Math53: Ordinary Differential Equations
Autumn 2004

Midterm I Solutions

Problem 1 (20pts)

Find the general solution $y=y(t)$ to the ODE

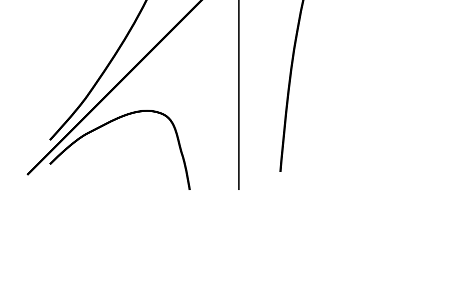
$$ty' = 7t - 6y.$$

Sketch at least five solution curves, on the same plot of the ty-plane, that indicate all possible types of behavior of solutions of this ODE.

This ODE is linear:

$$\begin{aligned}
ty' = 7t - 6y \implies y' + 6t^{-1}y = 7 \implies P(t) = e^{\int 6t^{-1}dt} = e^{6\ln|t|} = t^6 \\
\implies t^6(y'+6t^{-1}y) = t^6 \cdot 7 \implies (t^6y)' = 7t^6 \implies t^6y = \int 7t^6\,dt = t^7 + C \\
\implies \boxed{y(t) = t + Ct^{-6}}
\end{aligned}$$

If $C = 0$, the corresponding solution curve is $y = t$. If $C > 0$, there are two solution curves, corresponding to the intervals $(-\infty, 0)$ and $(0, \infty)$. On the latter interval, this curve lies above the line $y=t$, is asymptotic to it as $t \longrightarrow \infty$, and approaches the positive y-axis $t \longrightarrow 0$. If $C<0$, there are again two solution curves, corresponding to the intervals $(-\infty, 0)$ and $(0, \infty)$. On the latter interval, this curve lies below the line $y = t$, is asymptotic to it as $t \longrightarrow \infty$, and approaches the negative y-axis $t \longrightarrow 0$. Solution curves on the interval $(-\infty, 0)$ are obtained by reflecting solution curves on the interval $(0, \infty)$ about the origin. Under this reflection, C is replaced by $-C$.



Problem 2 (25pts)

(a; 7pts) *Show that the substitution $y = tv$, where $v = v(t)$ is a function of t, reduces the ODE*

$$y' = \frac{t^2 + y^2}{ty} \qquad \text{to} \qquad tv' = 1/v.$$

Plugging in tv instead of y into the first ODE and using the product rule, we get

$$(tv)' = \frac{t^2 + (tv)^2}{t \cdot (tv)} \quad \Longrightarrow \quad v + tv' = \frac{t^2 + t^2 v^2}{t^2 v} = \frac{1 + v^2}{v} \quad \Longrightarrow \quad tv' = \frac{1 + v^2}{v} - v = \frac{1}{v}.$$

(b; 8pts) *Find the general solution $v = v(t)$ to the second ODE in (a).*
The second ODE in (a) is separable:

$$t\frac{dv}{dt} = \frac{1}{v} \quad \Longrightarrow \quad v\,dv = \frac{dt}{t} \quad \Longrightarrow \quad \int v\,dv = \int \frac{dt}{t} \quad \Longrightarrow \quad \frac{1}{2}v^2 = \ln|t| + C$$

$$\Longrightarrow \quad \boxed{v^2 = 2\ln|t| + C \quad \text{or} \quad v(t) = \pm\sqrt{2\ln|t| + C}}$$

(c; 10pts) *Find the solution $y = y(t)$ to the initial value problem*

$$y' = \frac{t^2 + y^2}{ty}, \qquad y(1) = -1,$$

explicitly. Specify the interval of existence. Sketch the corresponding solution curve.
Since $y = tv$, $v(1) = -1$. Plugging this initial condition on v into the first expression in the box above, we get

$$(-1)^2 = 2\ln 1 + C \implies C = 1 \implies v(t) = -\sqrt{2\ln|t| + 1} \implies y(t) = t \cdot v(t) = -t\sqrt{2\ln|t| + 1}$$

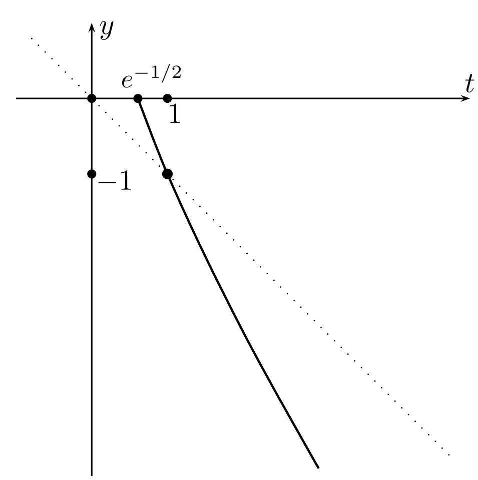
We must take the negative square root because $v(t) = -1$. The last expression above is defined if $t \neq 0$ and

$$2\ln|t| + 1 \geq 0 \iff \ln|t| \geq -1/2 \iff |t| \geq e^{-1/2} \iff t \in (-\infty, -e^{-1/2}), (e^{-1/2}, \infty).$$

The initial value of the parameter t is 1. It lies in the second interval. Thus,

$$\boxed{y(t) = -t\sqrt{2\ln t + 1}, \quad t \in (e^{-1/2}, \infty)}$$

The corresponding solution curve starts at $(e^{-1/2}, 0)$ and descends with ever, but slowly, increasing slope. It passes through the initial data point $(1, -1)$.



Problem 3 (30pts)

(a; 7pts) *Find the general solution $y=y(t)$ to the ODE*

$$y'' - 4y' + 3y = 0.$$

The characteristic polynomial for this ODE is

$$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3).$$

Since the two roots are $\lambda_1, \lambda_2 = 1, 3$, the general solution of this ODE is given by

$$\boxed{y(t) = C_1 e^t + C_2 e^{3t}}$$

(b; 15pts) *Find a solution $y=y(t)$ to the ODE*

$$y'' - 4y' + 3y = e^t.$$

Given the forcing term, we would normally look for a solution of the form $y_p(t) = Ae^t$. However, Ae^t is a solution of the homogeneous equation in this case. We instead try $y_p(t) = Ate^t$:

$$\begin{aligned} & y_p(t) = Ate^t \implies y_p'(t) = A\big(te^t + e^t\big) \implies y_p''(t) = A\big(te^t + 2e^t\big) \\ \implies & A\big((te^t + 2e^t) - 4(te^t + e^t) + 3te^t\big) = e^t \implies A \cdot \big(-2e^t\big) = e^t \implies A = -\frac{1}{2} \\ & \implies \boxed{y_p(t) = -\tfrac{1}{2} te^t} \end{aligned}$$

(c; 8pts) *Find the solution $y=y(t)$ to the initial value problem*

$$y'' - 4y' + 3y = -2e^t, \qquad y(0) = 0, \qquad y'(0) = 3.$$

Since $y = -\frac{1}{2}te^t$ is a solution of $y'' - 4y' + 3y = e^t$, a solution of

$$y'' - 4y' + 3y = -2e^t$$

is given by $y_p = -2 \cdot \big(-\frac{1}{2}te^t\big) = te^t$. Thus, the general solution of $y'' - 4y' + 3y = -2e^t$ is

$$y = C_1 e^t + C_2 e^{3t} + te^t,$$

using part (a). We need to find C_1 and C_2 such that

$$\begin{cases} y(0) = C_1 + C_2 = 0 \\ y'(0) = C_1 + 3C_2 + 1 = 3 \end{cases} \implies C_1 = -1, \quad C_2 = 1 \implies \boxed{y(t) = -e^t + e^{3t} + te^t, \ t \in (-\infty, \infty)}$$

Problem 4 (25pts)

(a; 5pts) *Sketch the graph of the function*

$$f(y) = (y+2)^2(y-1).$$

Label all its intercepts with the y-axis and the f(y)-axis.
See the first plot below.

(b; 20pts) *Find the equilibrium solutions of the ODE*

$$y' = (y+2)^2(y-1), \qquad y = y(t),$$

and sketch their graphs in the ty-plane. On the same plot, sketch at least one solution curve for this ODE in each region of the ty-plane cut out by the graphs of the equilibrium solutions. Indicate their asymptotic behavior, i.e. as $t \longrightarrow \pm\infty$. Explain your reasoning. Determine whether each of the equilibrium solutions is asymptotically stable or unstable. Draw the phase line.

The equilibrium, or constant solutions, are $y(t) = y^*$ such that $f(y^*) = 0$. In this case, the equilibrium solutions are $\boxed{y=-2,\ y=1}$ Their graphs are the horizontal lines $y=-2$ and $y=1$, shown in the last plot below.

Since no two solution curves can cross, no solution curve can cross the horizontal lines $y=-2$ and $y=1$. Thus, if $y(t_0) < -2$ for some t_0, $y(t) < -2$ for all t. It follows that in this case $y'(t) < 0$ for all t, as can be seen either from the graph of f or directly from its definition. Thus, the solution curves in the bottom region descend. They drop to $-\infty$ as t increases and approach the horizontal line $y = -2$ as $t \longrightarrow -\infty$. By the same reasoning, if $-2 < y(t_0) < 1$ for some t_0, $-2 < y(t) < 1$ and $y'(t) < 0$ for all t, and the solution curves in the middle region descend. They approach the horizontal lines $y=-2$ and $y=1$ as $t \longrightarrow \infty$ and $t \longrightarrow -\infty$, respectively. Finally, if $y(t_0) > 1$ for some t_0, $y(t) > 1$ and $y'(t) > 0$ for all t, and the solution curves in the top region ascend. They rise to ∞ as t increases and approach the horizontal line $y=1$ as $t \longrightarrow -\infty$; see the third plot below. The phase line, i.e. the middle plot below, encodes what happens to the solution curves in each region by arrows. An equilibrium solution is stable if both arrows around it point toward it. Since this is not the case for either of two equilibrium points, $\boxed{y=-2, 1 \text{ are unstable}}$