Math53: Ordinary Differential Equations
Autumn 2004

Midterm II from Winter 2004

Problem 1 (25pts)

(a; 5pts) Show that if $Y = Y(s)$ is the Laplace Transform of the function $y = y(t)$, then the Laplace Transforms of y'' and of y'''' are given by

$$\{\mathcal{L}y''\}(s) = s^2 Y(s) - sy(0) - y'(0), \qquad \{\mathcal{L}y''''\}(s) = s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0).$$

(b; 5pts) Show that if $y = y(t)$ is the solution to the initial value problem

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0,$$

then the Laplace Transform $Y = Y(s)$ of y is given by

$$Y(s) = \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}$$

(c; 15pts) Find the solution $y = y(t)$ to the initial value problem

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0.$$

Problem 2 (15pts)

(a; 8pts) Find all values of the constant c such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system

$$\mathbf{y}' = \begin{pmatrix} 5 & c \\ -c & 1 \end{pmatrix} \mathbf{y}.$$

Specify whether the origin is a spiral source or a spiral sink for the values of c you find.

(b; 7pts) Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain *at least two* solution curves. Explain your reasoning. *Hint:* You need to sketch two different phase-plane portraits. Do *not* solve the system.

Problems 3 and 4 are on the back

Problem 3 (30pts)

(a; 20pts) Find the *general* solution $(x, y) = (x(t), y(t))$ to the system of ODEs

$$\begin{cases} x' = 4y - x \\ y' = x + 2y \end{cases}$$

(b; 8pts) Sketch the phase-plane portrait, in the xy-plane, for the system of ODEs in (a). Clearly show all important qualitative information. In particular, indicate the slope of every relevant line, the flow directions, and the asymptotic behavior of the solution curves. You may want to state some of this information next to the sketch. Sketch *at least two* solution curves in each region cut out by the half-line solutions.

(c; 2pts) Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.

Problem 4 (30pts)

(a; 12pts) Let A be a square matrix. Write down the power-series definition of e^A and use it to show that

$$\text{if} \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \text{then} \qquad e^{tB} = \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} \quad \text{and} \quad e^{tC} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

(b; 6pts) Use any approach you can justify to show that

$$\text{if} \qquad A = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix}, \qquad \text{then} \qquad e^{tA} = \begin{pmatrix} e^{3t} & 0 \\ 2t\,e^{3t} & e^{3t} \end{pmatrix}.$$

(c; 12pts) Find the solution $\mathbf{y} = \mathbf{y}(t)$ to the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 3 & 0 \\ 2 & 3 \end{pmatrix} \mathbf{y} - \begin{pmatrix} 3 \\ 6t \end{pmatrix}, \qquad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
δ	1

$f(t)$	$F(s) = \{\mathcal{L}f\}(s)$
f'	$s \cdot F(s) - f(0)$
$t \cdot f(t)$	$-F'(s)$
$e^{at} f(t)$	$F(s-a)$
$H(t-a) f(t-a)$	$e^{-as} F(s)$

Laplace Transforms

Problem 1 (25pts)

(a; 5pts) *Show that if $Y = Y(s)$ is the Laplace Transform of the function $y = y(t)$, then the Laplace Transforms of y'' and of y'''' are given by*

$$\{\mathcal{L}y''\}(s) = s^2Y(s) - sy(0) - y'(0), \qquad \{\mathcal{L}y''''\}(s) = s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).$$

Since $\{\mathcal{L}f'\}(s) = s\{\mathcal{L}f\}(s) - f(0)$ by the second table,

$$\begin{aligned}\{\mathcal{L}y''\}(s) &= s\{\mathcal{L}y'\}(s) - y'(0) = s\big(s\{\mathcal{L}y\}(s) - y(0)\big) - y'(0) = s^2Y(s) - sy(0) - y'(0) \quad \Longrightarrow\\ \{\mathcal{L}y''''\}(s) &= s^2\{\mathcal{L}y''\}(s) - sy''(0) - y'''(0) = s^2\big(s^2Y(s) - sy(0) - y'(0)\big) - sy''(0) - y'''(0)\\ &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0).\end{aligned}$$

(b; 5pts) *Show that if $y = y(t)$ is the solution to the initial value problem*

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0,$$

then the Laplace Transform $Y = Y(s)$ of y is given by

$$Y(s) = \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}$$

Taking the Laplace Transform of both sides of the ODE, using (a) and the first table, and applying the initial conditions we obtain,

$$\begin{aligned}(s^4Y+3s) + 2s^2Y + Y = 9\frac{s}{s^2+4} \quad &\Longrightarrow \quad (s^4+2s^2+1)Y = \frac{9s}{s^2+4} - 3s\\ \Longrightarrow \qquad Y(s) &= \frac{9s}{(s^2+4)(s^4+2s^2+1)} - \frac{3s}{s^4+2s^2+1}\end{aligned}$$

(c; 15pts) *Find the solution $y = y(t)$ to the initial value problem*

$$y'''' + 2y'' + y = 9\cos 2t, \qquad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = -3, \quad y'''(0) = 0.$$

By part (b),

$$\begin{aligned}Y = \mathcal{L}y = 3s\frac{3-(s^2+4)}{(s^2+4)(s^2+1)^2} = 3s\frac{-1}{(s^2+4)(s^2+1)} = \frac{-3s}{4-1}\Big(\frac{1}{s^2+1} - \frac{1}{s^2+4}\Big) = -\frac{s}{s^2+1} + \frac{s}{s^2+4}\\ \Longrightarrow \qquad \boxed{y(t) = \cos 2t - \cos t}\end{aligned}$$

by the first table.

Problem 2 (15pts)

(a; 8pts) *Find all values of the constant c such that the origin in the xy-plane is a spiral sink or source for the solutions of the linear system*

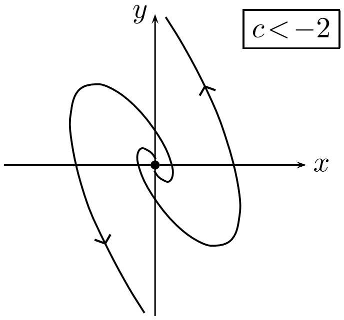
$$\mathbf{y}' = \begin{pmatrix} 5 & c \\ -c & 1 \end{pmatrix} \mathbf{y}.$$

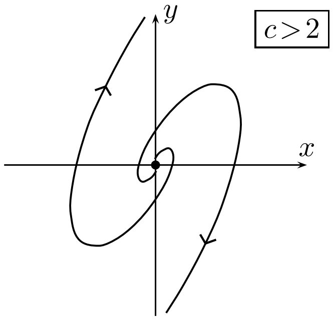
Specify whether the origin is a spiral source or a spiral sink for the values of c you find.
The origin is a spiral sink or source if and only if the eigenvalues of the matrix are complex, with nonzero real part. The characteristic polynomial in this case is

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - 6\lambda + (5+c^2) \quad \Longrightarrow \quad \lambda_1, \lambda_2 = 3 \pm \sqrt{3^2 - (5+c^2)} = 3 \pm \sqrt{4-c^2}.$$

Thus, we need $c^2 > 4$, or $\boxed{c \in (-\infty, -2), (2, \infty)}$ For these values of c, the origin is a $\boxed{\text{spiral source}}$ since the real part of the eigenvalues is positive.

(b; 7pts) *Sketch the phase-plane portraits, in the xy-plane, for the system of ODEs in (a) with the values of c you found. Clearly show all important qualitative information, including the direction of rotation. Each of your sketches should contain at least two solution curves. Explain your reasoning.*
Since the origin is a spiral *source*, the solution curves spiral *out from* the origin. The direction of rotation is determined by the matrix entry in the lower-left corner, i.e. $-c$. If this entry is positive, i.e. $c < 0$, the direction of rotation is positive, and vice versa.





Problem 3 (30pts)

(a; 20pts) *Find the general solution* $(x,y)=(x(t),y(t))$ *to the system of ODEs*

$$\begin{cases} x' = 4y - x \\ y' = x + 2y \end{cases}$$

This system of ODEs can be written as

$$\mathbf{y} = \begin{pmatrix} -1 & 4 \\ 1 & 2 \end{pmatrix}, \qquad \mathbf{y} = \mathbf{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The characteristic polynomial for this matrix is

$$\lambda^2 - (\operatorname{tr} A)\lambda + (\det A) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Thus, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. We first find an eigenvector $\mathbf{v}_1$ for λ_1:

$$\begin{pmatrix} -1-\lambda_1 & 4 \\ 1 & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} -4c_1 + 4c_2 = 0 \\ c_1 - c_2 = 0 \end{cases} \iff c_1 = c_2 \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

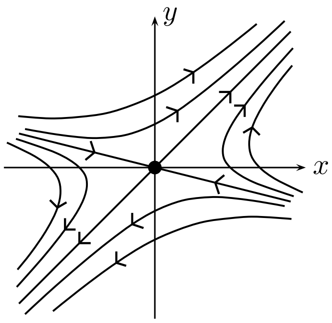
We next find an eigenvector $\mathbf{v}_2$ for λ_2:

$$\begin{pmatrix} -1-\lambda_2 & 4 \\ 1 & 2-\lambda_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} c_1 + 4c_2 = 0 \\ c_1 + 4c_2 = 0 \end{cases} \iff c_1 = -4c_2 \implies \mathbf{v}_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

Thus, the general solution to the system of ODEs is

$$\boxed{\mathbf{y}(t) = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} C_1 e^{3t} + 4C_2 e^{-2t} \\ C_1 e^{3t} - C_2 e^{-2t} \end{pmatrix}}$$

(b; 8pts) *Sketch the phase-plane portrait, in the xy-plane, for the system of ODEs in (a).*



slopes of half-lines: 1 and $-1/4$
other solution curves approach
$y = x$ as $t \longrightarrow \infty$
$y = -x/4$ as $t \longrightarrow -\infty$

(c; 2pts) *Determine whether the origin is a stable, asymptotically stable, or an unstable equilibrium point. Explain why.*

Since one of the eigenvalues, λ_1, is positive, some, and in fact nearly all, solution curves move away from the origin. Thus, the origin is an $\boxed{\text{unstable}}$ equilibrium.

Problem 4 (30pts)

(a; 12pts) *Let A be a square matrix. Write down the power-series definition of e^A and use it to show that*

$$\text{if}\quad B=\begin{pmatrix}0&0\\2&0\end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix}3&0\\0&3\end{pmatrix},\qquad\text{then}\qquad e^{tB}=\begin{pmatrix}1&0\\2t&1\end{pmatrix}\quad\text{and}\quad e^{tC}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}.$$

The power-series definition of e^A is

$$e^A=I+\frac{1}{1!}A+\frac{1}{2!}A^2+\frac{1}{3!}A^3+\ldots\;=\boxed{\sum_{k=0}^{k=\infty}\tfrac{1}{k!}A^k}$$

Applying this definition to tB and tC, we obtain

$$e^{tB}=I+tB+\frac{t^2}{2!}B^2+\frac{t^3}{3!}B^3+\ldots=\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&0\\2t&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}+\ldots=\begin{pmatrix}1&0\\2t&1\end{pmatrix}$$

$$e^{tC}=\sum_{k=0}^{k=\infty}\frac{t^k}{k!}\begin{pmatrix}3&0\\0&3\end{pmatrix}^k=\sum_{k=0}^{k=\infty}\frac{t^k}{k!}\begin{pmatrix}3^k&0\\0&3^k\end{pmatrix}=\begin{pmatrix}\sum\limits_{k=0}^{k=\infty}\frac{1}{k!}(3t)^k&0\\0&\sum\limits_{k=0}^{k=\infty}\frac{1}{k!}(3t)^k\end{pmatrix}=\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}.$$

(b; 6pts) *Use any approach you can justify to show that*

$$\text{if}\qquad A=\begin{pmatrix}3&0\\2&3\end{pmatrix},\qquad\text{then}\qquad e^{tA}=\begin{pmatrix}e^{3t}&0\\2t\,e^{3t}&e^{3t}\end{pmatrix}.$$

Since $tC=(3t)I$, $(tB)(tC)=(tC)(tB)$ and thus

$$e^{tA}=e^{tB+tC}=e^{tB}e^{tC}=\begin{pmatrix}1&0\\2t&1\end{pmatrix}\begin{pmatrix}e^{3t}&0\\0&e^{3t}\end{pmatrix}=\begin{pmatrix}e^{3t}&0\\2t\,e^{3t}&e^{3t}\end{pmatrix}.$$

(c; 12pts) *Find the solution $\mathbf{y}=\mathbf{y}(t)$ to the initial value problem*

$$\mathbf{y}'=\begin{pmatrix}3&0\\2&3\end{pmatrix}\mathbf{y}-\begin{pmatrix}3\\6t\end{pmatrix},\qquad\mathbf{y}(0)=\begin{pmatrix}1\\0\end{pmatrix}.$$

Since $t_0=0$,

$$\begin{aligned}\mathbf{y}(t)&=e^{tA}\big(\mathbf{y}(0)+\int_0^t e^{-sA}\mathbf{f}(s)\,ds\big)=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}1&0\\-2s&1\end{pmatrix}\begin{pmatrix}3\\6s\end{pmatrix}ds\right\}\\&=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}-\int_0^t e^{-3s}\begin{pmatrix}3\\0\end{pmatrix}ds\right\}=e^{tA}\left\{\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}e^{-3s}\Big|_{s=0}^{s=t}\right\}\\&=e^{3t}\begin{pmatrix}1&0\\2t&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}e^{-3t}=\boxed{\begin{pmatrix}1\\2t\end{pmatrix}}\end{aligned}$$