Math53: Ordinary Differential Equations
Autumn 2004

Problem Set 5 Solutions

Note: Even if you have done every problem, you are encouraged to look over these solutions, especially 8.2:13 and 9.2:1,4,24,26, where plots are discussed in detail. Such sketches will be central to most of the rest of course.

Section 7.2: 8,14 (6pts)

7.2:8; 2pts: *Consider the line L in $\mathbf{R}^2$ with the parametric equation*

$$\mathbf{y} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \end{pmatrix}.$$

Is L the solution set for a system of linear equations?

$$\mathbf{y} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \end{pmatrix} \quad\Longrightarrow\quad x = -2, \quad y = 3 - 3t.$$

Thus, L is the solution set for the one-equation system $\boxed{x = -2, \ \ (x, y) \in \mathbb{R}^2}$ in two variables.

7.2:14; 4pts: *Find a parametric representation for the solution set of the system of equations*

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 = 2 \\ x_2 - 3x_3 - x_4 = 3 \\ x_3 - x_4 = 0 \end{cases}$$

What is its dimension? How would you describe the solution set?

$$\begin{cases} x_1 + 2x_2 - 2x_3 + x_4 = 2 \\ x_2 - 3x_3 - x_4 = 3 \\ x_3 - x_4 = 0 \end{cases} \iff \begin{cases} x_1 + 2x_2 - x_4 = 2 \\ x_2 - 4x_4 = 3 \\ x_3 = x_4 \end{cases} \iff \begin{cases} x_1 = -4 - 7x_4 \\ x_2 = 3 + 4x_4 \\ x_3 = x_4 \end{cases}$$

$$\Longrightarrow \quad \boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 - 7x_4 \\ 3 + 4x_4 \\ x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 0 \\ 0 \end{pmatrix} + x_4\begin{pmatrix} -7 \\ 4 \\ 1 \\ 1 \end{pmatrix}}$$

The solution is a line in $\mathbb{R}^4$, and it has dimension 1.

Section 7.4: 16,20 (7pts)

Determine whether each of the matrices

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is singular. If not, find its inverse.
Since the determinant of A is zero, $\boxed{A \text{ is singular}}$ Since the matrix B is upper-triangular, $\det B$ is the product of the diagonal entries. Thus, $\det B = 1 \neq 0$ and $\boxed{B \text{ is nonsingular}}$ The inverse of any non-singular matrix can be computed in three steps, which for a 3×3 matrix are:

$$B \longrightarrow \begin{pmatrix} \det B_{11} & \det B_{12} & \det B_{13} \\ \det B_{21} & \det B_{22} & \det B_{23} \\ \det B_{31} & \det B_{32} & \det B_{33} \end{pmatrix} \longrightarrow \begin{pmatrix} \det B_{11} & -\det B_{21} & \det B_{31} \\ -\det B_{12} & \det B_{22} & -\det B_{32} \\ \det B_{13} & -\det B_{23} & \det B_{33} \end{pmatrix}$$
$$\longrightarrow \frac{1}{\det B} \begin{pmatrix} \det B_{11} & -\det B_{21} & \det B_{31} \\ -\det B_{12} & \det B_{22} & -\det B_{32} \\ \det B_{13} & -\det B_{23} & \det B_{33} \end{pmatrix}$$

where B_{ij} is the square matrix obtained from B by removing ith row and jth column of B. In this case, we get

$$B \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \boxed{\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}}$$

Section 7.5: 14,26 (9pts)

7.5:14; 7pts: *Check whether the vectors*

$$\mathbf{v}_1 = \begin{pmatrix} -8 \\ 9 \\ -6 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 0 \\ 7 \end{pmatrix} \quad \mathbf{v}_3 = \begin{pmatrix} 8 \\ -18 \\ 40 \end{pmatrix}$$

are linearly independent. If not, find a nontrivial linear combination that equals zero.
Since there are *three three*-vectors, we can simply check whether the determinant of the corresponding matrix is nonzero:

$$\det \begin{pmatrix} -8 & -2 & 8 \\ 9 & 0 & -18 \\ -6 & 7 & 40 \end{pmatrix} = (-2)\cdot 9 \cdot \det \begin{pmatrix} 4 & 1 & -4 \\ 1 & 0 & -2 \\ -6 & 7 & 40 \end{pmatrix}$$
$$= -18\big(4\cdot0\cdot40 + (-6)\cdot1\cdot(-2) + (-4)\cdot1\cdot7 - (-4)\cdot0\cdot(-6) - 4\cdot7\cdot(-2) - 40\cdot1\cdot1\big) = 0.$$

Thus, the three vectors are $\boxed{\text{linearly dependent}}$ We now need to find a solution to the linear system

$$\begin{pmatrix} 4 & 1 & -4 \\ 1 & 0 & -2 \\ -6 & 7 & 40 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 4c_1 + c_2 - 4c_3 = 0 \\ c_1 - 2c_3 = 0 \\ -6c_1 + 7c_2 + 40c_3 = 0 \end{cases} \iff \begin{cases} c_1 = 2c_3 \\ 4c_3 + c_2 = 0 \\ 28c_3 + 7c_2 = 0 \end{cases}$$
$$\iff \begin{cases} c_1 = 2c_3 \\ c_2 = -4c_3 \end{cases} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} \iff \boxed{2\mathbf{v}_1 - 4\mathbf{v}_2 + \mathbf{v}_3 = 0}$$

7.5:26; 2pts: *Find a basis for the nullspace of the matrix* $A = \begin{pmatrix} -3 & 5 \end{pmatrix}$.
We need to find a solution to

$$A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \iff -3c_1 + 5c_2 = 0 \iff c_2 = 3c_1/5 \implies \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Thus, the single-element set $\boxed{\left\{\mathbf{v}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}}$ is a basis for the nullspace of A.

Section 7.6: 14,28,44 (21pts)

7.6:14 (a; **5pts**) *Let A be an $n\times n$-matrix. If row i is a linear combination of the preceding rows, prove that the determinant of A is zero. State and prove a similar statement about the columns of A.*
We first state and prove the analogous statement for the columns of A:

If column j is a linear combination of the preceding columns, then $\det A = 0$.

Let $A = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, where $\mathbf{v}_j$ denotes the jth column. Suppose that the jth column is a linear combination of the columns that precede it, i.e.

$$\mathbf{v}_j = c_1\mathbf{v}_1 + \ldots + c_{j-1}\mathbf{v}_{j-1} \implies A\begin{pmatrix} c_1 \\ \vdots \\ c_{j-1} \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus, the equation $A\mathbf{x} = \mathbf{0}$ has a nonzero solution, and $\det A = 0$ by Corollary 6.3 on p385. On the other hand, if the ith row of A is a linear combination of its preceding rows, then the ith column of the transpose A^T of A is a linear combination of the preceding columns of A^T, and thus

$$\det A = \det A^T = 0.$$

(b; **3pts**) *Explain why the determinant of each of the following matrices is zero:*

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ -1 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 5 \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

In the first matrix, the third row is the sum of its first two rows. In the second matrix, the third column is the sum of its first two columns. In the third matrix, the third row equals twice its first row added to its second row. In the fourth matrix, the third column is the sum of twice its first column and three times its second column. By part (a) of the problem, it follows that all four matrices have zero determinant.

7.6:28; 8pts: *Compute the determinant of the matrix*

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

If the nullspace of this matrix is nontrivial, find a basis for it. Determine if the column vectors in the matrix are linearly independent.
To calculate the determinant, expand along the first row:

$$\det\begin{pmatrix}-1 & 0 & -1\\ 1 & 1 & 2\\ 2 & 1 & 3\end{pmatrix} = (-1)\det\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix} + (-1)\det\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix} = -1+1=0.$$

Since $\det A = 0$, the nullspace of A is nontrivial, and the column vectors of A are linearly dependent. We now find a basis for null(A):

$$\begin{pmatrix}-1 & 0 & -1\\ 1 & 1 & 2\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \iff \begin{cases}-c_1 - c_3 = 0\\ c_1 + c_2 + 2c_3 = 0\\ 2c_1 + c_2 + 3c_3 = 0\end{cases} \iff \begin{cases}c_3 = -c_1\\ c_2 - c_1 = 0\\ c_2 - c_1 = 0\end{cases}$$

$$\iff \begin{cases}c_2 = c_1\\ c_3 = -c_1\end{cases} \iff \begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix} = \begin{pmatrix}1\\ 1\\ -1\end{pmatrix} c_1.$$

Thus, $\{(1,1,-1)^T\}$ is a basis for null(A).

7.6:44; 5pts: *Compute the determinant of the matrix*

$$\begin{pmatrix}1 & 2 & -3\\ 0 & 6 & -2\\ -2 & 3 & 2\end{pmatrix}$$

Is there a nonzero vector in the nullspace?
To compute the determinant, expand along the first column:

$$\det\begin{pmatrix}1 & 2 & -3\\ 0 & 6 & -2\\ -2 & 3 & 2\end{pmatrix} = 1\cdot\det\begin{pmatrix}6 & -2\\ 3 & 2\end{pmatrix} - 2\cdot\det\begin{pmatrix}2 & -3\\ 6 & -2\end{pmatrix}$$
$$= \big(6\cdot 2 - (-2)3\big) - 2\big(2(-2) - (-3)6\big) = 18 - 28 = \boxed{-10}$$

Since $\det A \neq 0$, $\boxed{\text{the nullspace is trivial}}$ it consists only of the zero vector.

Section 8.1: 10 (4pts)

Show that the functions $x(t) = e^t$ and $y(t) = e^{-t}$ are solutions to the initial value problem

$$\begin{cases}x' = x^2 y\\ y' = -xy^2\end{cases} \qquad x(0) = 1, \quad y(0) = 1.$$

The initial conditions are satisfied, since $x(0) = e^0 = 1$ and $y(0) = e^{-0} = 1$. Both equations hold, since:

$$x'(t) = e^t = (e^t)^2 e^{-t} = (x(t))^2 y(t), \qquad y'(t) = -e^{-t} = -e^t(e^{-t})^2 = -x(t)(y(t))^2.$$

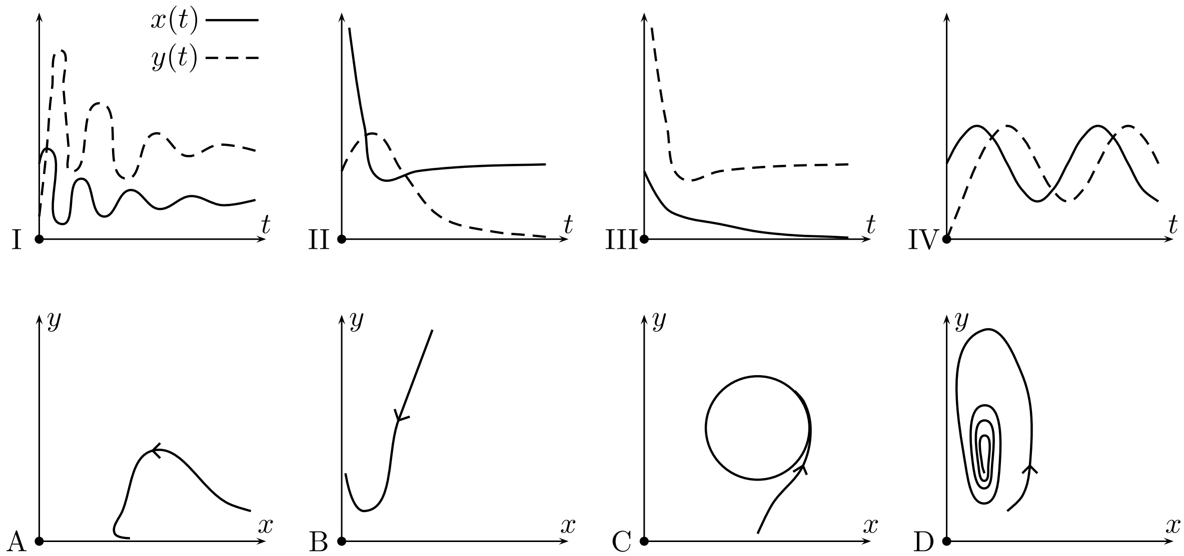

Figure 1: Component and Phase-Plane Plots for Problem 8.2:13

Section 8.2: 13 (7pts)

Match each component plot in the first row of Figure 1 with its phase-plane plot in the second row. Explain your reasoning.

I. x and y start, at $t = 0$, at a small nonzero value. Thus, the corresponding phase-plane sketch is D. Note also that x and y exhibit an oscillatory behavior that dies out as t becomes large. Thus, $(x(t), y(t))$ must spiral down to a point as $t \longrightarrow \infty$. This is the case only in D.

II. x starts out very large, while y starts at a small nonzero value. Thus, the corresponding phase-plane sketch is A. Note also that for t large, x ascends to a small nonzero value, while y decays to zero. In other words, $(x(t), y(t))$ approaches a nonzero point on the x-axis from the left and above. This is the case only in A.

III. x starts at a small nonzero value, while y starts out very large. Thus, the corresponding phase-plane sketch is B. Note also that for t large, x decays to zero, while y ascends to a small nonzero value. In other words, $(x(t), y(t))$ approaches a nonzero point on the y-axis from the right and below. This is the case only in B.

IV. x starts out at a small nonzero value, while y starts at zero. Thus, the corresponding phase-plane sketch is C. Note also that x and y oscillate around finite values without decay. Thus, $(x(t), y(t))$ must move along a curve around some region in the plane, except at the beginning. This is the case only in C.

In brief, $\boxed{\text{I} \longleftrightarrow \text{D}, \quad \text{II} \longleftrightarrow \text{A}, \quad \text{III} \longleftrightarrow \text{B}, \quad \text{IV} \longleftrightarrow \text{C}}$

Section 9.1: 6,54 (10pts)

9.1:6; 3pts: *Find the characteristic polynomial and eigenvalues for the matrix* $A = \begin{pmatrix} -2 & 5 \\ 0 & 2 \end{pmatrix}$.

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} -2-\lambda & 5 \\ 0 & 2-\lambda \end{pmatrix} = (-2-\lambda)(2-\lambda) = \boxed{\lambda^2 - 4}$$

The eigenvalues are the zeros of $p(\lambda)$: $\boxed{\lambda_1 = -2 \text{ and } \lambda_2 = 2}$

9.1:54; 7pts: *Diagonalize the matrix* $A = \begin{pmatrix} 6 & 0 \\ 8 & -2 \end{pmatrix}$ *by finding its eigenvalues and the corresponding eigenvectors.*
Since this matrix is lower-triangular, the eigenvalues are the diagonal entries: $\lambda_1 = 6$ and $\lambda_2 = -2$. Furthermore, an eigenvector for λ_2 is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We next find an eigenvector for λ_1:

$$(A - \lambda_1 I)\mathbf{v}_1 = 0 \iff \begin{pmatrix} 0 & 0 \\ 8 & -8 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff 8c_1 - 8c_2 = 0 \iff c_2 = c_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We have $A = VDV^{-1}$, where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}, \quad V = (\mathbf{v}_1\ \mathbf{v}_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \implies V^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Section 9.2: 1,4,10,24,26,30 (46pts)

9.2:1; 9pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} 2 & -6 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
Since the matrix A is upper-triangular in this case, the eigenvalues of A are the two diagonal entries, $\lambda_1 = 2$ and $\lambda_2 = -1$. Furthermore, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda_1 = 2$. We next find an eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2 = -1$:

$$\begin{pmatrix} 2 - \lambda_2 & -6 \\ 0 & -1 - \lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 3c_1 - 6c_2 = 0 \\ 0 = 0 \end{cases} \iff c_1 = 2c_2 \implies \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution to the ODE is thus given by

$$\boxed{\mathbf{y}(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{or} \quad x(t) = C_1 e^{2t} + 2C_2 e^{-t},\ y(t) = C_2 e^{-t}}$$

A phase-plane sketch is the first plot in Figure 2. The origin is a *saddle point*. The solutions move away from the origin along the two half-lines generated by the vectors $\mathbf{v}_1$ and $-\mathbf{v}_1$, since $\lambda_1 > 0$, and approach the origin along the two half-lines generated by the vectors $\mathbf{v}_2$ and $-\mathbf{v}_2$, since $\lambda_2 < 0$. Other solution curves approach one of the first two half-lines as $t \longrightarrow \infty$ and one of the latter two half-lines as $t \longrightarrow -\infty$.

9.2:4; 9pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
Since the matrix A is upper-triangular in this case, the eigenvalues of A are the two diagonal

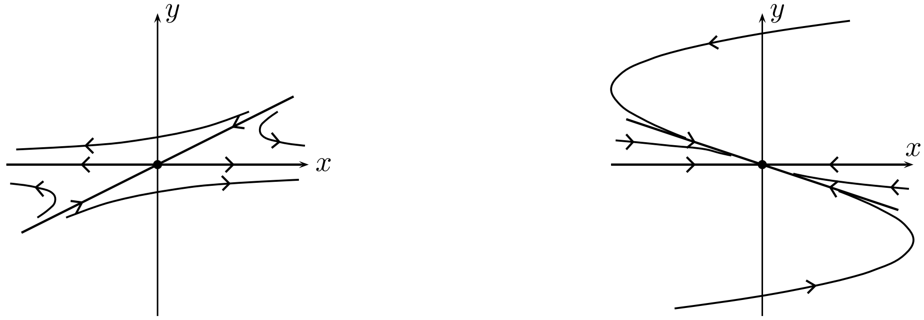

Figure 2: Phase-Plane Plots for Problems 9.2:1 and 9.2:4

entries, $\lambda_1 = -3$ and $\lambda_2 = -1$. Furthermore, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector for A with eigenvalue $\lambda_1 = -3$. We next find an eigenvector $\mathbf{v}_2$ corresponding to $\lambda_2 = -1$:

$$\begin{pmatrix} -3-\lambda_2 & -6 \\ 0 & -1-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -2c_1 - 6c_2 = 0 \\ 0 = 0 \end{cases} \iff c_1 = -3c_2 \implies \mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

The general solution to the ODE is thus given by

$$\boxed{\mathbf{y}(t) = C_1 e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{or} \quad x(t) = C_1 e^{-3t} - 3C_2 e^{-t},\ y(t) = C_2 e^{-t}}$$

A phase-plane sketch is the second plot in Figure 2. The origin is a *nodal sink*. The solutions approach the origin along the four half-lines generated by the vectors $\pm\mathbf{v}_1$ and $\pm\mathbf{v}_2$, since $\lambda_1, \lambda_2 < 0$. All other solution curves must also approach the origin as $t \longrightarrow \infty$. Their slope approaches that of the half-lines generated by $\pm\mathbf{v}_2$ as $t \longrightarrow \infty$ and that of the half-lines generated by $\pm\mathbf{v}_1$ as $t \longrightarrow -\infty$, since $\lambda_2 > \lambda_1$. However, none of these solution curves approaches a horizontal line as $t \longrightarrow -\infty$.

9.2:10; 5pts: *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By 9.2:4, it remains to find C_1 and C_2 such that

$$\mathbf{y}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \iff \begin{cases} C_1 - 3C_2 = 1 \\ C_2 = 1 \end{cases} \iff \begin{cases} C_1 = 4 \\ C_2 = 1 \end{cases}$$

Thus, the solution to the IVP is

$$\boxed{\mathbf{y}(t) = 4e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \quad \text{or} \quad x(t) = 4e^{-3t} - 3e^{-t},\ y(t) = e^{-t}}$$

9.2:24; 9pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

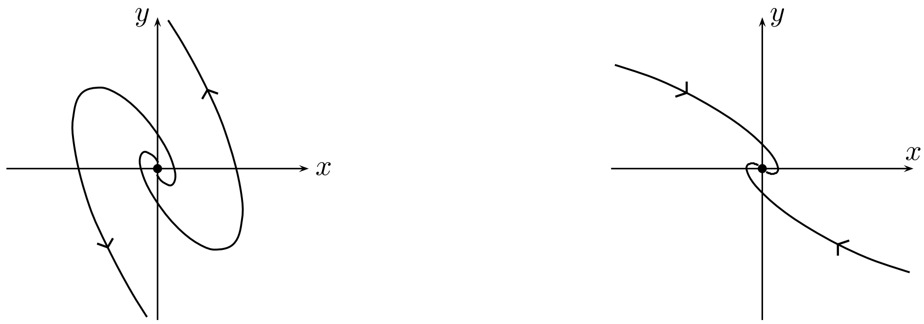

Figure 3: Phase-Plane Plots for Problems 9.2:24 and 9.2:26

Sketch the phase-plane portrait of solution curves.
The characteristic polynomial for this system is

$$\lambda^2 - (-1+3)\lambda + \big((-1)\cdot 3 - (-2)\cdot 4\big) = \lambda^2 - 2\lambda + 5.$$

Thus, the two eigenvalues are $\lambda_1, \lambda_2 = 1 \pm 2i$. We next find an eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 1+2i$:

$$\begin{pmatrix} -1-\lambda_1 & -2 \\ 4 & 3-\lambda_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} -(2+2i)c_1 - 2c_2 = 0 \\ 4c_1 + (2-2i)c_2 = 0 \end{cases}$$

$$\Longleftrightarrow \qquad c_2 = -(1+i)c_1 \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}.$$

Since our matrix is real, while $\mathbf{v}_1$ is complex, its complex conjugate

$$\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

must be an eigenvector with eigenvalue $\lambda_2 = \bar{\lambda}_1 = 1-2i$. Thus, the general solution to the ODE is

$$y(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^t \begin{pmatrix} C_1 e^{2it} + C_2 e^{-2it} \\ -(C_1 e^{2it} + C_2 e^{-2it}) - i(C_1 e^{2it} - C_2 e^{-2it}) \end{pmatrix}$$

$$= \boxed{e^t \begin{pmatrix} A_1 \cos 2t + A_2 \sin 2t \\ -(A_1 + A_2)\cos 2t + (A_1 - A_2)\sin 2t \end{pmatrix}}$$

A phase-plane sketch is the first plot in Figure 3. The origin is a *spiral source*. The solutions spiral away from the origin as t increases, since the real part of λ_1 and λ_2 is positive. They spiral out counterclockwise, since the entry in the lower-left corner of the matrix is positive.

9.2:26; 9pts: *Find the general solution to the system of linear ODEs*

$$\mathbf{y}' = \begin{pmatrix} 0 & 4 \\ -2 & -4 \end{pmatrix}\mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t).$$

Sketch the phase-plane portrait of solution curves.
The characteristic polynomial for this system is

$$\lambda^2 - (0-4)\lambda + \big((-4)\cdot 0 - 4\cdot(-2)\big) = \lambda^2 + 4\lambda + 8.$$

Thus, the two eigenvalues are $\lambda_1, \lambda_2 = 2(-1 \pm i)$. We next find an eigenvector $\mathbf{v}_1$ corresponding to $\lambda_1 = 2(-1+i)$:

$$\begin{pmatrix} 0-\lambda_1 & 4 \\ -2 & -4-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} 2(1-i)c_1 + 4c_2 = 0 \\ -2c_1 - 2(1+i)c_2 = 0 \end{cases}$$

$$\Longleftrightarrow \qquad c_1 = -(1+i)c_2 \quad \Longrightarrow \quad \mathbf{v}_1 = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}.$$

Since our matrix is real, while $\mathbf{v}_1$ is complex, its complex conjugate

$$\mathbf{v}_2 = \bar{\mathbf{v}}_1 = \begin{pmatrix} -1+i \\ 1 \end{pmatrix}$$

must be an eigenvector with eigenvalue $\lambda_2 = \bar{\lambda}_1 = 2(-1-i)$. Thus, the general solution to the ODE is

$$y(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t} \begin{pmatrix} -(C_1 e^{2it} + C_2 e^{-2it}) - i(C_1 e^{2it} - C_2 e^{-2it}) \\ C_1 e^{2it} + C_2 e^{-2it} \end{pmatrix}$$

$$= \boxed{e^{-2t} \begin{pmatrix} -(A_1+A_2)\cos 2t + (A_1-A_2)\sin 2t \\ A_1 \cos 2t + A_2 \sin 2t \end{pmatrix}}$$

A phase-plane sketch is the second plot in Figure 3. The origin is a *spiral sink*. The solutions spiral into the origin as t increases, since the real part of λ_1 and λ_2 is negative. They spin clockwise, since the entry in the lower-left corner of the matrix is negative.

9.2:30; 5pts: *Find the solution to the initial value problem*

$$\mathbf{y}' = \begin{pmatrix} -1 & -2 \\ 4 & 3 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

By 9.2:24, it remains to find A_1 and A_2 such that

$$\mathbf{y}(0) = \begin{pmatrix} A_1 \\ -(A_1+A_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{cases} A_1 = 0 \\ A_1 + A_2 = -1 \end{cases} \quad \Longleftrightarrow \quad \begin{cases} A_1 = 0 \\ A_2 = -1 \end{cases}$$

Thus, the solution to the IVP is

$$\boxed{\mathbf{y}(t) = e^t \begin{pmatrix} -\sin 2t \\ \cos 2t + \sin 2t \end{pmatrix}}$$

The same result can be obtained by finding the constants C_1 and C_2 in the complex form of the general solution.

Section 9.5: 8,12,14 (20pts)

9.5:8; 6pts: *Find* e^{tA} *for* $\quad A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$.

We split tA as

$$tA = \begin{pmatrix} ta & tb \\ 0 & ta \end{pmatrix} = at \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bt \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = atI + btB.$$

Since B^2 is the zero matrix and atI is diagonal,

$$e^{atI} = e^{at}I \qquad \text{and} \qquad e^{btB} = I + btB = \begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix}.$$

Since $(atI)(btB) = (btB)(atI)$,

$$e^{tA} = e^{atI+btB} = e^{atI}e^{btB} = e^{at}I\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix} = \boxed{e^{at}\begin{pmatrix} 1 & bt \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{at} & bt\,e^{at} \\ 0 & e^{at} \end{pmatrix}}$$

9.5:12; 8pts: *Compute* e^{tA} *for* $A = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix}$.

Since A is lower-triangular, the eigenvalues of A are its diagonal entries, $\lambda_1 = -2$ and $\lambda_2 = -3$. Furthermore, an eigenvector corresponding to λ_2 is $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We next find an eigenvector for λ_1:

$$\begin{pmatrix} -2-\lambda_1 & 0 \\ -3 & -3-\lambda_1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -3c_1 - c_2 = 0 \iff c_2 = -3c_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

These give us a diagonalization of A, $A = PDP^{-1}$, where:

$$D = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \implies P^{-1} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$\implies \quad e^{tA} = Pe^{tD}P^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 3e^{-3t} & e^{-3t} \end{pmatrix} = \boxed{\begin{pmatrix} e^{-2t} & 0 \\ -3e^{-2t} + 3e^{-3t} & e^{-3t} \end{pmatrix}}$$

9.5:14; 6pts: *Compute* e^{tA} *for* $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$.

We split tA as

$$tA = \begin{pmatrix} -t & 0 \\ t & -t \end{pmatrix} = -t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = (-t)I + tB.$$

Since B^2 is the zero-matrix and $(-t)I$ is diagonal,

$$e^{-tI} = e^{-t}I \qquad \text{and} \qquad e^{tB} = I + tB = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Since $(-tI)(tB) = (tB)(-tI)$,

$$e^{tA} = e^{(-t)I+tB} = e^{-tI}e^{btB} = e^{-t}I\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \boxed{e^{-t}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ t\,e^{-t} & e^{-t} \end{pmatrix}}$$