Math53: Ordinary Differential Equations
Autumn 2004

Midterm II Solutions

Problem 1 (25pts)

Throughout this problem, $y\!=\!y(t)$ *denotes the solution to the initial-value problem*

$$y''+y'-2y=e^t, \qquad y(0)=1, \quad y'(0)=2,$$

and $Y\!=\!Y(s)$ *is the Laplace Transform of* $y\!=\!y(t)$.

(a; 5pts) *Show that*

$$\mathcal{L}y' = sY-1 \qquad \text{and} \qquad \mathcal{L}y'' = s^2Y-s-2.$$

Since $\{\mathcal{L}f'\}(s)=s\{\mathcal{L}f\}(s)-f(0)$ by the second table,

$$\begin{aligned}\mathcal{L}y' &= s\mathcal{L}y-y(0)=sY-1\\ \mathcal{L}y'' &= s\mathcal{L}y'-y'(0)=s(sY-1)-2=s^2Y-s-2.\end{aligned}$$

(b; 4pts) *Show that*

$$Y(s)=\frac{1}{(s-1)^2(s+2)}+\frac{s+3}{(s-1)(s+2)}$$

Taking the Laplace Transform of both sides of the ODE defining y and using (a) and the first table, we obtain

$$\big(s^2Y-s-2\big)+\big(sY-1\big)-2Y=\frac{1}{s-1} \quad\Longrightarrow\quad (s^2+s-2)Y-(s+3)=\frac{1}{s-1}$$

$$\Longrightarrow \quad Y(s)=\frac{1}{(s-1)^2(s+2)}+\frac{s+3}{(s-1)(s+2)} \qquad \text{since} \qquad (s^2+s-2)=(s-1)(s+2).$$

(c; 16pts) *Find the solution* $y\!=\!y(t)$ *to the initial value problem*

$$y''+y'-2y=e^t, \qquad y(0)=1, \quad y'(0)=2.$$

By part (b),

$$Y=\mathcal{L}y=\frac{1}{(s-1)^2(s+2)}+\frac{s+2+1}{(s-1)(s+2)}=\frac{1}{(s-1)^2(s+2)}+\frac{1}{s-1}+\frac{1}{(s-1)(s+2)}.$$

Applying quick partial fractions to the last fraction and then twice to the first, we obtain

$$\frac{1}{(s-1)(s+2)}=\frac{1}{2-(-1)}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)=\frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)$$

$$\Longrightarrow \quad \frac{1}{(s-1)^2(s+2)}=\frac{1}{s-1}\cdot\frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)=\frac{1}{3}\frac{1}{(s-1)^2}-\frac{1}{9}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)$$

$$\Longrightarrow \quad Y=\mathcal{L}y=\frac{1}{3}\frac{1}{(s-1)^2}+\frac{11}{9}\frac{1}{s-1}-\frac{2}{9}\frac{1}{s+2} \quad\Longrightarrow\quad \boxed{y(t)=\tfrac{1}{3}te^t+\tfrac{11}{9}e^t-\tfrac{2}{9}e^{-2t}}$$

Problem 2 (15pts)

(a; 8pts) *Find all values of the constant c such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system*

$$\mathbf{y}' = \begin{pmatrix} c & -1 \\ 4 & -1 \end{pmatrix} \mathbf{y}.$$

Specify whether the origin is a spiral source or a spiral sink for the values of c you find.

The origin is a spiral sink or source if and only if the eigenvalues of the matrix are complex, with nonzero real part. The characteristic polynomial in this case is

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + (\det A) = \lambda^2 - (c-1)\lambda + (4-c)$$
$$\implies \quad \lambda_1, \lambda_2 = \frac{1}{2}\big((c-1) \pm \sqrt{(c-1)^2 - 4(4-c)}\big) = \frac{1}{2}\big((c-1) \pm \sqrt{c^2+2c-15}\big).$$

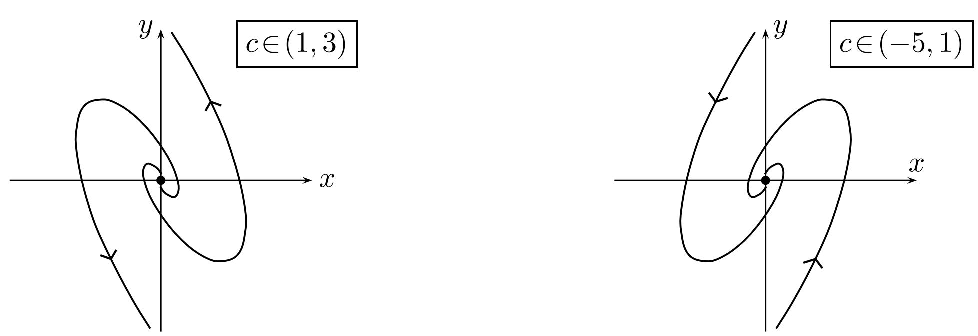
Thus, we need

$$c^2+2c-15=(c+5)(c-3) < 0 \quad \text{and} \quad c-1\neq 0 \qquad \implies \qquad c \in (-5,1), (1,3).$$

The origin is a $\boxed{\text{spiral sink if } c\in(-5,1)}$ since the real part of the eigenvalues is negative, and is a $\boxed{\text{spiral source if } c\in(1,3)}$ since the real part of the eigenvalues is positive.

(b; 7pts) *Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain at least two solution curves. Explain your reasoning.*

If $c \in (-5,1)$, the origin is a spiral sink and the solution curves spiral *in toward* the origin. If $c\in(1,3)$, the origin is a spiral source and the solution curves spiral *out from* the origin. In both cases, the direction of rotation is *positive*, i.e. counterclockwise, since the matrix entry in the lower-left corner is *positive* (4 in this case).



Problem 3 (30pts)

(a; 20pts) *Find the general solution* $(x, y) = (x(t), y(t))$ *to the system of ODEs*

$$\begin{cases} x' = 8x - y \\ y' = 7y - 2x \end{cases}$$

This system of ODEs can be written as

$$\mathbf{y}' = \begin{pmatrix} 8 & -1 \\ -2 & 7 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The characteristic polynomial for this matrix is

$$\lambda^2 - (\mathrm{tr}\, A)\lambda + (\det A) = \lambda^2 - 15\lambda + 54 = (\lambda - 6)(\lambda - 9).$$

Thus, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = 9$. We first find an eigenvector $\mathbf{v}_1$ for λ_1:

$$\begin{pmatrix} 8-\lambda_1 & -1 \\ -2 & 7-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2c_1 - c_2 = 0 \\ -2c_1 + c_2 = 0 \end{cases} \iff c_2 = 2c_1 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

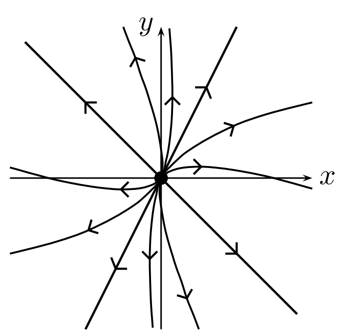
We next find an eigenvector $\mathbf{v}_2$ for λ_2:

$$\begin{pmatrix} 8-\lambda_2 & -1 \\ -2 & 7-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -c_1 - c_2 = 0 \\ -2c_1 - 2c_2 = 0 \end{cases} \iff c_2 = -c_1 \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, the general solution to the system of ODEs is

$$\boxed{\mathbf{y}(t) = C_1 e^{6t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{9t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{6t} + C_2 e^{9t} \\ 2C_1 e^{6t} - C_2 e^{9t} \end{pmatrix}}$$

(b; 8pts) *Sketch the phase-plane portrait, in the xy-plane, for the system of ODEs in (a).*
Since $\lambda_1 < \lambda_2$, the slopes of the solution curves corresponding to $C_1, C_2 \neq 0$ approach 2 as $t \longrightarrow -\infty$ and -1 as $t \longrightarrow \infty$, but the curves do not approach the line $y = -x$.



slopes of half-lines: 2 and -1
slopes of other solution curves approach
2 as $t \longrightarrow -\infty$
-1 as $t \longrightarrow \infty$

(c; 2pts) *Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.*
Since one of the eigenvalues is positive (in fact, both are), some solution curves move away from the origin (in fact, all do). Thus, the origin is an $\boxed{\text{unstable}}$ equilibrium.

Problem 4 (30pts)

(a; 15pts) *Show that*

$$\text{if} \qquad A=\begin{pmatrix}3&1\\0&3\end{pmatrix}, \qquad \text{then} \qquad e^{tA}=e^{3t}\begin{pmatrix}1&t\\0&1\end{pmatrix}.$$

In the given case, the quickest way to verify the claim is to show that the claimed expression for e^{tA} solves the initial value problem:

$$H'=AH, \qquad H(0)=I: \qquad\qquad H(0)=e^{3\cdot 0}\begin{pmatrix}1&0\\0&1\end{pmatrix}=I \ \checkmark$$

$$H'=3e^{3t}\begin{pmatrix}1&t\\0&1\end{pmatrix}+e^{3t}\begin{pmatrix}0&1\\0&0\end{pmatrix}=e^{3t}\begin{pmatrix}3&1+3t\\0&3\end{pmatrix}=e^{3t}\begin{pmatrix}3&1\\0&3\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix}=AH \ \checkmark$$

Here is a more direct approach. Since A has only one eigenvalue, $\lambda=3$, we write

$$A=\lambda I+(A-\lambda I)=\begin{pmatrix}3&0\\0&3\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}=B+C.$$

Since tB is diagonal, so is e^{tB} and the diagonal entries of e^{tB} are the exponentials of the diagonal entries of tB:

$$tB=\begin{pmatrix}3t&0\\0&3t\end{pmatrix} \qquad \Longrightarrow \qquad e^{tB}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}.$$

On the other hand, since the only eigenvalue of tC is zero, the power series for e^{tC} terminates after $n=2$ steps:

$$e^{tC}=I+tC+\frac{t^2}{2!}C^2+\frac{t^3}{3!}C^3+\ldots=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}+\ldots=\begin{pmatrix}1&t\\0&1\end{pmatrix}.$$

Finally, since $tB=3tI$ commutes with every matrix,

$$(tB)(tC)=(tC)(tB) \qquad \Longrightarrow \qquad e^{tA}=e^{tB+tC}=e^{tB}e^{tC}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}\begin{pmatrix}1&t\\0&1\end{pmatrix}=e^{3t}\begin{pmatrix}1&t\\0&1\end{pmatrix}.$$

Another way of finding e^{tA} is to use $e^{tA}=Y(t)Y(0)^{-1}$, where $Y=Y(t)$ is a fundamental matrix for the ODE $\mathbf{y}'=A\mathbf{y}$. In other words, $Y(t)=(\mathbf{y}_1(t)\ \mathbf{y}_2(t))$, where $\mathbf{y}_1(t)$ and $\mathbf{y}_2(t)$ are any two linearly independent solutions of the ODE. This approach works for *every* A.

(b; 15pts) *Find the solution* $\mathbf{y}=\mathbf{y}(t)$ *to the initial value problem*

$$\mathbf{y}'=\begin{pmatrix}3&1\\0&3\end{pmatrix}\mathbf{y}-\begin{pmatrix}3t\\3\end{pmatrix}, \qquad \mathbf{y}(0)=\begin{pmatrix}0\\1\end{pmatrix}.$$

Since $t_0=0$,

$$\begin{aligned}\mathbf{y}(t)&=e^{tA}\Big(\mathbf{y}(0)+\int_0^t e^{-sA}\mathbf{f}(s)\,ds\Big)=e^{tA}\left\{\begin{pmatrix}0\\1\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}1&-s\\0&1\end{pmatrix}\begin{pmatrix}3s\\3\end{pmatrix}ds\right\}\\&=e^{tA}\left\{\begin{pmatrix}0\\1\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}0\\3\end{pmatrix}ds\right\}=e^{tA}\left\{\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}e^{-3s}\Big|_{s=0}^{s=t}\right\}\\&=e^{3t}\begin{pmatrix}1&t\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}e^{-3t}=\boxed{\begin{pmatrix}t\\1\end{pmatrix}}\end{aligned}$$