Math53: Ordinary Differential Equations
Autumn 2004

Final Exam Practice Problems
(not a practice final exam)

Note: Guide to solutions to these practice problems, partial course overview, Final Exam from a previous quarter, and its solutions will be posted on the course website,

http://math.stanford.edu/∼azinger/math53,

on Friday evening, 12/3.

Problem 1

Find all solutions to the following ODEs:

(i) $y' = \cos t - y \cos t$; (ii) $y' = y(y+2)t$;
(iii) $y' = (1+y^2)e^t$; (iv) $ty' = \sin t - 2y$.

Sketch some solution curves in the ty-plane.

Problem 2

Find explicit solutions, including the interval of existence, to the following IVPs:

(i) $y' = -2t(1+y^2)/y$, $y(0)=1$; (ii) $y' = x/(1+2y)$, $y(-1)=0$;
(iii) $ty' = \sin t - 2y$, $y(\pi/2)=0$; (iv) $1 - y\sin t + (\cos t)y' = 0$, $y(0)=1$;
(v) $y' = \cos t - y\cos t$, $y(\pi)=3$; (vi) $y' = y(y+2)t$, $y(0)=2$.

Problem 3

Find the general solution $y=y(t)$ to each of the following ODEs:

(i) $(1 - y\sin t)dt + (\cos t)dy = 0$; (ii) $2t - y^2 + (y^3 - 2ty)y' = 0$.

Problem 4

Show that the following ODEs are not exact. Then find an integrating factor $\mu=\mu(t,y)$ of the specified form, and solve the equation:

(i) $y\,dt + (t^2y - t)\,dy = 0$, $\mu=\mu(t)$; (ii) $y^2 + 2ty - t^2y' = 0$, $\mu=\mu(y)$.

Problem 5

Find the general solutions to

(i) $y' = (t-y)/(t+y)$; (ii) $(t^2+y^2)y' - ty = 0$.

Problem 6

Show that the relation $t^2+y^2=C$ defines implicitly solutions $y=y(t)$ of the ODE $t+yy'=0$.

Problem 7

Sketch the graph of the function $f=f(y)$ below. Then find the equilibrium solutions of the ODE $y'=f(y)$. Draw the phase line and sketch solution curves, in the ty-plane:

(i) $f(y)=(y+3)^3(y-1)^2(y-3)$; (ii) $f(y)=(y+3)^2(y-1)(y-3)$.

Problem 8

For each of the initial value problems

(i) $|t|y'=\sqrt{|y|}, \quad y(t_0)=y_0$; (ii) $y'=\sqrt{|t-1|}\,y^{2/3}, \quad y(t_0)=y_0$,

determine for what values of t_0 and y_0
(a) the IVP is guaranteed by the Existence and Uniqueness Theorem to have a solution;
(b) the IVP is guaranteed by the Existence and Uniqueness Theorem to have a unique solution;
(c) the IVP has a solution;
(d) the IVP has a unique solution.

Problem 9

(i) A rocket ascends vertically with constant acceleration a m/s^2 for t_1 seconds. The rocket motor is then shut-off and the rocket continues upward under the influence of gravity. Find the maximum altitude y_m reached by the rocket and the total time T elapsed from the take-off until the rocket returns to the ground.
(ii) A tank contains V gallons of a salt-water solution at the concentration of ρ_0 pounds per gallon. Pure water is poured into the tank, and a drain at the bottom is adjusted so as to keep the volume of solution constant. At what rate r, gallons per minute, should the water be poured into the tank to lower the concentration to ρ_1 pounds per gallon in t_1 minutes?

Problem 10

Find the general solution to each of the following ODEs

(i)	$2y''-y'-y=0$;	(ii)	$y''+2y'+17y=0$;
(iii)	$y''-6y'+9y=0$;	(iv)	$y''+6y'+8y=-3e^{-t}$;
(v)	$y''+9y=\sin 2t$;	(vi)	$y''+5y'+6y=4-t^2$;
(vii)	$y''+3y'+2y=3e^{-4t}$;	(viii)	$y''-y=t-e^{-t}$;
(ix)	$y''+5y'+4y=te^{-t}$;	(x)	$y''-4y'+4y=24\sin 2t$.

Problem 11

Find the solution $y=y(t)$ to each of the following initial value problems:

(i) $4y''+y=0$, $y(1)=0$, $y'(1)=-2$;
(ii) $y''-4y'+4y=24\sin 2t$, $y(0)=0$, $y'(0)=0$;
(iii) $y''-y'-2y=t^2e^{2t}$, $y(0)=0$, $y'(0)=-1$
(iv) $y''+y=-2\sin t$, $y(0)=-1$, $y'(0)=1$;
(v) $y''+6y'+9y=te^{-3t}$, $y(0)=0$, $y'(0)=1$;

(vi) $y''''+2y''+y=9\cos 2t$, $y(0)=0$, $y'(0)=0$, $y''(0)=-3$, $y'''(0)=0$.

Problem 12

(i) Show that $y_1(t)=t^2$ is a solution to $t^2y''+ty'-4y=0$. Find the general solution of this ODE.
(ii) Show that $y_1(t)=t$ and $y_2(t)=t^{-3}$ are linearly independent solutions of $t^2y''+3ty'-3y=0$. Find the general solution to $t^2y''+3ty'-3y=t^{-1}$.
(iii) Show that $y_1(t)=t^{-1}$ and $y_2(t)=t^{-1}\ln t$ are linearly independent solutions of $t^2y''+3ty'+y=0$. Find the general solution to $t^2y''+3ty'+y=t^{-1}$.

Problem 13

For each of the following matrices A,
(a) find the general solution to $\mathbf{y}'=A\mathbf{y}$ and sketch the phase-plane portrait;
(b) compute e^{tA};
(c) find the general solution to $\mathbf{y}'=A\mathbf{y}+\begin{pmatrix}1\\t\end{pmatrix}$;
(d) find the solution to the initial value problem $\mathbf{y}'=A\mathbf{y}+\begin{pmatrix}1\\t\end{pmatrix}$, $\mathbf{y}(0)=\begin{pmatrix}1\\1\end{pmatrix}$.

(i) $A=\begin{pmatrix}2&-6\\0&-1\end{pmatrix}$
(ii) $A=\begin{pmatrix}-3&-6\\0&-1\end{pmatrix}$
(iii) $A=\begin{pmatrix}-1&-2\\4&3\end{pmatrix}$
(iv) $A=\begin{pmatrix}0&4\\-2&-4\end{pmatrix}$
(v) $A=\begin{pmatrix}-3&1\\-1&-1\end{pmatrix}$
(vi) $A=\begin{pmatrix}-2&-1\\4&2\end{pmatrix}$
(vii) $A=\begin{pmatrix}8&-1\\-2&7\end{pmatrix}$
(viii) $A=\begin{pmatrix}-1&1\\1&-1\end{pmatrix}$
(ix) $A=\begin{pmatrix}2&0\\0&2\end{pmatrix}$

Note: you should treat (a)-(d) as *separate* problems, and not as parts of the same problem. On the final exam you may get either of the four kinds of questions without the other three. The best approach to (a)-(d) depends on A. On the final exam, an analogue of (a), (c), or (d) may be stated as a system of ODEs, and it would be up to you to convert it to the matrix form.

Problem 14

Find all values of the constant c such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system

$$\mathbf{y}'=\begin{pmatrix}5&c\\-c&1\end{pmatrix}\mathbf{y}.$$

Specify whether the origin is a spiral source or a spiral sink for the values of c you find. Sketch the corresponding phase-plane portrait(s).

Problem 15

For each of the following matrices A, find the general solution to $\mathbf{y}' = A\mathbf{y}$ and determine whether the origin is an asymptotically stable, stable, or unstable equilibrium:

$$\text{(i) } A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{pmatrix} \qquad \text{(ii) } A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \qquad \text{(iii) } A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & 1 & -3 \end{pmatrix}$$

$$\text{(iv) } A = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{pmatrix} \qquad \text{(v) } A = \begin{pmatrix} -3 & 0 & -1 \\ 3 & 2 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

Problem 16

For each of the following autonomous systems of ODEs,
(a) find all equilibrium points and determine their type;
(b) find all cycles and determine their type (except for (ii));
(c) sketch the phase-plane portrait, showing important qualitative details (except for (ii)).

$$\text{(i) } \begin{cases} x' = x(6 - 2x - 3y) \\ y' = y(1 - x - y) \end{cases} \qquad \text{(ii) } \begin{cases} x' = y \\ y' = -\cos x - 0.5y \end{cases} \qquad \text{(iii) } \begin{cases} x' = -y - x^3 \\ y' = x \end{cases}$$

$$\text{(iv) } \begin{cases} x' = 1 - x^2 - y^2 \\ y' = x - y \end{cases} \qquad \text{(v) } \begin{cases} x' = -x(1-y) \\ y' = -4y(1+x) \end{cases} \qquad \text{(vi) } \begin{cases} x' = -y + x(\sqrt{x^2+y^2} - 3) \\ y' = x + y(\sqrt{x^2+y^2} - 3) \end{cases}$$

Problem 17

For each of the following autonomous systems of ODEs, find a conserved quantity and sketch the phase-plane portrait:

$$\text{(i) } \begin{cases} x' = y \\ y' = 3 - x \end{cases} \qquad \text{(ii) } \begin{cases} x' = y \\ y' = -2x + x^3 \end{cases}$$

Problem 18

Let $y = y(t)$ be the solution to the initial value problem $y' = t^{-1}y + 1$, $y(1) = 0$.
(a) Use the first-order Euler's numerical method with four steps to estimate $y(2)$.
(b) Use the second-order Runge-Kutta numerical method with two steps to estimate $y(2)$.
(c) Use the first-order Euler's numerical method with four steps to estimate $y(3)$.
(d) Use the second-order Runge-Kutta numerical method with two steps to estimate $y(3)$.
Note: Use simple fractions, i.e. p/q, to carry out your computations. No calculators.