranging (2) with $h(s)=f(s)g(s)$, we obtain

$$f(s)\cdot g'(s) = h'(s) - f'(s)\cdot g(s) \qquad \text{for all} \;\; s\in(a,b). \tag{10}$$

The integration-by-parts formula, i.e. (9), is obtained by integrating both sides of (10) and applying (1) to the middle term.

(i; **3pts**) *Suppose $a = a(t)$ is a smooth function, c is a real number,*

$$f(t) = \int_c^t a(s)ds, \qquad \text{and} \qquad h(t) = e^{f(t)}.$$

Compute $h'(t)$, using (a), (b), and (e).
We apply (b) to the functions $F(x) = e^x$ and $G(t) = f(t)$. Since $h(t) = F(G(t))$,

$$h'(t) = F'(G(t)) \cdot G'(t) = e^{G(t)} \cdot G'(t) = e^{f(t)} \cdot f'(t) = f'(t) \cdot e^{f(t)}. \tag{11}$$

The second equality in (11) is a consequence of (e). Since $f'(t) = a(t)$ by (a-ii),

$$h'(t) = a(t) \cdot e^{f(t)}. \tag{12}$$

(j; **1pt**) *Find a nontrivial first-order differential equation which is solved by the function $h = h(t)$ of (i).*
Since $h = e^f$, by (12),

$$h'(t) = a(t) \cdot h(t) \qquad \text{or} \qquad h' = a \cdot h, \quad h = h(t).$$

This is the simplest possible nontrivial ODE satisfied by h.

Note: There are a number of ways of phrasing (a)-(c) and (f)-(h).

Section 1.3, Problems 4 and 23 (14pts)

1.3: 4; 6pts: *Find the general solution of the differential equation*

$$y' = 2\sin 3t - \cos 5t.$$

Indicate the interval of existence and sketch at least two members of the family of solution curves.
By FTC, e.g. (a-i) of Problem 1,

$$y(t) = \int \big(2\sin 3t - \cos 5t\big)dt = \boxed{-\tfrac{2}{3}\cos 3t - \tfrac{1}{5}\sin 5t + C}$$

Since y is defined for all t, the interval of existence is $\boxed{(-\infty, \infty)}$ Three solution curves are shown in Figure 1. The most important feature here is that the three graphs differ by vertical shifts.

1.3: 23; 8pts: *Find the solution the initial value problem*

$$y' = \frac{t+1}{t(t+4)}, \qquad y(-1) = 0.$$

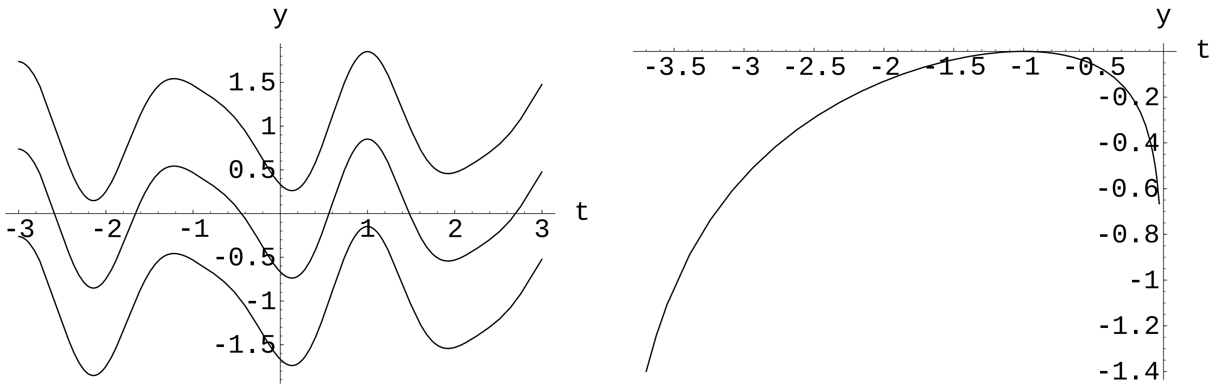

Figure 1: Plots for Problems 1.3:4 and 1.3:23

Indicate the interval of existence and sketch the solution. By FTC, e.g. (a-i) of Problem 1,

$$y(t) = y(-1) + \int_{-1}^{t} \frac{s+1}{s(s+4)} ds = 0 + \int_{-1}^{t} \frac{1}{4}\Big(\frac{1}{s} + \frac{3}{s+4}\Big) ds$$
$$= \frac{1}{4}\big(\ln|s| + 3\ln|s+4|\big)\Big|_{s=-1}^{s=t} = \frac{1}{4}\ln|t| + \frac{3}{4}\ln|t+4| - \frac{3}{4}\ln 3.$$

The last expression is defined on the intervals $(-\infty,-4)$, $(-4,0)$, and $(0,-\infty)$. Since the initial value of the parameter lies in the middle interval, the solution to the initial value problem is

$$\boxed{y(t) = \tfrac{1}{4}\ln|t| + \tfrac{3}{4}\ln(t+4) - \tfrac{3}{4}\ln 3, \qquad t \in (-4,0)}$$

The solution curve is shown in Figure 1. Note that y approaches $-\infty$ as t tends to -4 and 0, and the curve passes through the point $(-1,0)$, as required by the initial condition.

Section 2.1, Problems 8 and 18 (12pts)

2.1: 8 (a; **2pts**) *Use implicit differentiation to show that $t^2+y^2=C^2$ implicitly defines solutions of the differential equation $t+yy'=1$.*
We differentiate both sides of $t^2+y^2=C^2$ with respect to t. The derivative of RHS is 0. On the other hand, by the chain rule, i.e. (b) of Problem 1,

$$\frac{d}{dt}\big(t^2+y^2\big) = 2t + 2y \cdot y' = 2(t+yy').$$

Comparing the derivatives of the two sides, we obtain $t+yy'=0$, as needed.
(b; **2pts**) *Solve $t^2+y^2=C^2$ for y in terms of t to find explicit solutions. Show that these functions are also solutions of $t+yy'=0$.*
Solving for y, we obtain $y(t)=\pm\sqrt{C^2-t^2}$. By the chain rule,

$$y'(t) = \pm\frac{d}{dt}(C^2-t^2)^{1/2} = \pm\frac{1}{2}(C^2-t^2)^{1/2-1}\cdot(-2t) = \pm\frac{-t}{\sqrt{C^2-t^2}}$$
$$\Longrightarrow \qquad t + yy' = t + \big(\pm\sqrt{C^2-t^2}\big)\Big(\pm\frac{-t}{\sqrt{C^2-t^2}}\Big) = 0.$$

Figure 2: Plots for Problems 2.1:8 and 2.1:18

(c; **2pts**) *Discuss the interval of existence of each solution in part (b).*
We need $C^2 - t^2 \geq 0$. Thus, $t \in (-C, C)$ for $C > 0$.
(d; **1pt**) *Sketch the solutions in part (b) for $C = 1, 2, 3, 4$.*
The solution curves are circles of radii $1, 2, 3, 4$ centered at the origin as in the first diagram in Figure 2.

2.1: 18; 5pts: *Plot the direction field for the ODE $y' = y^2 - t$ by drawing short lines of the appropriate slope centered at the integer valued coordinates (t, y), where $-2 \leq t \leq 2$ and $-1 \leq y \leq 1$.*
The second plot in Figure 2 shows short lines of the slope $y' = y^2 - t$ at fifteen points. Notice that the top and bottom row look the same.

Section 2.2, Problems 4,12,14,18 (26pts)

2.2: 4; 5pts: *Find the general solution of the equation $y' = (1+y^2)e^x$.*
Write $y' = \frac{dy}{dx}$ and split the variables:

$$
\begin{aligned}
y' = (1+y^2)e^x \quad &\Longleftrightarrow \quad \frac{dy}{1+y^2} = e^x dx \quad \Longleftrightarrow \quad \int \frac{dy}{1+y^2} = \int e^x dx \\
&\Longleftrightarrow \quad \tan^{-1} y = e^x + C \quad \Longleftrightarrow \quad \boxed{y = \tan(e^x + C)}
\end{aligned}
$$

2.2: 12; 5pts: *Find the general solution of the equation $y' = (2xy + 2x)/(x^2 - 1)$.*
Write $y' = \frac{dy}{dx}$ and split the variables:

$$
\begin{aligned}
y' = \frac{2x(y+1)}{x^2-1} \quad &\Longleftrightarrow \quad \frac{dy}{y+1} = \frac{2x dx}{x^2-1} \quad \Longleftrightarrow \quad \int \frac{dy}{y+1} = \int \frac{2x}{x^2-1} dx \\
&\Longleftrightarrow \quad \ln|y+1| = \ln\left|x^2-1\right| + C \\
&\Longleftrightarrow \quad |y+1| = e^C |x^2-1| \quad \Longleftrightarrow \quad \boxed{y = -1 + C(x^2-1),\ x \neq \pm 1}
\end{aligned}
$$

2.2: 14; 8pts: *Find the exact solution, including the interval of existence, to the initial value problem*

$$
y' = -\frac{2t(1+y^2)}{y}, \qquad y(0) = 1.
$$

Write $y' = \frac{dy}{dx}$ and split the variables:

$$y' = -\frac{2t(1+y^2)}{y} \quad \Longleftrightarrow \quad \frac{ydy}{1+y^2} = -2tdt \quad \Longleftrightarrow \quad \int \frac{ydy}{1+y^2} = -\int 2tdt$$
$$\Longleftrightarrow \quad \frac{1}{2}\ln(1+y^2) = -t^2 + C \quad \Longleftrightarrow \quad 1+y^2 = e^{2C}e^{-2t^2}$$

Since $y(0)=1$, $1+1=e^{2C} \cdot 1$, $e^{2C}=2$. Thus,

$$\boxed{y = \sqrt{2e^{-2t^2}-1}, \quad t \in (-\sqrt{(\ln 2)/2}, \sqrt{(\ln 2)/2})}$$

We must take the positive square root in order to satisfy the initial condition.

2.2: 18; 8pts: *Find the exact solution, including the interval of existence, to the initial value problem*

$$y' = \frac{x}{1+2y}, \qquad y(-1) = 0.$$

Write $y' = \frac{dy}{dx}$ and split the variables:

$$y' = \frac{x}{1+2y} \quad \Longleftrightarrow \quad (1+2y)dy = xdx \quad \Longleftrightarrow \quad \int (1+2y)dy = \int x\,dx$$
$$\Longleftrightarrow \quad y + y^2 = \frac{1}{2}x^2 + C.$$

Since $y(-1)=0$, $0+0=(1/2)+C$, $C=-1/2$. Thus,

$$\boxed{y = \tfrac{1}{2}\big(-1+\sqrt{2x^2-1}\big), \quad x \in (-\infty, -1/\sqrt{2})}$$

In order to satisfy the initial condition, we must take the positive square root.

Section 2.3, Problem 4 (8pts)

A rocket ascends vertically with constant acceleration $a=100m/s^2$ for $t_1=1\,min$. The rocket motor is then shut-off and the rocket continues upward under the influence of gravity. Find the maximum altitude y_m reached by the rocket and the total time T elapsed from the take-off until the rocket returns to the ground.

The upward velocity $v=v(t)$ is described by

$$v'(t) = a \;\text{ if }\; t \in (0, t_1), \qquad v'(t) = -g \;\text{ if }\; t \in (t_1, T).$$

Integrating the two equations, we obtain

$$v(t) = v(0) + \int_0^t a\,ds = at \qquad \text{if } t \in (0, t_1),$$
$$v(t) = v(t_1) + \int_{t_1}^t (-g)\,ds = at_1 - g\cdot(t-t_1) = (a+g)t_1 - gt \qquad \text{if } t \in (t_1, T).$$

Since $y'(t)=v(t)$, integrating again, we obtain $y(t)=\frac{1}{2}at^2$ if $t\in(0,t_1)$, and

$$\begin{aligned} y(t) &= y(t_1) + \int_{t_1}^{t} v(t_1)\,ds = \frac{1}{2}at_1^2 + (a+g)t_1\cdot(t-t_1) - \frac{1}{2}g\cdot(t^2-t_1^2) \\ &= -\frac{1}{2}(a+g)t_1^2 + (a+g)t_1t - \frac{1}{2}gt^2 \qquad\qquad \text{if} \quad t\in(t_1,T). \end{aligned}$$

The maximum altitude is reached at the time $t_2\in(t_1,T)$ such that

$$v(t_2)=0 \quad\Longleftrightarrow\quad (a+g)t_1 - gt_2 = 0 \quad\Longleftrightarrow\quad t_2=\frac{a+g}{g}t_1 \Longrightarrow y_m = y(t_2) = \frac{a^2+ag}{2g}t_1^2.$$

The rocket returns to the ground at the time $T>t_1$ such that

$$y(T) = -\frac{1}{2}(a+g)t_1^2 + (a+g)t_1T - \frac{1}{2}gT^2 = 0 \quad\Longrightarrow\quad T = \frac{a+g+\sqrt{a(a+g)}}{g}t_1.$$

In order to satisfy the condition $T>t_1$, we must take the positive square root. *Can you check that the smaller root does not satisfy this inequality, if $a>0$?* Plugging in $a=100$, $g=9.8$, and $t_1=60$ into the above expressions for y_m and T, we conclude that

$$\boxed{y_m \approx 2,016,735 \ m., \qquad T \approx 1,314 \ sec.}$$

Section 2.4, Problems 2,6,13,14,18 (32pts)

2.4: 2; 5pts: *Find the general solution of the first-order linear ODE $y'-3y=5$.*
The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int(-3)dt} = e^{-3t}; \Longrightarrow \quad y'-3y=5 \quad\Longleftrightarrow\quad e^{-3t}y'-3e^{-3t}y=5e^{-3t} \quad\Longleftrightarrow\quad (e^{-3t}y)'=5e^{-3t} \\ \Longleftrightarrow\quad e^{-3t}y(t) = 5\int e^{-3t}dt = -\frac{5}{3}e^{-3t}+C \quad\Longleftrightarrow\quad \boxed{y(t) = -\tfrac{5}{3}+Ce^{3t}} \end{aligned}$$

2.4: 6; 5pts: *Find the general solution of the first-order linear ODE $tx'=4x+t^4$.*
First, rewrite this equation as $x'-4t^{-1}x=t^3$. The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t)=e^{\int(-4t^{-1})dt}=e^{-4\ln t}=t^{-4}; \Longrightarrow \quad x'-4t^{-1}x=t^3 \quad\Longleftrightarrow\quad t^{-4}x'-4t^{-5}x=t^{-1} \quad\Longleftrightarrow\quad \left(t^{-4}x\right)'=t^{-1} \\ \Longleftrightarrow\quad t^{-4}x(t) = \int t^{-1}dt = \ln|t|+C \quad\Longleftrightarrow\quad \boxed{x(t) = t^4\ln|t|+Ct^4} \end{aligned}$$

2.4: 13 (a; **4pts**) *Solve the ODE $y'+y\cos t=\cos t$, using the integrating factor approach.*
The integrating factor $P(t)$ is given by

$$P(t)=e^{\int \cos t dt}=e^{\sin t}; \Longrightarrow \quad y'+y\cos t=\cos t \iff e^{\sin t}(y'+y\cos t)=e^{\sin t}\cos t$$
$$\iff (e^{\sin t}y)'=e^{\sin t}\cos t \iff e^{\sin t}y=\int e^{\sin t}\cos t dt$$
$$\iff \boxed{y(t)=1+Ce^{-\sin t}}$$

(b; **4pts**) *Solve the ODE $y'+y\cos t=\cos t$, using the separation of variables approach. Discuss any discrepancies between this solution and the solution found in part (a).*

$$y'+y\cos t=\cos t \iff \frac{dy}{dt}=(1-y)\cos t$$
$$\frac{dy}{1-y}=\cos x dx \iff -\ln|1-y|=\sin x+C$$
$$\iff |1-y|=e^{-\sin x-C} \iff 1-y=\pm e^{-C}e^{-\sin x}$$

Now let $A=\pm e^{-C}$ vary over all real numbers except zero, and note that after separating the variables we divided by $y-1$, which hints that $y=1$ (the constant function) is another solution. *Check it!* So we can let A vary over all real numbers, including zero, to get the solution in closed form:

$$\boxed{y(t)=1-Ae^{-\sin t}}$$

Note that this is the same result as the one in part (a), since A and C vary over the set of reals.

2.4: 14; 6pts: *Find the solution to the initial value problem $y'=y+2te^{2t}$, $y(0)=3$.*
First, rewrite the ODE as $y'-y=2te^{2t}$. The integrating factor $P(t)$ is given by

$$P(t)=e^{\int(-1)dt}=e^{-t}; \Longrightarrow \quad y'-y=2te^{t} \iff e^{-t}y'-e^{-t}y=2te^{t} \iff (e^{-t}y)'=2t$$
$$\iff e^{-t}y(t)=2\int te^{t}dt=2\big(te^{t}-\int e^{t}dt\big)=2te^{t}-2e^{t}+C.$$

Since $y(0)=3$, $1\cdot 3=-2+C$, and $C=5$. Thus, $\boxed{y(t)=2te^{2t}-2e^{2t}+5e^{t}}$

2.4: 18; 8pts: *Find the solution, including the interval of existence, to the initial value problem $ty'+2y=\sin t$, $y(\pi/2)=0$, and sketch it.*
Rewrite the ODE as $y'+2t^{-1}y=t^{-1}\sin t$. The integrating factor $P(t)$ is given by

$$P(t)=e^{\int 2t^{-1}dt}=e^{2\ln|t|}=t^2 \Longrightarrow \quad y'+2t^{-1}y=t^{-1}\sin t \iff t^2y'+2ty=t^2\sin t \iff (t^2y)'=t\sin t$$
$$\iff t^2y(t)=\int t\sin t\,dt=-t\cos t+\int\cos t\,dt=-t\cos t+\sin t+C.$$

Since $y(\pi/2)=0$, $(-\pi/2)\cdot 0+1+C=0$, and $C=-1$. Thus,

$$\boxed{y(t)=\frac{\sin t-t\cos t-1}{t^2}}$$

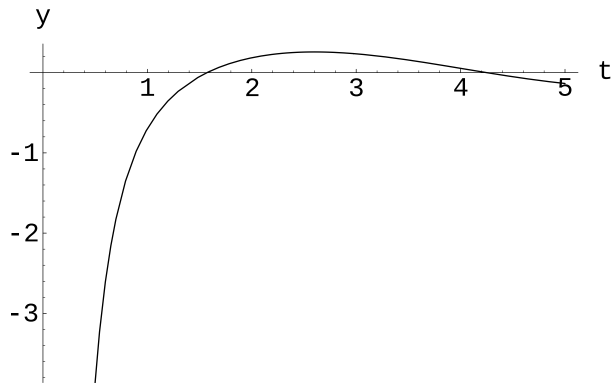

Figure 3: Solution Curve for Problem 2.4:18

Since the solution cannot be extended to zero, the interval of existence is $\boxed{(0,\infty)}$ Figure 3 shows the solution curve. Its main features are that the curve approaches $-\infty$ as t tends to 0, passes through the point $(\pi/2, 0)$, as required by the initial condition, and is close to the x-axis for large values of t.

Section 2.5, Problem 4 (8pts)

A tank contains $V = 500$ Gls of a salt-water solution at the concentration $\rho_0 = .05$ lbs/Gl. Pure water is poured into the tank, and a drain at the bottom is adjusted so as to keep the volume of solution constant. At what rate r should the water be poured into the tank to lower the concentration to $\rho_1 = .01$ lbs/Gl in $t_1 = 1$ Hr.

Let $y(t)$ be the amount of salt at time t and $\rho(t) = y(t)/V$ be the salt concentration. Then,

$$y'(t) = 0 - \rho(t) \cdot r,$$

since no salt is coming in, while it is leaving at the rate of $\rho(t) \cdot r$. Thus,

$$\rho' = -\frac{r}{V}\rho.$$

Since r/V is constant, the general solution to this equation is

$$\rho(t) = Ce^{-(r/V)t}.$$

Since $\rho(0) = \rho_0$ and $\rho(t_1) = \rho_1$,

$$C = \rho_0 \quad \Longrightarrow \quad \rho_1 = \rho_0 e^{-(r/V)t_1} \quad \Longrightarrow \quad r = \frac{V}{t_1}\ln(\rho_0/\rho_1).$$

Plugging in $t_1 = 60$, $\rho_0 = .05$, $V = 500$, we obtain $\boxed{r \approx 13.4 \; Gls/min}$

Section 2.6, Problems 10,14,26,36 (25pts)

2.6: 10; 6pts: *Determine whether the equation* $(1 - y\sin x)dx + (\cos x)dy = 0$ *is exact, and, if it is, solve it.*
For $P(x, y) = 1 - y\sin x$ and $Q(x, y) = \cos x$, we get:

$$\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x}$$

so the equation is exact. We solve it by setting

$$F(x, y) = \int P(x)dx = \int (1 - y\sin x)dx = x + y\cos x + \phi(y)$$

$$Q(x, y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y)$$

so that $\phi' = 0$, ϕ is constant, and the solution is $\boxed{F(x, y) = x + y\cos x = C}$

2.6: 14; 3pts: *Determine whether the equation* $dy/dx = x/(x - y)$ *is exact, and, if it is, solve it.*
Rewrite the equation as $(x)dx + (y - x)dy = 0$, so that $P(x, y) = x$ and $Q(x, y) = y - x$. Then:

$$\frac{\partial P}{\partial y} = 0 \neq -1 = \frac{\partial Q}{\partial x}$$

so the given equation is not exact.

2.6: 26; 8pts: *The equation* $y\,dx + (x^2y - x)\,dy = 0$ *is not exact. Suppose it has an integrating factor that is a function of* x *alone. Find the integrating factor and use it to solve the equation.*
Let $\mu(x)$ be the integrating factor, so the equation becomes

$$\mu(x)y\,dx + \mu(x)(x^2y - x)\,dy = 0$$

In order for this equation to be exact, we need:

$$\begin{aligned} &\frac{\partial}{\partial y}(\mu(x)y) = \frac{\partial}{\partial x}(\mu(x)(x^2y - x)) \\ \Longrightarrow\ &\mu(x) = \mu'(x)(x^2y - x) + \mu(x)(2xy - 1) \\ \Longrightarrow\ &2\mu(x)(1 - xy) = x\mu'(x)(xy - 1) \\ \Longrightarrow\ &\mu(x) = -\frac{1}{2}x\mu'(x) \end{aligned}$$

so $\mu(x) = 1/x^2$ is an integrating factor. After multiplying the equation by $\mu(x)$, we get the exact equation

$$\begin{aligned} &\frac{y}{x^2}\,dx + \left(y - \frac{1}{x}\right)\,dy = 0 \\ \Longrightarrow\ &F(x, y) = \int \frac{y}{x^2}dx + \phi(y) = -\frac{y}{x} + \phi(y) \end{aligned}$$

To find ϕ, differentiate F with respect to y:

$$\begin{aligned}
&\frac{\partial F}{\partial y}(x,y) = y - \frac{1}{x} = -\frac{1}{x} + \phi'(y) \\
\implies\ &\phi'(y) = y \implies \phi(y) = \frac{y^2}{2} + C \\
\implies\ &F(x,y) = -\frac{y}{x} + \frac{y^2}{2} \implies \boxed{-\tfrac{y}{x} + \tfrac{y^2}{2} + C = 0}
\end{aligned}$$

2.6: 36; 8pts: *Solve the homogeneous equation* $(x+y)dx + (y-x)dy = 0$
After making the substitution $y = xv$ we get

$$\begin{aligned}
(x+y)dx + (y-x)dy = 0 \iff\ & (1+v)x\,dx + (v-1)x(v\,dx + x\,dv) = 0 \\
\iff\ & (1+v^2)x\,dx + (v-1)x^2\,dv = 0 \iff \frac{dx}{x} + \frac{(v-1)dv}{1+v^2} = 0 \\
\iff\ & \int \frac{dx}{x} + \int \frac{v\,dv}{1+v^2} - \int \frac{dv}{1+v^2} = 0 \\
\implies\ & \ln|x| + \frac{1}{2}\ln|1+v^2| - \arctan v = C \\
\implies\ & \ln|x| + \frac{1}{2}\ln\left(\frac{x^2+y^2}{x^2}\right) - \arctan\left(\frac{y}{x}\right) + C = 0
\end{aligned}$$

Section 2.7, Problems 2,4,6,26 (16pts)

2.7: 2; 4pts: *Does the initial value problem* $y' = \sqrt{y}$, $y(4) = 0$ *satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?*
The equation is of the form $y' = f(t,y) = \sqrt{y}$. f is defined only on the half-plane $\{(t,y) : y \geq 0\}$ and it is continuous there. But $\partial f/\partial y = 1/(2\sqrt{y})$ is continuous only on the open half-plane $\{(t,y) : y > 0\}$. Hence, any rectangle in the (y,t)-plane containing the initial point $(4,0)$ contains points where $\partial f/\partial y$ is discontinuous, so the conditions of the theorem are not satisfied.

2.7: 4; 4pts: *Does the initial value problem* $\omega' = \omega\sin\omega + s$, $\omega(0) = -1$ *satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?*
The equation is of the form $\omega' = f(s,\omega) = \omega\sin\omega + s$. Function f is continuous in the whole plane, and so is its partial $\partial f/\partial\omega = \sin\omega + \omega\cos\omega$. In particular, any rectangle around the initial value point will satisfy the conditions of the theorem.

2.7: 6; 4pts: *Does the initial value problem* $y' = (y/x) + 2$, $y(0) = 1$ *satisfy the conditions of the Theorem on the uniqueness of solutions (Theorem 7.16 in the textbook)?*
The equation is of the form $y' = f(x,y) = (y/x) + 2$. Function f is continuous outside the line $x = 0$. The initial value point is $(0,1)$, so there is no rectangle containing it in which f is continuous, and the conditions for uniqueness of solution are not satisfied.

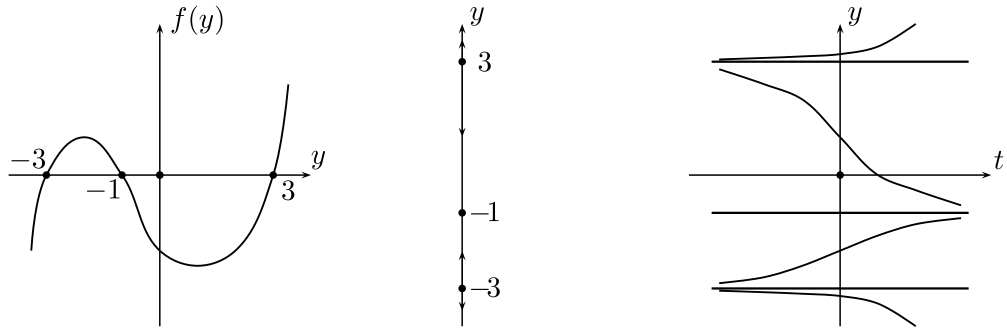

Figure 4: Plots for Problem 2.9:20: (i),(ii),(iii)

2.7: 26; 4pts: *Is it possible to find a function $f(t,x)$ that is continuous and has continuous partial derivatives such that the functions $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ are both solutions to the equation $x' = f(t,x)$ near $t = \pi/2$?*
Since f is continuous and has continuous partial derivatives in the entire (t,x)-plane, the equation $x' = f(t,x)$ satisfies the conditions of the uniqueness theorem. Notice that $x_1(\pi/2) = x_2(\pi/2) = 0$, so the curves $x_1(t) = \cos t$ and $x_2(t) = 1 - \sin t$ have a common point $(\pi/2, 0)$, so if they were both solutions of our equation, by the uniqueness theorem they would have to agree on any rectangle containing $(\pi/2, 0)$. Since they do not, they cannot both be solutions of the equation $x' = f(t,x)$.

Section 2.9, Problems 20,26,28 (23pts)

2.9: 20; 9pts: *For the autonomous differential equation $y' = f(y) = (y+1)(y^2-9)$, sketch a graph of $f(y)$ and use it to develop a phase line and classify each equilibrium point as either unstable or asymptotically stable. Sketch the equilibrium solutions in the (t,y)-plane and at least one solution trajectory in each plane region bounded by these equilibrium solutions.*
Function $f(y)$ factors as $f(y) = (y+3)(y+1)(y-3)$, so the equilibrium solutions are $y = -3$, $y = -1$ and $y = 3$, and the phase line is sketched by looking at the graph of f; see Figure 4. Since $f'(-3) = 12 > 0$, $f'(-1) = -8 < 0$, $f'(3) = 24 > 0$, the middle equilibrium is stable, and the other two are unstable. Solution trajectories in each of the four regions are sketched below.

2.9: 26; 9pts: *Solve the initial value problem $y' = (3 + y)(1 - y)$, $y(0) = 2$ and describe the behavior of the solution when $t \to \infty$.*
What is the long-term behavior of the solution?

$$y' = (3+y)(1-y) \iff \frac{dy}{(3+y)(1-y)} = dt \iff \frac{1}{4}\left(\frac{1}{3+y} + \frac{1}{1-y}\right) dy = dt$$
$$\iff \ln\frac{|3+y|}{|1-y|} = 4t + C \iff \frac{3+y}{1-y} = Ae^{4t}$$

From $y(0) = 2$ we get that $A = Ae^{4\cdot 0} = (3+2)/(1-2) = -5$, and we continue solving for y:

$$3 + y = -5(1-y)e^{4t} \iff y = \frac{3+5e^{4t}}{5e^{4t}-1} = \frac{3e^{-4t}+5}{5-e^{-4t}}$$

It follows that $\lim_{t\to\infty} y = (0+5)/(5-0) = 1$. Note that of the two equilibrium points, namely zeros -3 and 1 of $f(y) = (3+y)(1-y)$, 1 is asymptotically stable, which provides with another method of concluding that as $t \longrightarrow \infty$, $y \longrightarrow 1$.

2.9: 28; 5pts: *Determine the stability of the equilibrium solutions of* $x' = x(x-1)(x+2)$.
The equilibrium points for $x' = f(x) = x(x-1)(x+2)$ are zeros of f, so they are -2, 0 and 1. Since $f'(x) = 3x^2 + 2x - 2$, we have that:

$$\begin{aligned} f'(-2) = 6 > 0 &\implies x = -2 \text{ is unstable.} \\ f'(0) = -2 < 0 &\implies x = 0 \text{ is asymptotically stable.} \\ f'(1) = 3 > 0 &\implies x = 1 \text{ is unstable.} \end{aligned}$$

Section 3.1, Problem 12 (8pts)

A population is observed to obey the logistic equation with eventual population $20,000$. *The initial population is* 1000, *and* 8 *hours later, the observed population is* 1200. *Find the reproductive rate and the time required for the population to reach three quarters of its carrying capacity.*
In the logistic model of population growth, we assume that the death and birth rates vary with population P according to formulas $d = d_0 + aP$ and $b = b_0 - cP$. Then we get:

$$P(t+\Delta t) - P(t) \approx (b - cP(t)P(t)\Delta t - (d + aP(t))P(t)\Delta t = (b - d - (a+c)P(t))P(t)\Delta t$$

$$P'(t) = \lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = (b - d - (a+c)P(t))P(t)$$

Set $r = b - d$ and $a + c = r/K$ where K is a new constant, to get the logistic equation

$$P' = r(1 - P/K)P$$

This equation is autonomous, with equilibrium points $P_1 = 0$ and $P_2 = K$. Since $P' = f(P) > 0$ if $0 < P < K$ and $P' = f(P) < 0$ if $P > K$ or $P < 0$, P_1 is an unstable, and P_2 a stable equilibrium. If $P(t)$ is any solution with positive population, then the population must stay positive.
To solve the logistic equation, note that it is autonomous, hence separable.

$$\begin{aligned} \frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) &\iff \frac{K\,dP}{P(K-P)} = r\,dt \\ &\iff \left(\frac{1}{P} + \frac{1}{K-P}\right) dP = r\,dt \iff \ln|P| - \ln|K-P| = rt + C \\ &\iff \frac{P}{K-P} = Ae^{rt} \iff P(t) = \frac{KAe^{rt}}{1 + Ae^{rt}} \end{aligned}$$

If P_0 is population at time t_0, then

$$Ae^{rt_0} = \frac{P_0}{K - P_0}$$

enables us to eliminate A, and we get:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-r(t-t_0)}}$$

Note that when $t \to \infty$, $e^{-r(t-t_0)} \to 0$, so that $\lim_{t\to\infty} P(t) = K$ and K is called the carrying capacity.
In this particular problem, we have that the carrying capacity is $K = 20,000$, $P_0 = 1000$ and $P(8) = 1200$, where time is expressed in hours. We want to find t such that $P(t) = 15,000$. Applying the derived formula gives:

$$1200 = \frac{20,000 \cdot 1000}{1000 + 18,000 \cdot e^{-8r}} \implies r \approx 0.0241$$

Time t at which the population is $P(t) = 15,000$ then satisfies:

$$15,000 = \frac{20,000 \cdot 1000}{1000 + 18,000 \cdot e^{-rt}} \implies e^{-rt} = \frac{1}{57} \implies t \approx 167.671$$

Section 3.4, Problem 14 (8pts)

Solve the general IVP modeling the LR circuit,

$$L\frac{dI}{dt} + RI = E, \qquad I(0) = I_0,$$

where L, R, and E are constants.
Rewrite the ODE as $I' + (R/L)I = (E/L)$. The integrating factor $P(t)$ is given by

$$\begin{aligned} P(t) = e^{\int (R/L)dt} = e^{(R/L)t}; \implies \quad & I' + \frac{R}{L}I = \frac{E}{L} \iff e^{(R/L)t}I' + \frac{R}{L}e^{(R/L)t}I = \frac{E}{L} \cdot e^{(R/L)t} \\ \implies \quad & \left(e^{(R/L)t}I\right)' = \frac{E}{L} \cdot e^{(R/L)t} \\ \implies \quad & e^{(R/L)t}I(t) = \frac{E}{L}\int e^{(R/L)t}dt = \frac{E}{R}e^{(R/L)t} + C. \end{aligned}$$

Since $I(0) = I_0$, $1 \cdot I_0 = \frac{E}{R} \cdot 1 + C$, and $C = I_0 - \frac{E}{R}$. Thus,

$$\boxed{I(t) = \tfrac{E}{R} + \left(I_0 - \tfrac{E}{R}\right)e^{-(R/L)t}}$$