$$\boxed{\begin{array}{c}\text{MAT 644: Complex Curves and Surfaces}\\ \text{Notes for 04/29/20}\end{array}}$$

**Last time:** Detecting Rational Surfaces ($S$ birational to $\mathbb{P}^2$)

avoidable

**Prp (Crl of Noether's Lemma):** $S$ = projective surface with $h^1(\mathcal{O}_S), h^0(K_S)=0$.

If $\exists$ irred. curve $C\subset S$ with $g(C)=0$ and $C^2\geq 0$, then $S$ is rational.

(If $S$ is minimal, $K_S\cdot C<0 \Rightarrow C^2\geq 0$)

**Castelnuovo-Enriques Thm** Projective surface $S$ is rational iff $h^1(\mathcal{O}_S), h^0(K_S^{\otimes 2})=0$ $\Longrightarrow \boxed{\begin{array}{c}\chi(S)>0\\ \chi(\mathcal{O}_S)>0\end{array}}$

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Thm proved for $S$ minimal in 3 cases: $K_S^2 =, >, < 0$

Which can actually occur? Only $\boxed{K_S^2>0}$

**Last week:** $S$ = minimal rational $\Rightarrow S=\mathbb{P}^2$ or $\mathbb{F}_k\equiv \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(k))$ with $k=0,2,3,\ldots$

On $\mathbb{P}^2$: $K=-3H \Rightarrow K^2=9>0$

On $\mathbb{F}_k$: $K=-2E_0+(k-2)F,\ E_0^2=k,\ E_0\cdot F=1,\ F^2=0 \Rightarrow K^2=4k-4(k-2)=8>0$

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**Asides:** (1) C-E Thm holds with projective $\longrightarrow$ cmpt complex

Barth-Hulek-Peters-van de Ven, Chapter 6:

$S$ = cmpt $\mathbb{C}$-surface, non-algebraic $\Rightarrow h^1(\mathcal{O}_S), h^0(K_S^2)=0$ not possible

(2) Prp holds in symplectic category w/o $h^1(\mathcal{O}_S), h^0(K_S)=0$

McDuff '90; application of Gromov '85 pseudoholom. curves

(3) C-E Thm holds with projective $\longrightarrow$ cmpt symplectic

$h^1(\mathcal{O}_S)=0 \longrightarrow b_1(S)\leq 1$

$h^0(K_S^{\otimes 2})=0 \longrightarrow \underbrace{c_1^2<0}_{K_S^2<0}$ or $\underbrace{c_1\cdot\omega>0}_{K_S\cdot\omega<0}$

Tian-Jun Li '15: survey

surface

Crl (Luroth's Thm) If $f: S_{\text{rational}} \to S'_{\text{projective}}$ holom. and onto, then $S'$ is also rational.

Pf: Show $h^{1,0}(S)=0 \Rightarrow h^{1,0}(S')=0$ ; $P_n(S)=0 \Rightarrow P_n(S')=0$

$\equiv h^0(K_S^{\otimes n})$

$\omega' \in H^{1,0}(S')$ holom. 1-form $\Rightarrow f^*\omega' \equiv \omega' df \in H^{1,0}(S)$

$\Rightarrow \omega' df = 0$ on $S$

$\Rightarrow \underbrace{Jac(f)=0}_{\text{proper subvariety if } f \text{ is onto}}$ on $S - \underbrace{f^{-1}(\omega'^{-1}(0))}_{\text{proper subvariety if } \omega' \not\equiv 0 \text{ and } f \text{ is onto}}$ $\square$

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Analogue for Curves: $f: \mathbb{P}^1 \to \Sigma_g$ holom., non-const.

$\Sigma_g$ cmpt. conn. Riemann surface

$\Longrightarrow g=0 \Longleftrightarrow \Sigma_g = \mathbb{P}^1$

Luroth's Thm does not extend to higher dim (Clemens-Griffiths '72)

$\exists S, S'$ smooth projective 3-fold, $S$ rational, $S'$ not rational and $f: S \to S'$ holom. surjective

$\therefore$ unirational $\not\Rightarrow$ rational in dim $\geq 3$

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C-E Thm is sharp: neither $h^1(\mathcal{O}_S)=0$ nor $h^0(K_S^{\otimes 2})=0$ can be dropped

(1) $S = \Sigma_g \times \mathbb{P}^1$ $g \geq 1$ $\Rightarrow h^1(\mathcal{O}_S) = h^{0,1}(\Sigma_g) = g \Rightarrow S$ not rational

$K_S = \pi_1^* K_\Sigma + \pi_2 K_{\mathbb{P}^1} \Longrightarrow (pt \times \mathbb{P}^1)\cdot K_S = -2 < 0 \ \forall pt \in \Sigma_g$

$\Rightarrow H^0(K_S^n) = 0 \ \forall n \in \mathbb{Z}^+ \Rightarrow P_2(S)=0$, $\kappa(S) = -1$

Kodaira dimension

$K_S^2 = 2\underbrace{\langle K_\Sigma, \Sigma \rangle}_{2g-2}\underbrace{\langle K_{\mathbb{P}^1}, \mathbb{P}^1 \rangle}_{-2} = -8(g-1) \leq 0$

$\chi(S) = \chi(\Sigma)\chi(\mathbb{P}^1) = -4(g-1) \rightsquigarrow \boxed{\chi(S), K_S^2 \leq 0}$

(2) $S=$ Enriques surface: $\pi_1(S)=\mathbb{Z}_2 \Rightarrow h^1(\mathcal{O}_S)=0$

$K_S\neq 0,\ 2K_S=0 \Rightarrow p_g(S)\equiv P_1(S)\equiv h^0(K_S)=0,$

$P_2(S)\equiv h^0(K_S^{\otimes 2})=1 \Rightarrow S$ not rational

These exist (G&H, later?) $\quad \kappa(S)=0$

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(3) $S=$ Godeaux surface (HW6 #2): $\pi_1(S)=\mathbb{Z}_5 \Rightarrow h^1(\mathcal{O}_S)=0$

$P_1(S)\equiv h^0(K_S)=0;\quad K_S\to S$ positive $\Rightarrow P_n(S)\sim n^2$

$\Longrightarrow S$ is of general type, $\quad \kappa(S)=2$

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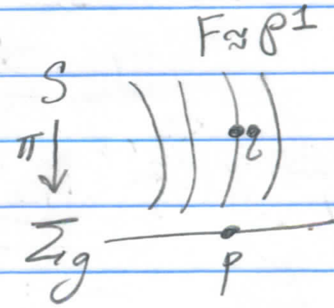
Generalization of (1): Irrational Ruled Surfaces

$S=\mathbb{P}E \xrightarrow{\pi} \Sigma_g,\quad E\to\Sigma_g$ rk 2 holom. v.b., $g\geq 1$

4/20/20 $\longrightarrow \Longleftrightarrow \pi: S\to\Sigma_g$ holom. s.t. $\pi^{-1}(p)\approx\mathbb{P}^1\ \forall p\in\Sigma_g$




$\pi^*: H^{1,0}(\Sigma_g)\to H^{1,0}(S),\ \omega\to\pi^*\omega=\omega\circ d\pi,\quad$ isom.

- injective b/c $d_q\pi$ onto $\forall q\in S$
- onto b/c $\tilde{\omega}|_{T_qF}\equiv 0\ \ \forall\tilde{\omega}\in H^{1,0}(S),\ q\in S$

$\Rightarrow h^1(\mathcal{O}_S)=h^1(\mathcal{O}_{\Sigma_g})=g \Rightarrow S$ is not rational

But $P_n(S)\equiv h^0(K_S^{\otimes n})=0\ \ \forall n\in\mathbb{Z}^+$ b/c $K_S^{\otimes n}\cdot F=-n(\overset{\shortparallel\,2}{\deg TF}+\overset{\shortparallel\,0}{\deg \mathcal{N}_SF})=-2n<0$

$\Rightarrow \forall\omega\in H^0(K_S^{\otimes n}),\ \omega|_F\equiv 0\ \forall$ fibers $F$

$\chi(S)=-4(g-1)\leq 0,\quad K_S^2=-8(g-1)\leq 0$

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$S=$ minimal projective surface

Thm: If either $\chi(S)<0$ or $K_S^2<0$, then $S$ is irrational ruled.

Crl: If $\chi(\mathcal{O}_S)<0$, then — ॥ —

Pf: Noether's Formula $\chi(\mathcal{O}_S)=\frac{1}{12}\left(\chi(S)+K_S^2\right)$

$\therefore\ \chi(\mathcal{O}_S)<0\Rightarrow\chi(S)<0$ or $K_S^2<0$

**Lemma 0:** $S =$ cmpt conn. $\mathbb{C}$-surface, $\Sigma =$ cmpt conn. $\mathbb{C}$-curve, $f\colon S \longrightarrow \Sigma$ holomor. onto

$\exists$ branch cover $\sigma\colon \widetilde{\Sigma} \longrightarrow \Sigma$ and holomor. $\widetilde{f}\colon S \longrightarrow \widetilde{\Sigma}$

s.t. $f = \widetilde{f}\circ\sigma$ and $\widetilde{f}^{-1}(\widetilde{z})$ is conn. $\forall\, \widetilde{z}\in\widetilde{\Sigma}$.

$$\begin{tikzcd} S \arrow[r, "\widetilde{f}"] \arrow[dr, "f"'] & \widetilde{\Sigma} \arrow[d, "\sigma"] \\ & \Sigma \end{tikzcd}$$

***Proof.*** For generic $z\in\Sigma$, the fiber $F_z \equiv f^{-1}(z)$ is smooth

$$\Longrightarrow F_z = \sum_i F_{z,i} \quad \text{with } F_{z,i}\subset S \text{ smooth conn. curve.}$$

index $i$ defined locally, but not globally

$F_{z,i}$ with $z\in\Sigma$ generic limit to curves $F_{z',i}\subset f^{-1}(z')$ with $z'\in\Sigma$ non-generic

$F_{z,i}\cap F_{z,j} = \emptyset$ if $i\neq j \implies$ either $F_{z',i}\cap F_{z',j}=\emptyset$ or $F_{z',i}=F_{z',j}$

Take $\widetilde{\Sigma} \equiv \big\{(z,i)\colon z\in\Sigma,\ F_{z,i}\in\pi_0(f^{-1}(z))\big\}$, $\quad \sigma\colon \widetilde{\Sigma}\longrightarrow\Sigma,\ \ \sigma(z,i)=z$

$\widetilde{f}\colon S\longrightarrow\widetilde{\Sigma},\ \ F_{z,i}\ni x\longrightarrow (z,i)\in\widetilde{\Sigma}$

$\sigma\colon\widetilde{\Sigma}\longrightarrow\Sigma$ is a covering projection over $\Sigma - B$, $B\subset\Sigma$ finite

$\implies \exists!$ holomor. str on $\widetilde{\Sigma}$ s.t. $\sigma\colon\widetilde{\Sigma}\longrightarrow\Sigma$ holomor $\implies \widetilde{f}\colon S\longrightarrow\widetilde{\Sigma}$ holomor.

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**Crl 1:** $S =$ cmpt conn. $\mathbb{C}$-surface. If $h^{1,0}(S)=1$, then $\chi(S)\geqslant 0$.

***Proof.*** Fix $p\in S$. Define $\mu_p\colon S\longrightarrow \mathrm{Alb}(S)\equiv H^{1,0}(S)^*/\Lambda_S \approx \mathbb{C}/\Lambda$

$$\mu_p(p') = \int_p^{p'}\cdot\, \colon H^{1,0}(S)\longrightarrow\mathbb{C}, \qquad \Lambda_S\equiv\Big\{\int_\gamma\cdot\,\colon \gamma\in H_1(S;\mathbb{Z})\Big\}\subset H^{1,0}(S)^* \text{ lattice}$$

integration along a path from $p\in S$ fixed to $p'\in S$

Lemma 0 $\implies \exists$ branch cover $\sigma\colon\widetilde{\Sigma}\longrightarrow\mathbb{C}/\Lambda$ and holomor. $\widetilde{\mu}_p\colon S\longrightarrow\widetilde{\Sigma}$

s.t. $\mu_P = \widetilde{\mu}_p\circ\sigma$ and $\widetilde{\mu}_p^{-1}(\widetilde{z})$ is conn. $\forall\,\widetilde{z}\in\widetilde{\Sigma}$

$\widetilde{f}^*\colon H^{1,0}(\widetilde{\Sigma})\longrightarrow H^{1,0}(S)$ injective, $\ g(\widetilde{\Sigma})\geqslant g(\mathbb{C}/\Lambda)=1$, $\ h^{1,0}(S)=1 \implies \chi(\widetilde{\Sigma})=0$

$\chi(S) = \chi(\widetilde{\Sigma})\chi(\widetilde{F})$ + corrections from singular fibers

$\widetilde{F}\equiv$ generic fiber of $\widetilde{f}$ $\qquad$ $\widetilde{F}$ conn. $\Longrightarrow$ corrections$\geqslant 0$ ($\chi$ of singular fiber $\geqslant\chi(\widetilde{F})$)

partial pf on pp508-10

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**Crl 2:** $S =$ cmpt conn. Kähler surface. If $\chi(S)<0$, then $h^{1,0}(S)\geqslant 2$.

***Proof.*** $\chi(S) = 2+b_2(S)-4h^{1,0}(S) < 0 \implies h^{1,0}(S)\geqslant 1$

Crl 1 $\implies h^{1,0}(S)\neq 1$ $\qquad\square$

*Note:* The first statement in the above proof uses $h^{0,1}(S)=h^{1,0}(S)$. If $S$ is a cmpt conn. $\mathbb{C}$-surface and $h^{0,1}(S)\neq h^{1,0}(S)$, then $h^{0,1}(S)=h^{1,0}(S)+1$ (see HW4 #3). Thus, the conclusions of the first line of the proof and of Crl 2 hold for cmpt conn. $\mathbb{C}$-surfaces.

## Proof of Thm

**Lemma 1:** If $S$ minimal $\xrightarrow{\pi} \Sigma$ holom. and $\pi^{-1}(p) \approx \mathbb{P}^1$ for a generic $p \in \Sigma$, then $S \xrightarrow{\pi} \Sigma$ is a $\mathbb{P}^1$-bundle.

**Lemma 2:** If $S$ is minimal, $\chi(S) < 0$, and $\exists\, \omega_1, \omega_2 \in H^{1,0}(S)$ lin. indep. $\underbrace{\text{s.t. } \omega_1 \wedge \omega_2 \equiv 0 \in H^0(K_S)}_{\text{linearly dependent pointwise}}$,

then $S$ is irrational ruled ($g \geq 2$)

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**Lemma 3:** If $S$ = cmpt Kähler and $\chi(S) < 0$, then $\exists\, \omega_1, \omega_2 \in H^{1,0}(S)$ linearly indep. s.t. $\omega_1 \wedge \omega_2 \equiv 0 \in H^0(K_S)$.

Pf: Crl on p4 $\Rightarrow \dim H^{1,0}(S) \geq 2$

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$\therefore$ (i) $\exists\, \omega_1, \omega_2 \in H^{1,0}(S)$ lin. independ.

(ii) $\exists$ subgroup of $G \subset \pi_1(S)$ of index 5 (b/c Abel($\pi_1(S)$) contains a $\mathbb{Z}$-factor)

$\Rightarrow \exists$ 5:1 covering $\tilde{S} \xrightarrow{p} S \Rightarrow \underbrace{\chi(\tilde{S})}_{= 2 + 2h^0(K_{\tilde{S}}) + h^{1,1}(\tilde{S}) - 4h^1(\mathcal{O}_{\tilde{S}})} = 5\chi(S) \leq -5 \Bigg\}\ h^0(K_{\tilde{S}}) \leq 2h^1(\mathcal{O}_{\tilde{S}}) - 4$

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$\Rightarrow \operatorname{codim}\left(\operatorname{Ker} \Lambda^2 H^{1,0}(\tilde{S}) \to H^{2,0}(\tilde{S})\right) \leq 2h^1(\mathcal{O}_{\tilde{S}}) - 4$

$\dim(\text{decomposables in } \Lambda^2 H^{1,0}(\tilde{S})) = \dim G(2, h^1(\mathcal{O}_{\tilde{S}})) + 1 = 2h^1(\mathcal{O}_{\tilde{S}}) - 3$

$\Longrightarrow$ decomposables $\cap \operatorname{Ker} \neq \{0\} \Rightarrow \exists\, \tilde{\omega}_1, \tilde{\omega}_2 \in H^{1,0}(\tilde{S})$ lin. indep. s.t. $\tilde{\omega}_1 \wedge \tilde{\omega}_2 \equiv 0$

Lemma 2 $\Rightarrow \tilde{\pi}: \tilde{S} \to \tilde{\Sigma}$ is (irrational) ruled $\Rightarrow H^{1,0}(\tilde{S}) = \tilde{\pi}^* H^0(K_{\tilde{\Sigma}})$

$\Rightarrow \tilde{\omega}_1' \wedge \tilde{\omega}_2' \equiv 0\ \forall\, \tilde{\omega}_1', \tilde{\omega}_2' \in H^{1,0}(\tilde{S})$, e.g. $(p^*\omega_1) \wedge (p^*\omega_2) \equiv 0 \Rightarrow \omega_1 \wedge \omega_2 \equiv 0 \in H^0(K_S)$ $\square$

**Lemma 1:** If $S$ minimal $\xrightarrow{\pi} \Sigma$ holom. and $\pi^{-1}(p) \simeq \mathbb{P}^1$ for a generic $p \in \Sigma$, then $S \xrightarrow{\pi} \Sigma$ is a $\mathbb{P}^1$-bundle (all fibers are $\mathbb{P}^1$).

**Pf:** $F \in H_2(S)$ fiberclass $\Rightarrow F^2 = 0$, $F \cdot K_S = -\chi(\mathbb{P}^1) = -2$

Suppose $\pi^{-1}(p) = \sum_{i=1}^{K} m_i C_i$, $K \geq 2$, $m_i \in \mathbb{Z}^+$, $C_i \subset S$ irred.

$\Longrightarrow C_i \cdot K_S < 0$ for some $i$

$0 = F \cdot C_i = \underbrace{\sum_{j \neq i} m_j C_j \cdot C_i}_{\geq 0 \text{ b/c } \pi^{-1}(p) \text{ is conn.}} + C_i^2 \Rightarrow C_i^2 < 0 \Rightarrow C_i \subset S$ exceptional

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**Lemma 2:** If $S$ is minimal, $\chi(S) < 0$, and $\exists \omega_1, \omega_2 \in H^{1,0}(S)$ lin. indep. s.t. $\omega_1 \wedge \omega_2 \equiv 0 \in H^0(K_S)$, then $S$ is irrational ruled ($g \geq 2$).

**Pf:** $\omega_1 \wedge \omega_2 \equiv 0 \in H^{2,0}(S) \Rightarrow \exists f : S \dashrightarrow \mathbb{C}$ merom. s.t. $\omega_1 = f \omega_2$

$f \neq$ const b/c $\omega_1, \omega_2$ lin. indep.

**Claim:** $\pi : S \dashrightarrow \mathbb{P}^1$, $p \to [1, f(p)]$, extends to $S \to \mathbb{P}^1$ holom.

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**Pf of Claim:** $\mathcal{U}_p \subset S$ small neighb. of $p$; define $\Psi : \mathcal{U}_p \to \mathbb{C}^2$; $\Psi(\xi) = \left(\int_p^{\xi} \omega_1, \int_p^{\xi} \omega_2\right)$

$p \rightsquigarrow \xi \subset S$

$\Psi^*(dz_1 \wedge dz_2) = d\Psi_1 \wedge d\Psi_2 = \omega_1 \wedge \omega_2 = 0$

$\Rightarrow \operatorname{Im} \Psi \subset \mathbb{C}^2$ is a curve, $= (g)$ for some $g : \mathbb{C}^2 \to \mathbb{C}$ holom.

$g \circ \Psi = 0 \Rightarrow f = \frac{d\Psi_1}{d\Psi_2} = -\frac{\partial g / \partial z_2}{\partial g / \partial z_1} \circ \Psi$

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\Psi"] \arrow[rr, bend right, "\pi"'] & \operatorname{Im} \Psi \arrow[r, dashed] & \left[1, -\frac{\partial g / \partial z_2}{\partial g / \partial z_1}\right] \in \mathbb{P}^1 \end{tikzcd}$$

$-\frac{\partial g / \partial z_2}{\partial g / \partial z_1}$ merom. function on $\operatorname{Im} \Psi$

$\operatorname{Im} \Psi \subset \mathbb{C}^2$ is a curve $\Rightarrow \operatorname{Im} \Psi \dashrightarrow \mathbb{P}^1$ extends to holom.,

**Also:** $\omega_1, \omega_2 |_{T(\text{fiber of } \Psi)} \equiv 0 \Rightarrow \omega_1, \omega_2 |_{T(\text{fiber of } \pi)} = 0$

$\therefore$ got $\pi: S \to \mathbb{P}^1$ holom.

Lemma 0 $\Rightarrow$ $\exists$ branch cover $\sigma: \tilde{\Sigma} \to \mathbb{P}^1$ and holomor. $\tilde{\pi}: S \to \tilde{\Sigma}$

s.t. $\pi = \tilde{\pi} \circ \sigma$ and $\tilde{\pi}^{-1}(\tilde{z})$ is conn. $\forall \tilde{z} \in \tilde{\Sigma}$

$\omega_1, \omega_2|_{T\pi^{-1}(z)} \equiv 0 \;\; \forall z \in \mathbb{P}^1 \Rightarrow \omega_1, \omega_2|_{T\tilde{\pi}^{-1}(\tilde{z})} \equiv 0 \;\; \forall \tilde{z} \in \tilde{\Sigma}$

$\tilde{\pi}^{-1}(\tilde{z})$ conn. $\forall \tilde{z} \in \tilde{\Sigma}$ $\Rightarrow$ $\omega_i = \pi_i^* \eta_i$ for some $\eta_i \in H^{1,0}(\tilde{\Sigma})$

$\omega_1, \omega_2$ lin. indep. $\Rightarrow \eta_1, \eta_2$ lin. indep. $\Rightarrow g(\tilde{\Sigma}) \geq 0 \Rightarrow \chi(\tilde{\Sigma}) < 0$

$\therefore$ $0 > \chi(S) = \chi(\tilde{\Sigma})\chi(\tilde{F}) + $ (corrections from singular fibers)

$\tilde{F}$ = generic fiber of $\tilde{\pi}$ $\quad$ $\tilde{F}$ conn. $\Rightarrow$ corrections $\geq 0$

$\Rightarrow \chi(\tilde{F}) > 0 \Rightarrow$ generic fiber of $\tilde{\pi}$ is $\mathbb{P}^1$

b/c $\chi(\tilde{\Sigma}) < 0$ $\quad \Rightarrow$ Lemma 2

Lemma 1