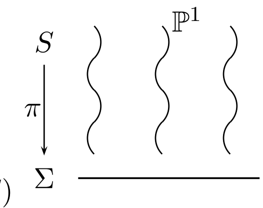
MAT 644: Complex Curves and Surfaces
Notes for 05/06/20

Last 2 times: irrational ruled surfaces

Dfn: $\mathbb{C}$-surface S is ruled
if $\exists$ cmpt conn Riemann surf. Σ and holomor. $\pi\colon S\longrightarrow\Sigma$
s.t. $\pi^{-1}(z)\approx\mathbb{P}^1$ for all $z\in\Sigma$
$\Longleftrightarrow$ $S=\mathbb{P}E$ for some holomor. rk 2 v.b. $E\longrightarrow\Sigma$
$\Longrightarrow$ $\pi^*\colon H^0(K_\Sigma)\longrightarrow H^0(T^*S)$, $\pi^*\eta=\eta\circ\mathrm{d}\pi$, isom. $\Longrightarrow$ $g(\Sigma)=h^{1,0}(S)$
Also: $\chi(S)=-4(g-1)$, $K_S^2=-8(g-1)$



Lemma 0: $S=$ minimal projective surface.
If $\exists$ cmpt conn Riemann surf. Σ and holomor. $\pi\colon S\longrightarrow\Sigma$
s.t. $\pi^{-1}(z)\approx\mathbb{P}^1$ for *generic* $z\in\Sigma$, then S is ruled.

Thm: $S=$ minimal projective surface.
If $\chi(S)<0$, or $K_S^2<0$, or $\chi(\mathcal{O}_S)<0$, *then* S is irrational ruled ($g\geqslant 2$).

$\Longrightarrow$ *If* S is a projective surface with either $\chi(S)<0$ or $\chi(\mathcal{O}_S)<0$,
then S is a blowup of irrational ruled surface with $g\geqslant 2$
b/c $\chi(S)$ decreases, $\chi(\mathcal{O}_S)$ unchanged under blowdown

$\chi(S)<0$ and $K_S^2<0$ cases $\Longrightarrow$ $\chi(\mathcal{O}_S)<0$ case b/c of Noether's Formula, $\chi(\mathcal{O}_S)=\frac{1}{12}\big(\chi(S)+K_S^2\big)$
$\chi(S)<0$ case done Wed; $K_S^2<0$ case almost done Mon

Main Tool: Albanese variety $\mathrm{Alb}(S)$ of S and Albanese map $\Psi_p\colon S\longrightarrow\mathrm{Alb}(S)$, $p\in S$ fixed

$\gamma\subset S$ loop $\rightsquigarrow$ $\int_\gamma\cdot\colon H^0(T^*S)\longrightarrow\mathbb{C}$ $\rightsquigarrow$ $\int_\gamma\cdot\in H^0(T^*S)^*$

$$\Lambda_S\equiv\left\{\int_\gamma\cdot\in H^0(T^*S)^*\colon[\gamma]\in H_1(S;\mathbb{Z})/\mathrm{Tor}\right\}\subset H^0(T^*S)^*$$

Claim: $\Lambda_S\subset H^0(T^*S)^*$ is a lattice ($\Longleftrightarrow$ $\Lambda_S\otimes_{\mathbb{Z}}\mathbb{R}=H^0(T^*S)^*$); same pf as for curves

$\Longrightarrow$ (a) $\mathrm{Alb}(S)\equiv H^0(T^*S)^*/\Lambda_S\approx\mathbb{T}^{2h^{1,0}(S)}$
(b) $\forall\,p\in S$, $\Psi_p\colon S\longrightarrow\mathrm{Alb}(S)$, $\Psi_p(p')=\int_p^{p'}\cdot\in H^0(T^*S)^*/\Lambda_S$, is well-defined holomor.
(c) if $h^{1,0}(S)\geqslant 1$ and $P_1(S)\equiv h^0(K_S)=0$, $\mathrm{Im}\Psi_p\subset\mathrm{Alb}(S)$ is a curve:
if $\mathrm{d}_{p'}\Psi_p$ is injective for $p'\in S$, $\exists\ \omega\in H^0\big(\Lambda^2(T^*\mathrm{Alb}(S))\big)$ s.t. $(\Psi_p^*\omega)_{p'}\equiv\omega_{\Psi_p(p')}\circ\mathrm{d}_{p'}\Psi_p\neq 0$
(d) $\Psi_{p*}\colon H_1(S;\mathbb{Z})/\mathrm{Tor}\longrightarrow H_1(\mathrm{Alb}(S);\mathbb{Z})=\pi_1(S)=\Lambda_S$ is isom.
(e) $\Psi_p^*\colon H^0(T^*\mathrm{Alb}(S))\longrightarrow H^0(T^*S)$ is isom.

Prp 1: $S =$ cmpt conn. $\mathbb{C}$-surface.

If $\mathrm{Im}\Psi_p \subset \mathrm{Alb}(S)$ is a curve with normalization $\pi\colon \Sigma \longrightarrow \mathrm{Im}\Psi_p$, then

$$\begin{array}{ccccc} & & \Sigma & & \\ & \overset{\widetilde{\Psi}_p}{\nearrow} & \downarrow \pi & & \\ S & \xrightarrow{\Psi_p} & \mathrm{Im}\Psi_p & \hookrightarrow & \mathrm{Alb}(S) \end{array}$$

(i) $h^{1,0}(S) = g(\Sigma)$ (ii) $\exists\ \widetilde{\Psi}_p\colon S \longrightarrow \Sigma$ holomor. s.t. $\Psi_p = \pi \circ \widetilde{\Psi}_p$

(iii) $\widetilde{\Psi}_p^{-1}(z)$ is conn. $\forall\, z \in \Sigma$

(iv) if S is minimal projective, $K_S^2 \leqslant 0$, and $\exists$ effective D on S with $K_S \cdot D < 0$, *then* $\widetilde{\Psi}_p^{-1}(z) \approx \mathbb{P}^1$ for $z \in \Sigma$ generic (Lemma 0 $\Longrightarrow$ S is ruled).

proved Mon assuming Lemma below

Lemma 1: $S =$ minimal projective surface s.t. $\Psi_p(S) \subset \mathrm{Alb}(S)$ is a curve

$\Sigma =$ normalization of $\Psi_p(S)$, $\widetilde{\Psi}_p\colon S \longrightarrow \Sigma$ lift of Ψ_p

If $C \subset S$ is an irred. curve with $K_S \cdot C < 0$ and $|\widetilde{\Psi}_p(C)| = 1$, then $\widetilde{\Psi}_p^{-1}(z) \approx \mathbb{P}^1$ for $z \in \Sigma$ generic.

Proof. *Suppose* $C \subset F_z \equiv \widetilde{\Psi}_p^{-1}(z)$ and $F_z = \sum_i m_i F_i$ with $m_i \in \mathbb{Z}^+$, $F_i \subset S$ irred.

$\Longrightarrow$ $K_S \cdot F_i < 0$ for some i, $0 = F_z \cdot F_i = \sum_{j \neq i} m_j (F_j \cdot F_i) + m_i F_i^2$

if $k \geqslant 2$, then $\sum > 0$ (b/c F_z conn.) $\Longrightarrow$ $F_i^2 < 0$ $\rightsquigarrow$ impossible (S is minimal)

if $k = 1$, then $0 > K_S \cdot F_z = 2(a(F_z) - 1)$ $\Longrightarrow$ $a(F_z) = 0$

$\Longrightarrow$ generic fiber of $\widetilde{\Psi}_p$ is $\mathbb{P}^1$ $\square$

Prp 2: $S =$ projective surface with $K_S^2 < 0$. (i) if S minimal, $P_n(S) \equiv h^0(K_S^{\otimes n}) = 0\ \forall\, n \in \mathbb{Z}^+$

(ii) $\exists$ effective divisor D on S with $K_S \cdot D < 0$

Prp 1,2: $S =$ minimal projective surface with $K_S^2 < 0$ $\Longrightarrow$ generic fiber of $\widetilde{\Psi}_p\colon S \longrightarrow \Sigma$ is $\mathbb{P}^1$ (Mon)

Proof of (i). Suppose $n \in \mathbb{Z}^+$ and $nK_S \sim \sum_i m_i C_i$ with $m_i \in \mathbb{Z}^+$, $C_i \subset S$ irred.

$K_S^2 > 0$ $\Longrightarrow$ $K_S \cdot C_i < 0$ for some i

$0 > K_S \cdot C_i = \sum_{j \neq i} m_j (C_j \cdot C_i) + m_i C_i^2$, $\sum \geqslant 0$ $\Longrightarrow$ $C_i^2 < 0$ $\rightsquigarrow$ impossible (S is minimal) $\square$

Note (HW5 #2)*:* $S =$ min. proj. surf., $\exists$ effective D on S with $K_S \cdot D < 0$ $\Longrightarrow$ $P_n(S) = 0\ \forall\, n \in \mathbb{Z}^+$

Proof of (ii). $H^{1,1}(S;\mathbb{Z}) \otimes H^{1,1}(S;\mathbb{Z}) \longrightarrow \mathbb{Z}$, $\alpha \otimes \beta \longrightarrow \int_S \alpha\beta$, has pos. eigenvalue

$K_S^2 < 0$ $\Longrightarrow$ $\exists$ divisor D' on S s.t. $K_S \cdot D' = 0$, $D'^2 > 0$

Riemann-Roch $\Longrightarrow$ $h^0(nD') + h^0(K_S - nD') \geqslant \chi(nD') = \chi(\mathcal{O}_S) + \frac{1}{2}(n^2 D'^2 - K_S \cdot nD')$

$\Longrightarrow$ $\forall\ n$ large, nD' or $K_S - nD'$ is effective

if $K_S - nD'$ is effective, take $D = K_S - nD'$ $\Longrightarrow$ $K_S \cdot D = K_S^2 < 0$

if nD' is effective, take $D = K_S + nD'$ $\Longrightarrow$ $h^2(K_S + nD') = h^0(-nD') = 0$

Riemann-Roch $\Longrightarrow$ $h^0(D) \geqslant \chi(D) = \chi(\mathcal{O}_S) + \frac{1}{2}(n^2 D'^2 + K_S \cdot nD') > 0$ if n is large $\square$

Recap: $S=$ minimal projective surface $\quad \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & 1+b_2^- & & p_g \\ & q & & q & \\ & & 1 & & \end{array} \quad P_n(S)\equiv h^0(nK_S),\ P_1(S)=p_g(S)$

Thm 1: $P_2(S), q(S)=0 \iff S$ is rational
$\iff S=\mathbb{P}^2$ or $S=\mathbb{F}_k\equiv\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k)\big)$ with $k=0,2,3,\ldots$
$\implies \kappa(S)=-\infty$ (i.e. $P_n(S)=0\ \forall\, n\in\mathbb{Z}^+$), $\ \chi(S)>0,\ K_S^2>0$

Thm 2: $\chi(S)<0 \iff K_S^2<0 \iff S$ is irrational ruled with $q(S)\geqslant 2$
$\iff S=\mathbb{P}E$ for some rk 2 holomor. v.b. $E\longrightarrow\Sigma_g$
$\Sigma_g=$ Riemann surf. of genus $g\geqslant 2$
$\implies \kappa(S)=-\infty$

$S=$ minimal projective surface

Thm 3: $\chi(S), K_S^2\geqslant 0$ and S is not rational or ruled $\implies$ either $P_4(S)\neq 0$ or $P_6(S)\neq 0$
Crl 1: $\kappa(S)=-\infty \iff S=\mathbb{P}^2$ or $S=\mathbb{P}E$ for some rk 2 holomor. v.b. $E\longrightarrow\Sigma$
$\Sigma=$ cmpt conn. Riemann surf., $E\not\approx\mathcal{O}_{\mathbb{P}^1}(k)\oplus\mathcal{O}_{\mathbb{P}^1}(k+1)\ \forall\, k\in\mathbb{Z}$
$\iff P_4(S), P_6(S)=0$

Note: $\exists$ non-projective minimal $\mathbb{C}$-surfaces with $\kappa(S)=-\infty$, e.g. $\underbrace{\text{Hopf surfaces } S\approx S^1\times S^3}_{c_1(S)=0,\ nK_S\not\approx\mathcal{O}_S\ \forall\, n\neq 0}$

see Barth-Hulek-Peters-van de Ven, p244

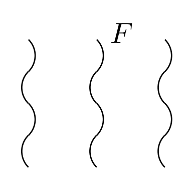
Lemma: $\chi(S), K_S^2\geqslant 0$ and S is not rational $\implies$ either $P_4(S)\neq 0$, or $P_6(S)\neq 0$,
or $\chi(S), K_S^2=0,\ q(S)=1$

Proof. Can assume $P_1(S)=0 \implies 1-q=\chi(\mathcal{O}_S)=\frac{1}{12}\big(\chi(S)+K_S^2\big)\geqslant 0 \implies q\in\{0,1\}$
$q=0$, S not rational $\implies P_2(S)\neq 0$ by Thm 1
$q=1 \implies \chi(S), K_S^2=0 \implies h^{1,1}(S)=2$ □

Prp 3: $S=$ minimal projective surface with $q(S)=1$, $P_2(S)=0$, and $\chi(S)=0$ ($\iff K_S^2=0$).
If S is not ruled, $\exists$ holomor. $\pi\colon S\longrightarrow\mathbb{P}^1$ s.t. $\pi^{-1}(\lambda)$ is smooth of genus 1 for generic $\lambda\in\mathbb{P}^1$
(S is elliptic with base $\mathbb{P}^1$).

generic fiber E of π is smooth conn., $g(E)=1$
$q(S)=1\implies$ Albanese map $\widetilde{\Psi}_p=\Psi_p\colon S\longrightarrow\Sigma=\mathrm{Im}\Psi_p=\mathrm{Alb}(S)\approx\mathbb{T}^2$, generic fiber F of Ψ_p is smooth
Prp 1(iii) $\implies$ all fibers of $\widetilde{\Psi}_p$ are connected $\qquad$ S is not ruled $\implies g(F)\geqslant 1$

$0=\chi(S)=\chi(\mathbb{P}^1)\chi(E)+$(contr. from singular fibers of π)
$\quad=\chi(\mathbb{T}^2)\chi(F)+$(contr. from singular fibers of Ψ_p)
fibers conn. $\implies$ contr$\geqslant 0\implies$ contr$=0$
$\implies$ all fibers are smooth of same genus
but could be multiple if of genus 1 (e.g. for π)



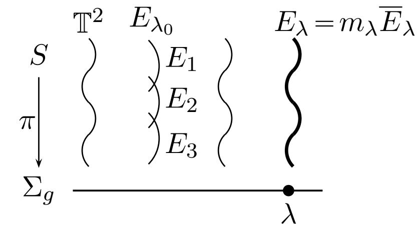
$$\begin{array}{ccc} S & \xrightarrow{\ \pi\ } & \mathbb{P}^1 \\ {\scriptstyle\Psi_p}\downarrow & & \\ \mathbb{T}^2 & & \end{array}$$

Prp 3 + Crl 3 $\implies$ Thm 3

Dfn. $\mathbb{C}$-surface S is elliptic if $\exists$ holomor. $\pi\colon S \longrightarrow \Sigma_g$, where $\Sigma_g =$ Riemann surf. of genus g,
s.t. $E_\lambda \equiv \pi^{-1}(\lambda)$ is smooth of genus 1 for generic $\lambda \in \Sigma_g$.
Non-generic fibers could be singular and/or multiple.

G&H pp569-72: if all multiple fibers are irred., then

$$K_S = \pi^* L + \sum_{\lambda\in\Sigma} (m_\lambda - 1)\overline{E}_\lambda$$



where $E_\lambda \equiv m_\lambda \overline{E}_\lambda$, $\overline{E}_\lambda \subset S$ irred. curve
$L \longrightarrow \Sigma$ holomor. l.b., $\deg L = 2(g-1) + \chi(\mathcal{O}_S)$

follows from failure of exactness of $0 \longrightarrow TS^\vee \longrightarrow TS \xrightarrow{\mathrm{d}\pi} \pi^* T\Sigma \longrightarrow 0$
need only when all fibers are irred.

$m_\lambda \overline{E}_\lambda = E_\lambda \equiv \pi^* \mathcal{O}_\Sigma(\lambda) \implies$ if $m_\lambda | n \ \forall\, \lambda \in \Sigma$, then

$$nK_S = \pi^* \widetilde{L} \qquad \text{with} \qquad \widetilde{L} = nL + \sum_{\lambda\in\Sigma} \mathcal{O}_\Sigma\big((n/m_\lambda)(m_\lambda - 1)\lambda\big) \longrightarrow \Sigma,$$

$$\deg \widetilde{L} = n\left(2(g-1) + \chi(\mathcal{O}_S) + \sum_\lambda \frac{m_\lambda - 1}{m_\lambda}\right) \equiv n\,\delta(S).$$

Crl 2: $P_n(S) \equiv h^0(nK_S)$ $\begin{cases} O(n), & \text{if } \delta(S) > 0,\ m_\lambda | n \ \forall\, \lambda \in \Sigma; \\ \leqslant 1, & \text{if } \delta(S) = 0; \\ 0, & \text{if } \delta(S) < 0. \end{cases}$

E.g. $\pi\colon S \longrightarrow \Sigma_g$ elliptic with $\chi(\mathcal{O}_S) = 0$

$\boxed{g=0}$ $P_n(S) \neq 0$ for some $n \implies \delta(S) = -2 + \sum_{\lambda\in\mathbb{P}^1} \frac{m_\lambda - 1}{m_\lambda} \geqslant 0 \implies$ # multiple fibers $(m_\lambda \geqslant 3) \geqslant 3$

$\implies$ $4K_S$ or $6K_S$ is effective

e.g. $\# = 3$, $m_1, m_2, m_3 \geqslant 3 \implies 3K_S = \sum_i (m_i - 3)\overline{E}_i$ is effective

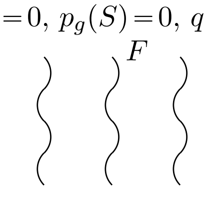
$\boxed{g=1}$ # multiple fibers $\geqslant 1 \implies P_2(S) \neq 0$:

$$2K_S = \big(2\pi^* L + \sum_i m_i \overline{E}_i\big) + \sum_i (m_i - 2)\overline{E}_i \equiv \pi^* \widetilde{L} + \sum_i (m_i - 2)\overline{E}_i$$

$$\widetilde{L} = 2L + \sum_i \lambda_i,\ \deg L = 0 \implies \deg \widetilde{L} > 0 \implies h^0(\widetilde{L}) \neq 0 \implies k^0(2K_S) \neq 0$$

Crl 3: $S \longrightarrow \mathbb{P}^1$ minimal projective elliptic surface with $\chi(S) = 0$, $p_g(S) = 0$, $q(S) = 1$.
If S is not ruled, *then* $P_4(S) \neq 0$ or $P_6(S) \neq 0$.

every fiber E of π is smooth conn., $g(E) = 1$
every fiber F of Ψ_p is smooth conn., $g(F) \geqslant 1$



$$\begin{array}{ccc} S & \xrightarrow{\pi} & \mathbb{P}^1 \\ \Big\downarrow{\scriptstyle \Psi_p} & & \\ \mathbb{T}^2 & & \end{array}$$

Proof. $S \longrightarrow \mathbb{P}^1$ elliptic, all fibers irred., $\chi(\mathcal{O}_S) = 0$

$$\implies K_S = \pi^* L + \sum_\lambda (m_\lambda - 1)\overline{E}_\lambda, \quad \deg L = -2$$

$\pi\colon F \longrightarrow \mathbb{P}^1$ $k{:}1$ for some $k \in \mathbb{Z}^+ \implies \overline{E}_\lambda \cdot F = k/m_\lambda$ b/c $m_\lambda \overline{E}_\lambda = E_\lambda \equiv \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$

$$\implies 0 \leqslant 2(g(F) - 1) = \deg K_F = K_S \cdot F = k\left(-2 + \sum_\lambda \frac{m_\lambda - 1}{m_\lambda}\right) = k\,\delta(S)$$

$\boxed{g=0} \implies P_4(S) \neq 0$ or $P_6(S) \neq 0$. $\square$

Prp 3: $S =$ minimal projective surface with $q(S)=1$, $P_2(S)=0$, and $\chi(S)=0$ ($\Longleftrightarrow K_S^2=0$).
If S is not ruled, $\exists$ holomor. $\pi\colon S\longrightarrow\mathbb{P}^1$ s.t. $\pi^{-1}(\lambda)$ is smooth of genus 1 for generic $\lambda\in\mathbb{P}^1$
(S is elliptic with base $\mathbb{P}^1$).

Lemma: $S =$ projective surface s.t. $K_S\cdot C\geqslant 0$ and $g(\widetilde{C})\geqslant 1$ for all irred. curves $C\subset S$.
If $E\equiv\sum_i m_iE_i$ is an effective divisor on S s.t. $E^2, K_S\cdot E=0$,
then E_i is smooth of genus 1, $E_i^2, K_S\cdot E_i=0$, and $E_i\cap E_j=\varnothing$ for all $i\neq j$.

Proof. $K_S\cdot E=0,\ K_S\cdot E_i\geqslant 0\ \forall\ i\implies K_S\cdot E_i=0\ \forall\ i$

$$E^2=0,\ 1\leqslant g(\widetilde{E}_i)\leqslant a(E_i)=1+\frac{1}{2}\big(E_i^2+K_S\cdot E_i\big)\ \forall\ i\implies 1=g(\widetilde{E}_i)=a(E_i)\ \forall i,\ E_i\cap E_j=\varnothing\ \forall\ i\neq j$$
$$\implies E_i=\widetilde{E}_i \text{ is smooth of genus 1 } \forall\ i \qquad \square$$

Proof of Prp 3. $q(S)=1,\ \chi(S)=0$, Prp 1(iii) $\implies$ all fibers F of $\Psi_p\colon S\longrightarrow \mathrm{Alb}(S)\approx\mathbb{T}^2$ are conn. smooth of same genus

S not ruled $\implies g(F)\geqslant 1\implies g(\widetilde{C})\geqslant 1$ for all irred. curves $C\subset S$
$K_S^2=0$, Prp 1(iv) $\implies K_S\cdot C\geqslant 0$ for all irred. curves $C\subset S$

Claim. $\exists$ a pencil $\{E_\lambda\}_{\lambda\in\mathbb{P}^1}$ of curves on S s.t. (i) $E_\lambda^2, K_S\cdot E_\lambda=0$ (ii) $F\cdot E_\lambda>0$ (iii) $E_0\cap E_\infty=\varnothing$

(i) + Lemma $\implies$ each E_λ splits into disjoint $\mathbb{T}^2$
(iii) $\implies \pi\colon S\longrightarrow\mathbb{P}^1$, $E_\lambda\ni x\longrightarrow\lambda$, is well-defined, holomor.
(ii) $\implies$ (iv) $\pi\colon F\longrightarrow\mathbb{P}^1$ is a branch cover
Lemma 0 on 04/27 $\implies \exists$ branch cover $\sigma\colon\widetilde{\Sigma}\longrightarrow\mathbb{P}^1$ and holomor. $\widetilde{\pi}\colon S\longrightarrow\widetilde{\Sigma}$
s.t. $\pi=\widetilde{\pi}\circ\sigma$ and $\widetilde{\pi}^{-1}(\widetilde{z})$ is conn. $\forall\,\widetilde{\lambda}\in\widetilde{\Sigma}$.
$\widetilde{\Sigma}=\big\{$connected components of E_λ with $\lambda\in\mathbb{P}^1\big\}$.
If $\eta\in H^0(K_{\widetilde{\Sigma}})-\{0\}$, $\widetilde{\pi}^*\eta|_{TF}\not\equiv 0$ b/c of (iv); impossible b/c $\omega|_{TF}\equiv 0\ \forall\,\omega\in H^0(T^*S)$
$\implies\widetilde{\pi}\colon S\longrightarrow\widetilde{\Sigma}=\mathbb{P}^1$ is a fibration with all fibers smooth, conn., of genus 1. $\square$

Crl 4. $S =$ minimal projective surface. $\kappa(S)=0,1\implies\chi(S)\geqslant 0,\ K_S^2=0$
$\kappa(S)=0\implies$ either (a) $(p_g(S),q(S))=(0,0)\implies K_S\neq\mathcal{O}_S,\ 2K_S=\mathcal{O}_S$ (Enriques surface)
(b) $(p_g(S),q(S))=(0,1)\implies S\longrightarrow\mathbb{P}^1$ is elliptic with smooth fibers
(c) $(p_g(S),q(S))=(1,0)\implies K_S=\mathcal{O}_S$ (K3 surface)
(d) $(p_g(S),q(S))=(1,1)\implies$ impossible (G&H p585)
(e) $(p_g(S),q(S))=(1,2)\implies S\approx\mathbb{T}^2$ abelian surface (G&H p583-4)

Partial proof. $\kappa(S)\geqslant 0$ + Thm 2 $\implies\chi(S), K_S^2\geqslant 0$

$$\kappa(S)=0,1\ +\ h^0(nK_S)+h^0(-(n-1)K_S)\geqslant\chi(nK_S)=\chi(\mathcal{O}_S)+\binom{n}{2}K_S^2\implies K_S^2=0$$

$\kappa(S)=0\implies\{h^0(nK_S)\colon n\in\mathbb{Z}^+\}=\{0,1\}\implies(p_g(S),q(S))\in\{(0,0),(0,1),(1,0),(1,1),(1,2)\}$
$(p_g(S),h^0(2K_S),h^0(6K_S))=(0,1,1)\implies h^0(3K_S)=0$
$(\kappa(S),q(S))=(0+,0)$ + Thm 1 $\implies h^0(2K_S)\neq 0$
$(\kappa(S),p_g(S),q(S))=(0+,0,0)\ +\ h^0(3K_S)+h^0(-2K_S)\geqslant\chi(3K_S)=\chi(\mathcal{O}_S)=1$
$\implies K_S\not\approx\mathcal{O}_S,\ h^0(\pm 2K_S)\neq 0\implies 2K_S\approx\mathcal{O}_S$
$(\kappa(S),p_g(S),q(S))=(0,0,1)$ + pf of Prp 3 $\implies S\longrightarrow\mathbb{P}^1$ is elliptic with smooth fibers (only)
$(\kappa(S),p_g(S),q(S))=(0,1,0)\ +\ h^0(2K_S)+h^0(-K_S)\geqslant\chi(2K_S)=\chi(\mathcal{O}_S)=2\implies h^0(-K_S)\neq 0$
$\implies K_S\approx\mathcal{O}_S$

Proof of Claim

(1) $\exists$ curve $E\subset S$ satisfying (i) and (ii) with $E_\lambda = E$

$g(F)\geqslant 2$: comes from the linear system $|2K_S+F_z-F_{z_0}|$ for some $z\in\mathbb{T}^2$ depending on $z_0\in\mathbb{T}^2$ fixed

$K_S\cdot F = 2(g(F)-1)>0 \implies h^0(-nK_S)=0\ \forall\, n\in\mathbb{Z}^+ \implies h^0(2K_S+F_z)\geqslant \chi(2K_S+F_z)=3g-3\ \forall\, z\in\mathbb{T}^2$
Fix $z_0\in\mathbb{T}^2$. Exact sequence

$$0\longrightarrow \mathcal{O}_S\big(2K_S+F_z-F_{z_0}\big)\longrightarrow \mathcal{O}_S\big(2K_S+F_z\big)\longrightarrow \mathcal{O}_S\big(2K_S+F_z\big)\big|_{F_{z_0}} = \mathcal{O}_{F_{z_0}}\big(2K_{F_{z_0}}\big)\big|_{F_{z_0}}\longrightarrow 0$$

of sheaves over S gives exact sequence

$$H^0\big(2K_S+F_z-F_{z_0}\big)\longrightarrow \underbrace{H^0\big(2K_S+F_z\big)}_{\dim\geqslant 3g-3}\xrightarrow{r_z}\underbrace{H^0\big(2K_{F_{z_0}}\big)}_{\dim=3g-3}$$

If r_z is not injective for some z, then $h^0(2K_S+F_z-F_{z_0})\neq 0$ as needed.

If r_z is isomorphism for all z, fix $D\in|2K_{F_{z_0}}| \implies \forall\, z\in\mathbb{T}^2,\ \exists!\ D_z\in|2K_S+F_z|$ s.t. $D_z\cap F_{z_0}=D$.
$D_z\neq D_{z'}$ for $z,z'\in\mathbb{T}^2$ generic $\implies \{(z,p)\in\mathbb{T}^2\times S\colon D_z\ni p\}\longrightarrow S$ is onto
Take any $p\in F_{z_0}-D$, $z\in\mathbb{T}^2$ s.t. $p\in D_z \implies D_z\cap F_{z_0}-D\neq\varnothing \implies F_{z_0}\subset D_z \implies r_z$ is not injective.

$g(F)=1$: $P_2(S)=0$, $\boxed{g=1} \implies \Psi_p$ has no multiple fibers $\implies [K_S]=0\in H^2(S;\mathbb{Z})$
Take curve $H\subset S$ s.t. $m\equiv F\cdot H>0$. For $z\in\mathbb{T}^2$, let $D_z\equiv\{p\in F_z\colon mp\sim H\cap F_z$ on $F_z\}$.
$F_z\longrightarrow \mathrm{Pic}^0(F_z)\approx F_z$, $p\longrightarrow[mp-mp_0]$ unbranched $m^2\!:\!1$ cover $\implies D_z$ consists of m^2 distinct pts
$\implies E\equiv\bigcup_{z\in\mathbb{T}^2}D_z\longrightarrow\mathbb{T}^2$ is unbranched $m^2\!:\!1$ cover $\implies$ every component $E_i\subset E$ is smooth of genus 1
$[K_S]=0\in H^2(S;\mathbb{Z}) \implies K_S\cdot E_i \implies E_i^2=0$ □

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(2) $\exists$ curve $E'\subset S-E$ satisfying (i) and (ii) with $E_\lambda=E'$

$K_S\cdot F=2(g(F)-1)\geqslant 0$, $E\cdot F>0 \implies$ (a) $h^2(nK_S+(n-1)E)=h^0(-(n-1)K_S-(n-1)E)=0\ \forall\, n\geqslant 2$
$\implies$ (b) $h^2(nK_S+nE)=h^0(-(n-1)K_S-nE)=0\ \forall\, n\geqslant 2$
E smooth of genus 1 $\implies (nK_S+nE)|_E=nK_E\equiv\mathcal{O}_E$

Exact sequence

$$0\longrightarrow \mathcal{O}_S\big(nK_S+(n-1)E\big)\longrightarrow \mathcal{O}_S\big(nK_S+nE\big)\longrightarrow \mathcal{O}_S\big(nK_S+nE\big)\big|_E=\mathcal{O}_E\longrightarrow 0$$

of sheaves over S gives exact sequence

$$H^1\big(nK_S+nE\big)\longrightarrow \underbrace{H^1\big(\mathcal{O}_E\big)}_{\dim=1}\longrightarrow H^2\big(nK_S+(n-1)E\big)$$

(a) $\implies$ (c) $h^1(nK_S+nE)\geqslant 1$ (b) + (c) $\implies h^0(nK_S+nE)\geqslant\chi(nK_S+nE)+1=1$
$\implies \exists$ effective $E_2\sim 2(K_S+E)$ and $E_3\sim 3(K_S+E)$
If $E_2=m_2E$ and $E_3=m_3E$ with $m_2,m_3\in\mathbb{Z}$, then $m_2<m_3$ (b/c $E_2\cdot F<E_3\cdot F$)
$\implies K_S\approx(m_3-m_2-1)$ is effective $\rightsquigarrow$ impossible (b/c $P_1(S)=0$)
$\implies E_2$ or E_3 contains component $E'\neq E$ Lemma $\implies E'$ smooth of genus 1, $E'^2, K_S\cdot E'=0$
$E_n\cdot E=0$, $F\cdot E>0 \implies E'\cap E=\varnothing$, $E'\cdot E>0$ □

(3) $\therefore\ \exists$ irred. curves $E, E' \subset S$ s.t. $E \cap E' = \varnothing$, $E^2, E'^2 = 0$, $K_S \cdot E, K_S \cdot E' = 0$, $F \cdot E, F \cdot E' > 0$

$K_S \cdot F = 2(g(F)-1) \geqslant 0$, $E \cdot F, E' \cdot F > 0 \implies$ (a) $h^2(2K_S+E+E') = h^0(-K_S-E-E') = 0$

$\implies$ (b) $h^2(2K_S+2E+2E') = h^0(-K_S-2E-2E') = 0$

Lemma $\implies$ E, E' are smooth of genus 1 $\implies$ $(K_S+E+E')|_{E \sqcup E'} = K_{E \sqcup E'} = \mathcal{O}_{E \sqcup E'}$

Exact sequence

$$0 \longrightarrow \mathcal{O}_S\big(2K_S+E+E'\big) \longrightarrow \mathcal{O}_S\big(2(K_S+E+E')\big) \longrightarrow \mathcal{O}_S\big(2(K_S+E+E')\big)\big|_{E \sqcup E'} = \mathcal{O}_E \oplus \mathcal{O}_{E'} \longrightarrow 0$$

of sheaves over S gives exact sequence

$$H^1\big(2(K_S+E+E')\big) \longrightarrow \underbrace{H^1(\mathcal{O}_E) \oplus H^1(\mathcal{O}_{E'})}_{\dim=2} \longrightarrow H^2\big(2K_S+E+E'\big)$$

(a) $\implies$ (c) $h^1(2(K_S+E+E')) \geqslant 1$ (b) + (c) $\implies$ $h^0(2(K_S+E+E')) \geqslant \chi(2(K_S+E+E'))+2 = 2$

$\implies$ linear system $|2(K_S+E+E')|$ contains a pencil $\{E_\lambda\}_{\lambda \in \mathbb{P}^1}$ of curves on S satisfying (i), (ii)

Lemma $\implies$ base locus $C \subset S$ of $\{E_\lambda\}_{\lambda \in \mathbb{P}^1}$ is a curve

Lemma + $F \cdot E_\lambda > 0$ $\implies$ $F_z \not\subset E_\lambda$ $\forall\, z \in \mathbb{T}^2$, $\lambda \in \mathbb{P}^1$

$\implies$ $\{E_\lambda - C\}_{\lambda \in \mathbb{P}^1}$ is a pencil satisfying (i)-(iii) with E_λ replaced by $E_\lambda - C$ $\square$

Notes on Crl 4: $\kappa(S) = 1 \implies S \longrightarrow \Sigma$ elliptic fibration (G&H p575)

$(\kappa(S), p_g(S), q(S)) = (0,0,1) \implies S = (E_1 \times E_2)/G$ with $E_1, E_2 =$ smooth elliptic curves

$G =$ finite group acting freely on $E_1 \times E_2$ (G&H pp585-8)