Math53: Ordinary Differential Equations
Autumn 2004

Unit 1 Summary

First-Order Ordinary Differential Equations

Extremely Important: what it means for a function to solve an ODE/IVP and how to check this; implicitly defined solutions.

Very Important: finding solutions to linear and first-order ODEs and IVPs; geometric implications of the existence and uniqueness theorem; descriptive analysis of first-order ODEs, including sketches as in Figure 1 below; structure of solutions of inhomogeneous linear equations, i.e. $y=y_p+y_h$; solution curves and direction fields.

Important: interval of existence; solving exact ODEs, including exactness check and sketches of solution curves; reducing other ODEs to linear, separable, or exact ODEs by substitution or by multiplying by an integrating factor; application problems.

Finding Solutions of Special First-Order ODEs

(1) The simplest first-order ODEs to solve are those of the form

$$y' = f(t), \qquad y = y(t). \tag{1}$$

These ODEs are solved by taking the indefinite integral of both sides:

$$\boxed{y' = f(t), \quad y = y(t) \qquad \Longrightarrow \qquad y = \int f(t)dt}$$

The *solution curves* of (1) differ by vertical shifts. An initial-value problem for (1) is solved by

$$\boxed{y' = f(t), \quad y(t_0) = y_0 \qquad \Longrightarrow \qquad y(t) = y_0 + \int_{t_0}^{t} f(s)ds}$$

(2) *Linear* first-order ODEs are the equations of the form

$$y' + a(t)\cdot y = f(t), \qquad y = y(t). \tag{2}$$

Note that only the first powers of the function y and its t-derivative appear in (2). For example, there are no terms y^2, yy', $\sin y$, etc. Equation (2) can be reduced to (1) by multiplying by an integrating factor

$$P_a = P_a(t) = e^{\int a(t)dt}.$$

We need only one such integrating factor. Its key property is that

$$P_a'(t) = a(t)\cdot P_a(t) \qquad \Longrightarrow \qquad (P_a y)' = P_a y' + a\cdot P_a y. \tag{3}$$

Equation (2) is solved by multiplying both sides by P_a and using the second identity in (3):

$$\boxed{P_a = P_a(t) = e^{\int a(t)dt}} \qquad \boxed{y' + a(t)\cdot y = f(t), \quad y = y(t) \quad \Longrightarrow \quad (P_a y)' = P_a(t) f(t)}$$

The last equation above is solved by integrating both sides with respect to t. An initial-value problem for (2) is solved by

$$\boxed{P_a(t) = e^{\int_{t_0}^{t} a(s)ds}} \qquad \boxed{y' + a(t)y = f(t), \;\; y(t_0) = y_0 \quad \Longrightarrow \quad P_a(t)y(t) = y_0 + \int_{t_0}^{t} P_a(s)f(s)ds}$$

For this choice of P_a, $P_a(t_0)=1$. Alternatively, one can first find the general solution and then find the constant C by plugging in the initial conditions.
Caution: Before computing the integrating factor, you need to put the ODE into the form (2), which is *not* its normal form; see (19) below.

(3) *Separable* first-order ODEs are the equations of the form

$$y' = f(y)\cdot g(t), \qquad y = y(t). \tag{4}$$

Equation (4) is solved by writing $y' = \frac{dy}{dt}$, moving all expressions involving y to LHS and all expressions involving t to RHS, and integrating both sides:

$$\boxed{\frac{dy}{dt} = f(y)\cdot g(t), \quad y = y(t) \quad \Longrightarrow \quad \frac{dy}{f(y)} = g(t)dt \quad \Longrightarrow \quad \int \frac{dy}{f(y)} = \int g(t)dt}$$

Once the two integrals are computed, one obtains a relation between y and t of the form

$$F(y) = G(t) + C \qquad \Longleftrightarrow \qquad F(y) - G(t) = C. \tag{5}$$

These relations define solutions $y=y(t)$ of (4) *implicitly*. In some cases, it is possible to solve (5) for $y=y(t)$. An initial-value problem for (4) is solved by

$$\boxed{\frac{dy}{dt} = f(y)\cdot g(t), \quad y(t_0) = y_0 \quad \Longrightarrow \quad \frac{dy}{f(y)} = g(t)dt \quad \Longrightarrow \quad \int_{y_0}^{y} \frac{dz}{f(z)} = \int_{t_0}^{t} g(s)ds}$$

Alternatively, one can first find the general solution and then find the constant C by plugging in the initial conditions. It is the easiest to find C as soon as it appears, i.e. plug in the initial conditions into (5) and then solve for $y=y(t)$, instead of first solving (5) for $y=y(t)$ and then solving for C.

Caution: (i) This separation-of-variables method involves division by $f=f(y)$ and may miss some of the constant solutions of (4). Such solutions are necessarily of the form $y=y^*$, where y^* is a real number such that $f(y^*)=0$.
(ii) If you are solving an IVP and it is possible to solve for $y=y(t)$ explicitly, make sure you take the correct branch, if there is more than one, of the appropriate level curve of $H=F-G$, e.g. the positive or negative square root, and not both. The correct branch is the one satisfying the initial condition $y(t_0)=y_0$.

(4) The first-order ODE

$$P(t,y) + Q(t,y)y' = 0 \quad \text{or} \quad P(t,y)dx + Q(t,y)dy = 0, \qquad y = y(t), \tag{6}$$

is *exact* if there exists a smooth function $H=H(t,y)$ such that

$$H_t \equiv \frac{\partial H}{\partial t} = P \quad \text{and} \quad H_y \equiv \frac{\partial H}{\partial y} = Q, \quad \text{or} \quad \vec{\nabla} H = P\hat{\imath} + Q\hat{\jmath}, \quad \text{or} \quad dH \equiv H_t dt + H_y dy = P dt + Q dy.$$

These three conditions are exactly the same. The equality of mixed partial derivatives, $H_{yt}=H_{ty}$, implies that

$$\boxed{\text{If} \quad P(t,y)+Q(t,y)y'=0 \quad \text{is exact,} \quad \text{then } P_y=Q_t} \tag{7}$$

In particular, if $P_y \neq Q_t$, (6) is not exact. On the other hand, if P and Q are defined on a rectangle R, the converse of (7) is true as well:

$$\boxed{P_y=Q_t \quad \text{and} \quad H(t,y)=\int_{t_0}^t P(s,y_0)ds+\int_{y_0}^y Q(t,z)dz \quad \Longrightarrow \quad H_t=P \text{ and } H_y=Q} \tag{8}$$

If (6) is exact, it is *implicitly* solved by

$$\boxed{P(t,y)+Q(t,y)y'=0, \quad y=y(t) \quad \Longrightarrow \quad H(t,y)=C \quad \text{if} \quad H_t=P \text{ and } H_y=Q}$$

An initial-value problem for (6) is solved implicitly by

$$\boxed{P(t,y)+Q(t,y)y'=0, \quad y(t_0)=y_0 \quad \Longrightarrow \quad H(t,y)\equiv\int_{t_0}^t P(s,y_0)ds+\int_{y_0}^y Q(t,z)dz=0 \;\text{ if }\; P_y=Q_t}$$

Alternatively, one can first find the general solution and then find the constant C by plugging in the initial conditions. As above, it is the easiest to find C as soon as it appears.

Caution: (i) While we are looking for H such that $H_t=P$ and $H_y=Q$, the derivative test for exactness is $P_y=Q_t$, i.e. the derivatives are taken in the "opposite" way.
(ii) *Check* the conclusion in (8). You'll see that the assumption $P_y=Q_t$ is critical.
(iii) The assumption that P and Q are defined on a rectangle is essential for the validity of (8), though it is also true for some other domains as well.
(iv) Note that in the constructions of H above, the two integrands are $P(s,y_0)$ and $Q(t,z)$, and not $Q(t_0,z)$. If you have taken Math52, you might recognize $H(t,y)$ as the line integral of $Pdt+Qdy$ along the horizontal line $s \longrightarrow (s,y_0)$, with $t_0 \le s \le t$, followed by the vertical line $z \longrightarrow (t,z)$, with $y_0 \le z \le y$. Due to our assumptions on P and Q, the line integral of $Pdt+Qdy$ depends only on the end points, (t_0,y_0) and (t,y), and not the path between them.
(v) The method for finding the function $H=H(t,y)$ described above is different from the one described in lecture and in the text. The method described above is more direct and quicker, but it is also less safe, as it defines H whether or not $P_y=Q_t$. However, if $P_y \neq Q_t$, we will not have $H_t=P$. If you use the method described previously to find H, you will get an equation of the form

$$\phi'(y)=f(t,y),$$

with f depending on t if $P_y \neq Q_t$. In such a case, this equation has no solution.

(5) While many first-order ODEs are neither linear, separable, nor exact, it may be possible to reduce some of them to linear, separable, or exact ODEs by making a change of variables or by multiplying by a nonzero function. For example, the ODE

$$y'=\frac{y}{t+y}, \qquad y=y(t), \tag{9}$$

is neither linear, separable, nor exact. However, if we set $z = z(t) = y(t)/t$ or $y = t \cdot z$, (9) becomes

$$z' = -\frac{z^2}{1+z} t^{-1}, \qquad z = z(t). \tag{10}$$

Please check this! Equation (10) is separable and can be solved implicitly as $H(t,z) = 0$, for some function H. Plugging in $z = y/t$, we obtain implicit solutions $y = y(t)$ of (9). Another example is

$$ty + (t^2 + y^2)y' = 0, \qquad y = y(t). \tag{11}$$

This equation is again neither linear, separable, nor exact. However, multiplying both sides of (11) by y, we obtain

$$ty^2 + (t^2 y + y^3)y' = 0, \qquad y = y(t).$$

This equation is equivalent to (11) and is exact. *Please check this!*

Autonomous First-Order ODEs

(1) An *autonomous* first-order ODE is an ODE of the form

$$y' = f(y), \qquad y = y(t). \tag{12}$$

This equation is of course separable. Thus, we can solve it implicitly for $y = y(t)$ as $F(y) = t + C$. However, a lot of descriptive information about (12) can be obtained without solving it. First of all, since RHS of (12) does not involve t, the direction field of (12) does not change under horizontal shifts. Thus, a *horizontal* shift of a solution curve is again a solution curve. Furthermore, if y^* is a real number such that $f(y^*) = 0$, the constant function $y(t) = y^*$ is a solution of (12). Such a number y^* is an *equilibrium point* for (12) and $y(t) = y^*$ is an *equilibrium solution* of (12). The corresponding solution curve is the horizontal line $y = y^*$ in (t,y)-plane. The horizontal graphs of the equilibrium solutions of (12) partition the (t,y)-plane into horizontal bands $y_1^* < y < y_2^*$. In each band, the function $f(y)$ does not change sign. Thus, in each single band, all solution curves of (12) either descend and approach the line $y = y_1^*$ or ascend and approach the line $y = y_2^*$ as t approaches ∞. The equilibrium point y^* and the equilibrium solution $y = y^*$ are *stable* if the solution curves in the two bands surrounding the horizontal line $y = y^*$ approach $y = y^*$ as t approaches ∞. Otherwise, they are *unstable*.

(2) Here is an example. The equilibrium solutions of

$$y' = (y+3)^2(y+1)(y-3) \tag{13}$$

are $y = -3$, $y = -1$, and $y = 3$. The graphs of these solutions are the horizontal lines $y = -3$, $y = -1$, and $y = 3$, shown in the third plot above. These lines partition the ty-plane into horizontal bands $y_1^* < y < y_2^*$. Since solution curves of the ODE (13) do not intersect, no solution curve can cross the graphs of the equilibrium solutions. For example, if $y = y(t)$ is a solution of (13) such that $y(t_0) \in (-1, -3)$ for some t_0, then $y(t) \in (-1, -3)$ for all t. In each band, the function $f(y)$ does not change sign. Thus, in each single band, all solution curves of (13) either descend or ascend. Furthermore, each solution curve must approach either an equilibrium solution curve or $\pm\infty$ as

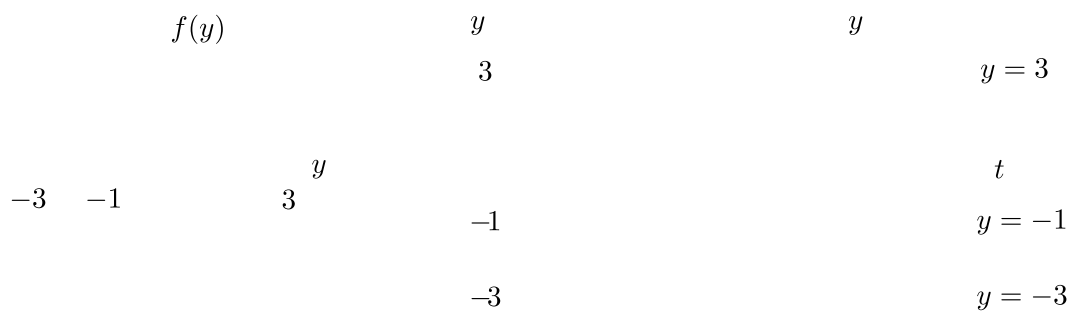

Figure 1: Plots for ODE $\quad y' = f(y) = (y+3)^2(y+1)(y-3)$

$t \longrightarrow \pm\infty$. The best way to tell whether the solution curves in the given band descend or ascend is by sketching the graph of the function $f = f(y)$, as done for the ODE (13) in the first plot above. Note that the y-intercepts of this graph correspond to the equilibrium solutions of the ODE. The equilibrium point y^* and the equilibrium solution $y = y^*$ are *stable* if the solution curves in the two bands surrounding the horizontal line $y = y^*$ approach $y = y^*$ as t approaches ∞. Otherwise, they are *unstable*. The (*first-*) *derivative test* can often, but not always, be used to determine the type of an equilibrium solution. However, it is usually simpler to first draw the *phase line* for the ODE by looking at the graph of $f = f(y)$, as done in the middle plot above for the ODE (13). The phase line shows the equilibrium points for the ODE (13), or the y-intercepts of the graph of f. It also indicates, using arrows, whether the solution curves in each band cut out by the horizontal equilibrium-solution lines ascend and descend. The arrow corresponding to a segment of the phase line points up (down) if $f(y)$ is positive (negative) on the this segment. An equilibrium point y^* of (13) is *stable* if on the phase line *both* arrows surrounding y^* point toward y^* and is *unstable* otherwise. In this case, $y = -1$ is a stable solution of (13), while $y = -3$ and $y = 3$ are not.

Qualitative Descriptions

(1) There is no problem with existence and uniqueness of solutions for initial value problems involving first-order linear ODEs. In other words, every IVP

$$y' = a(t) \cdot y + f(t), \qquad y(t_0) = y_0, \tag{14}$$

has a *unique* solution, provided that a and f are defined and continuous near t_0. Furthermore, the interval of existence for the solution of (14) is the largest interval containing t_0 on which a and f are defined and continuous.

(2) More generally, the *Existence and Uniqueness Theorem* for first-order ODEs *guarantees* a solution to IVP

$$y' = Q(t, y), \qquad y(t_0) = y_0, \tag{15}$$

if the function Q is continuous near (t_0, y_0). If in addition $\partial Q / \partial y$ is defined and continuous near (t_0, y_0), this theorem *guarantees* that there is only *one* solution to this IVP near t_0. Thus, this theorem's applicability depends on the ODE *and* the initial condition. The *most important* implication of this theorem is that no two solutions curves of

$$y' = Q(t, y), \qquad y = y(t),$$

intersect and there is a solution curve through every point in the (t, y)-plane provided that Q is smooth. This means that if a system starts in a certain state, it can be in only one possible state at a later time.

Here is an example. IVP

$$y' = \sqrt{|y|}/|t|, \qquad y(0) = 0,$$

is not *guaranteed* to have a solution at all, because $\sqrt{|y|}/|t|$ is not continuous near $(0, 0)$, but it *may* have a solution or even lots of solutions. We have to determine this in some other way, e.g. by trying to solve this IVP. On the other hand, IVP

$$y' = \sqrt{|y|}/|t|, \qquad y(2) = 0,$$

is guaranteed to have a solution, because $\sqrt{|y|}/|t|$ continuous near $(2, 0)$. However, this IVP *may* have many solutions, because $\partial(\sqrt{|y|}/|t|)/\partial y$ is not continuous near $(2, 0)$. On the other hand, IVP

$$y' = \sqrt{|y|}/|t|, \qquad y(2) = 1,$$

is guaranteed to have a *unique* solution because $\partial(\sqrt{|y|}/|t|)/\partial y$ is continuous near $(2, 1)$.

Caution: (i) It makes sense to talk about existence and uniqueness of solutions only for IVPs, such as (15). Otherwise, there will be lots of solutions, which we usually describe by the constant C.
(ii) Note that the uniqueness statement involves only the partial $\partial Q/\partial y$.

(2) A *homogeneous* linear first-order ODE is an ODE of the form

$$y' = a(t)y, \qquad y = y(t). \tag{16}$$

If y_1 and y_2 are solutions of (16), so is $C_1y_1+C_2y_2$, for any real numbers C_1 and C_2. *Please check this directly, without solving the equation!* This property of the set of all solutions of homogeneous linear ODEs, of any order, makes it a *vector space*, i.e. the sum of two solutions is again a solution and any multiple of a solution is also solution. This is not the case for other ODEs. The general solution of any linear equation

$$y' = a(t)y + f(t), \qquad y = y(t), \tag{17}$$

has the form $y=y_h+y_p$, where y_p is a fixed *particular* solution of (17) and y_h is the general solution of the corresponding homogeneous equation, i.e. (16) with the same $a=a(t)$ as in (17). In order to check this claim, you need to show two things. The first one is that if y_p is a solution of (17) and y_h is a solution of (16), then y_h+y_p is a solution of (17). The second statement is that if y_p and y are solutions of (17), then $y-y_p$ is a solution of (16). *Please check these two statements directly, without solving the two equations!*

Terminology

(1) A *first-order ordinary differential equation* is a relation of the form

$$R(t, y, y') = 0, \qquad y = y(t), \tag{18}$$

that cannot be simplified, through algebraic means, to a relation $\tilde{R}(t,y)=0$. In (18), R is a function of three variables. The *normal form* of a first-order ODE is an expression

$$y' = Q(t,y), \qquad y = y(t), \tag{19}$$

where Q is a function of two variables. Most first-order ODEs arising in applications can be put into the normal form. An initial-value problem, for a first-order ODE, is a set of conditions:

$$R(t,y,y') = 0 \quad \text{or} \quad y' = Q(t,y), \qquad y = y(t), \qquad y(t_0) = y_0. \tag{20}$$

The last condition in (20) is the *initial-value requirement* for (20).

(2) A *solution* of (18), or of (19), is a function $y=y(t)$ that satisfies (18), or (19). A *solution* of the initial-value problem (20) is a function $y=y(t)$ that satisfies the ODE *and* the initial-value requirement in (20). Typically, but not always, (20) will have a unique solution. A *solution curve* for the first-order ODE (18), or for (19), is the graph, in ty-plane, of a solution $y=y(t)$ of (18), or of (19). Typically, but not always, solution curves for the same first-order ODE will not intersect. A *solution curve* for the initial-value problem (20) is the graph of a solution $y=y(t)$ of (20). Such a graph *must* pass through the point (t_0,y_0). The *direction field* for the ODE (19) is usually thought of as a diagram, in the ty-plane, consisting of short line segments of slope $y'=Q(t,y)$ through a number of points (t,y). Since the derivative of a function $y=y(t)$ is the slope of the tangent line to the graph of y, a solution curve for (19) is everywhere tangent to the direction field. In particular, if the direction field is drawn at sufficiently many points, one can pretty much see solution curves. *Caution:* While the solution curves for the simplest ODEs, i.e. (1), differ by vertical shifts, this is *not* the case for other ODEs.

(3) The *interval of existence* of a solution of an ODE is the largest interval on which the solution is defined. If you are asked to find all solution of an ODE, you may end up with several intervals of existence for the same expression for $y=y(t)$. For example,

$$\boxed{y' = \tfrac{2}{t(t+2)} \qquad \Longrightarrow \qquad y(t) = \ln|t| - \ln|t+2| + C, \qquad t \in (-\infty,-2),(-2,0),(0,\infty)}$$

Please check this! In this case, there are three solutions, and thus three intervals of existence, for each constant C. In some cases, if the range for C is not all real numbers, you should be specify it. For example,

$$\boxed{t + yy' = 0 \qquad \Longrightarrow \qquad y(t) = \pm\sqrt{C-t^2}, \quad t \in (-\sqrt{C},\sqrt{C}), \quad C > 0}$$

Please check this! For an initial value problem, the interval of existence *must* contain the initial value of the parameter. For example,

$$\boxed{y' = \tfrac{2}{t(t+2)}, \quad y(-1) = 1 \qquad \Longrightarrow \qquad y(t) = \ln|t| - \ln|t+2| + 1, \qquad t \in (-2,0)}$$

In some cases, you may need to pick the correct branch of an implicitly defined solution. For example,

$$\boxed{t + yy' = 0, \quad y(0) = -2 \qquad \Longrightarrow \qquad y(t) = -\sqrt{4-t^2}, \quad t \in (-2,2)}$$