MAT 644: Complex Curves and Surfaces
Notes for 04/06/20

Part I: study C = cmpt conn. $\mathbb{C}$-curves ($\dim_{\mathbb{C}} C = 1$)

Hodge diamond

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

$g \equiv h^0(K_C)$

$g=0$: $\mathbb{P}^1 \iff h^0(mK_C)=0\ \forall\, m\in\mathbb{Z}^+$

$g=1$: $\mathbb{C}/\Lambda \iff h^0(mK_C)=1\ \forall\, m\in\mathbb{Z}^+$

$g\geq 2$: less clear
$\dim \mathrm{Im}\big(\iota_{K_C}: C\longrightarrow \mathbb{P}(H^0(K_C))\big) = \dim C$

Part III: study S = cmpt conn. $\mathbb{C}$-surfaces ($\dim_{\mathbb{C}} S = 2$)

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

$p_g \equiv h^0(K_S)$

$\kappa(S)=-\infty \iff h^0(mK_S)=0\ \forall\, m\in\mathbb{Z}^+$ (easiest)

$\kappa(S)=0 \iff \limsup\{h^0(mK_S): m\in\mathbb{Z}^+\}\in\mathbb{R}^+$
(easy) $\iff \{h^0(mK_S): m\in\mathbb{Z}^+\}=\{0,1\}$

$\kappa(S)=1 \iff \limsup\{h^0(mK_S)/m: m\in\mathbb{Z}^+\}\in\mathbb{R}^+$

$\kappa(S)=2 \iff \limsup\{h^0(mK_S)/m^2: m\in\mathbb{Z}^+\}\in\mathbb{R}^+$ (general type)

S = surface $\Longrightarrow$ can get a new surface $\widetilde{S}\equiv \mathrm{Bl}_p S$ by blowing up S at $p\in S$:
replace $p\in S$ by $E_p\equiv \mathbb{P}(T_pS)\approx\mathbb{P}^1\subset\widetilde{S}$, exceptional divisor
replace $\mathbb{C}^2\subset S$ with $C^2\ni 0=p\in S$ by $\gamma=\{(\ell,v)\in\mathbb{P}^1\times\mathbb{C}^2: v\in\ell\}\subset\widetilde{S}$
normal bundle $\mathcal{N}_{\widetilde{S}}E_p$ of E_p in $\widetilde{S}$ is $\gamma\longrightarrow\mathbb{P}^1$, tautological line bundle
$\Longrightarrow E_p\cdot E_p=\langle e(\mathcal{N}_{\widetilde{S}}E_p),E_p\rangle=-1$, homology self-intersection number of E_p in $\widetilde{S}$
$\therefore$ if $\pi:\widetilde{S}\longrightarrow S$ is the blowup of S at some $p\in S$,
then $\widetilde{S}$ contains a smooth curve E s.t. $E\approx\mathbb{P}^1$ and $E\cdot E=-1$ (exceptional curve)

Want to ignore blown up surfaces $\rightsquigarrow$ consider only minimal surfaces

Castelnuovo-Enriques Criterion

If $\widetilde{S}$ is a projective surface and $E\subset\widetilde{S}$ is a smooth curve s.t. $E\approx\mathbb{P}^1$ and $E\cdot E=-1$,
then $\widetilde{S}$ is the blowup of a projective surface S at some $p\in S$ so that $E=E_p$ is the exceptional divisor.

Idea of proof: Find line bundle $\widetilde{L}\longrightarrow\widetilde{S}$ s.t. $\iota_{\widetilde{L}}:\widetilde{S}\longrightarrow\mathbb{P}(H^0(\widetilde{L})^*)$
is well-defined, embedding on $\widetilde{S}-E$, and
maps E to a smooth point of $S\equiv\iota_{\widetilde{L}}(\widetilde{S})$.

Proof. $\widetilde{S}$ projective $\Longrightarrow \exists$ l.b. $L\longrightarrow\widetilde{S}$ s.t. $\iota_L:\widetilde{S}\longrightarrow\mathbb{P}(H^0(L)^*)$ is embedding
$\Longrightarrow L$ is positive
$\Longrightarrow$ (1) $m\equiv\deg(L|_E)>0 \Longrightarrow L|_E=\mathcal{O}_E(m)$ with $m\in\mathbb{Z}^+$
(2) $H^1(\widetilde{S};L^\mu)=0\ \forall\,\mu\gg 0$ (Kodaira Vanishing) $\Longrightarrow$ can assume $H^1(\widetilde{S};L)=0$

$[E]|_E=\mathcal{O}_E(-1) \overset{(1)}{\Longrightarrow} H^1(E;L(kE)|_E)=0 \quad \forall\, k\le m+1$

$$0\longrightarrow \mathcal{O}_{\widetilde S}\big(L(kE-E)\big)\longrightarrow \mathcal{O}_{\widetilde S}\big(L(kE)\big)\longrightarrow \mathcal{O}_{\widetilde S}\big(L(kE)\big)\big|_E\longrightarrow 0$$

$\Longrightarrow H^1(\widetilde S;L((k-1)E))\longrightarrow H^1(\widetilde S;L(kE))$ is onto $\forall\, k\le m+1$

$\overset{(2)}{\Longrightarrow}$ (3) $H^1(\widetilde S;L(kE))=0\ \forall\, k\le m+1$. Use $k=m-2,m-1$ below.

Take $\widetilde L=L(mE) \overset{(3)}{\Longrightarrow}$ (4) restriction $H^0(\widetilde S;\widetilde L(-E))\longrightarrow H^0(E;\mathcal{O}_E(1))$ onto
(5) restriction $H^0(\widetilde S;\widetilde L)\longrightarrow H^0(E;\mathcal{O})$ onto

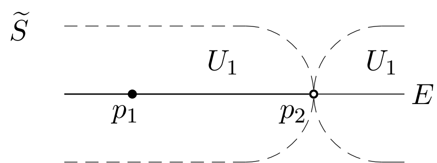
$\overset{(5)}{\Longrightarrow} \iota_{\widetilde L}\colon \widetilde S\longrightarrow \mathbb{P}(H^0(\widetilde S;\widetilde L)^*)$ is well-defined on E

$\iota_{\widetilde L}|_E\equiv$const: $\iota_{\widetilde L}(x)\equiv\big\{s\in H^0(\widetilde S;\widetilde L)\colon s(x)=0\big\}$ is independent of $x\in E$
$=H^0(\widetilde S;\widetilde L(-E))$

$\iota_L\colon \widetilde S\longrightarrow \mathbb{P}(H^0(\widetilde S;L)^*)$ is embedding $\Longrightarrow \iota_{\widetilde L}\colon \widetilde S-E\longrightarrow \mathbb{P}(H^0(\widetilde S;\widetilde L)^*)$ is embedding, $\iota_{\widetilde L}|_E\equiv$const
$\iota_{\widetilde L}(\widetilde S-E)\not\ni \iota_{\widetilde L}(E)=H^0(\widetilde S;\widetilde L(-E))$

Remains to show $\iota_{\widetilde L}(E)\in\iota_{\widetilde L}(\widetilde S)\subset\mathbb{P}^N$ is a smooth point of $S\equiv\iota_{\widetilde L}(\widetilde S)$.
If $\xi_0,\ldots,\xi_N\in H^0(\widetilde S;\widetilde L)$ is basis, $\iota_{\widetilde L}(p)=[\xi_0(p),\ldots,\xi_N(p)]$ for all $p\in\widetilde S$.
Choose ξ_0 s.t. $\xi_0|_E$ does not vanish $\Longleftrightarrow$ basis for $H^0(E;\mathcal{O})$
$\xi_1,\xi_2\in H^0(\widetilde S;\widetilde L(-E))\subset H^0(\widetilde S;\widetilde L)$ s.t. $\xi_1|_E,\xi_2|_E$ is a basis for $H^0(E;\mathcal{O}_E(1))$
$\xi_3,\ldots,\xi_N\in H^0(\widetilde S;\widetilde L(-2E))\subset H^0(\widetilde S;\widetilde L)$ is a basis
$\Longrightarrow$ Near E, $\iota_{\widetilde L}=\left(z_1\equiv\dfrac{\xi_1}{\xi_0},\ldots,z_N\equiv\dfrac{\xi_N}{\xi_0}\right)\colon \widetilde S\longrightarrow\mathbb{C}^N$, $E=\{\xi_1,\xi_2=0\}\longrightarrow(0,\ldots,0)$

Let $\{p_1\}=\xi_1^{-1}(0)\cap E$, $\{p_2\}=\xi_2^{-1}(0)\cap E$
$U_1=$ neighborhood of $E-\{p_2\}$ in $\widetilde S$, $U_2=$ neighborhood of $E-\{p_1\}$ in $\widetilde S$



Claim. $(w_1,w_2)\equiv\left(\dfrac{\xi_1}{\xi_2},z_2=\dfrac{\xi_2}{\xi_0}\right)$ is a chart on U_1 if U_1 is sufficiently small
$\Longrightarrow \iota_{\widetilde L}\colon U_1\longrightarrow\mathbb{C}^N$, $(w_1w_2,w_2,f_1(w_1,w_2))$ for some $f_1\colon U_1\longrightarrow\mathbb{C}^{N-2}$ vanishing on $w_2=0$
similarly for U_2

Claim $\Longrightarrow$ (z_1,z_2) is a chart on $\iota_{\widetilde L}(\widetilde S)$ around $\iota_{\widetilde L}(E)=(0,\ldots,0)\in\mathbb{C}^N$
$\Longrightarrow \iota_{\widetilde L}(E)\in\iota_{\widetilde L}(\widetilde S)$ is a smooth point, $\iota_{\widetilde L}\colon U_1\cup U_2\longrightarrow\mathbb{C}^2$ is blowdown map, $\iota_{\widetilde L}(E)=(0,0)$. □

Pf of Claim. ξ_1/ξ_2 is defined on $E-\{p_2\}\Longrightarrow$ on U_1 if U_1 is sufficiently small
$\mathrm{d}(\xi_1/\xi_2)\neq0$ on TE b/c $\xi_1|_E,\xi_2|_E\in H^0(E;\mathcal{O}(1))$ is a basis
$\mathrm{d}(\xi_2/\xi_0)\big|_{\mathcal{N}_{\widetilde S}E}=(\mathrm{d}\xi_2)/\xi_0\big|_{\mathcal{N}_{\widetilde S}E}$ does not vanish on $E-\{p_2\}$
$\Longrightarrow \mathrm{Jac}(\xi_1/\xi_0,\xi_2/\xi_0)$ has full rank along $E-\{p_2\}$. □

Changes under Blowups

$S =$ cmpt conn. $\mathbb{C}$-surface, $p\in S$, $\widetilde{S}\equiv \mathrm{Bl}_p S$: replace $p\in\mathbb{D}^4\subset S$ by $\gamma\longrightarrow\mathbb{P}^1\subset\widetilde{S}$
$E_p\equiv\mathbb{P}^1$ exceptional divisor, $\pi\colon\widetilde{S}\longrightarrow S$ blowdown map, $\pi^{-1}(p)=E_p$, $\pi\colon\widetilde{S}-E\longrightarrow S-\{p\}$ biholom.

(1) $\pi_*\colon\pi_1(\widetilde{S})\longrightarrow\pi_1(S)$ is an isomorphism:
any loop in S can be homotoped off p; any loop in $\widetilde{S}$ can be homotoped off E

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(2) Mayer-Vietoris for $\widetilde{S}=(S-\{p\})\cup\gamma$ and $S=(S-\{p\})\cup\mathbb{D}^4$ give

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H_i(S^3) & \longrightarrow & H_i(S-\{p\})\oplus H_i(E_p) & \longrightarrow & H_i(\widetilde{S}) & \longrightarrow & \ldots\\
 & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\
\ldots & \longrightarrow & H_i(S^3) & \longrightarrow & H_i(S-\{p\})\oplus H_i(\{p\}) & \longrightarrow & H_i(S) & \longrightarrow & \ldots
\end{array}$$

$\Longrightarrow$ $0\longrightarrow H_i(E_p)\longrightarrow H_i(\widetilde{S})\xrightarrow{\pi_*}H_i(S)\longrightarrow 0$ is exact for all $i\ge 1$

$$\Longrightarrow H_i(\widetilde{S})=\begin{cases}H_i(S), & \text{if } i\neq 2;\\ H_2(S)\oplus\mathbb{Z}\{[E_p]\}, & \text{if } i=2;\end{cases}\qquad h^{p,q}(\widetilde{S})=\begin{cases}h^{p,q}(S), & \text{if } (p,q)\neq(1,1);\\ h^{1,1}(S)+1 & \text{if } (p,q)=(1,1).\end{cases}$$

only the center of the Hodge diamond changes

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(3) Divisors on S vs. $\widetilde{S}$: $\mathrm{Div}(\widetilde{S})=\mathrm{Div}(S)\oplus\mathbb{Z}\{E_p\}$

divisor $=\displaystyle\sum_{i=1}^{k}a_iC_i$ with $a_i\in\mathbb{Z}$, $C_i\subset S,\widetilde{S}$ irred. curve

$\big\{$irred. curve $C\subset S-\{p\}\big\}\overset{\pi^{-1}}{\approx}\big\{$irred. curve $\widetilde{C}\subset\widetilde{S}-E_p\big\}$

What if irred. curve $C\subset S$ passes thr. p?
Take a chart $(z_1,z_2)\colon(U,p)\longrightarrow(\mathbb{C}^2,0)$
$\Longrightarrow$ $C\cap U=(f)$ with $f(z_1,z_2)=\displaystyle\sum_{k=m}^{\infty}f_k(z_1,z_2)$, $f_k=$ homogen. of degree k, $f_m\not\equiv 0$, $m=\mathrm{ord}_pC\ge 1$
charts on $\widetilde{U}\equiv\{(\ell,z)\in\mathbb{P}^1\times\mathbb{C}^2\colon z\in\ell\}\subset\widetilde{S}$: $\widetilde{U}_i\equiv\{([u_1,u_2],z)\in\widetilde{U}\colon u_i\neq 0\}$
$\mathbb{C}^2\longleftrightarrow\widetilde{U}_1$, $(z_1,w_2=u_2/u_1)\longleftrightarrow([1,w_2],(z_1,w_2z_1))$

$$\pi^{-1}(C)\cap\widetilde{U}_1=(f\circ\pi|_{\widetilde{U}_1})\equiv\widetilde{f}_1,\ \widetilde{f}_1(z_1,w_2z_1)=\sum_{k=m}^{\infty}z_1^kf_k(1,w_2)=z_1^mg_1(z_1,w_2)\text{ with } g_1|_{\mathbb{P}^1\cap\widetilde{U}_1}\not\equiv 0$$

$\therefore$ $\pi^{-1}(C)\cap\widetilde{U}_1=\big\{z_1^mg_1(z_1,w_2)=0\big\}=m(E_p\cap\widetilde{U}_1)+\overline{C}\cap\widetilde{U}_1$
$\pi^{-1}(C)\cap\widetilde{U}_2=\big\{z_2^mg_2(z_2,w_1)=0\big\}=m(E_p\cap\widetilde{U}_2)+\overline{C}\cap\widetilde{U}_2$
$\overline{C}\equiv$ the closure of $C-\{p\}$ in $\widetilde{S}\equiv$ the proper transform of C under π (or in $\widetilde{S}$)
Crl. If $C\subset S$ is a curve, then $\pi^*C=\overline{C}+mE_p$, where $m=\mathrm{ord}_pC\Longrightarrow\overline{C}\cdot E_p=m$

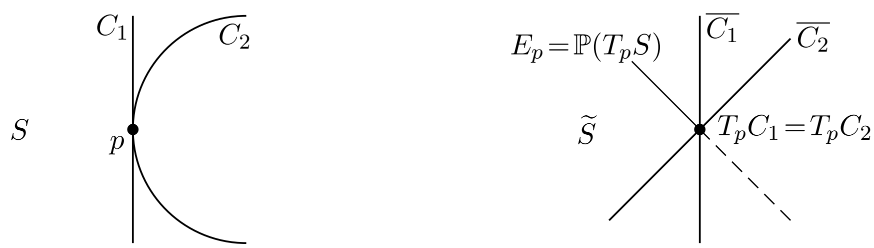
$\therefore$ irred. curve $C\subset S\rightsquigarrow\overline{C}+mE_p\subset\widetilde{S}$ with $\overline{C}$ and E_p irred.
$\Longrightarrow$ if $\widetilde{C}\subset\widetilde{S}$ is irred. curve, then either $\widetilde{C}=E_p$
or $\widetilde{C}\not\supset E_p\Longrightarrow\widetilde{C}\cdot E\equiv m\ge 0\Longrightarrow\widetilde{C}=\overline{\pi(C)}=\pi^*(\pi(C))-mE_p$
$\Longrightarrow$ $\mathrm{Div}(\widetilde{S})=\mathrm{Div}(S)\oplus\mathbb{Z}\{E_p\}$

Side Note 1. As smooth manifolds, $\mathrm{Bl}_p S = S \# \overline{\mathbb{P}^2}$ and $E_p = \mathbb{P}^1$ linearly embedded into $\mathbb{P}^2$. Since the normal bundle of $\mathbb{P}^1$ in $\mathbb{P}^2$ with the standard complex orientation is γ^*, the normal bundle of $\mathbb{P}^1$ in $\overline{\mathbb{P}^2}$ is γ (which is isomorphic to γ^* as a smooth vector bundle, but has the opposite orientation). By definition, $\mathrm{Bl}_p S$ is obtained by gluing the complement $S - \mathbb{D}^4$ of the open ball $\mathbb{D}^4$ around p in S with the disk bundle of γ; the latter is the complement of the open ball $\mathbb{D}^4$ around a point in $\mathbb{P}^2 - \mathbb{P}^1$. This implies the claim above.

Side Note 2. The blowup of S at p separates curves intersecting transversally at p:

If $(z_1, z_2)\colon (U, p) \longrightarrow (\mathbb{C}^2, 0)$ is a coordinate chart, we could take $C_1 = (z_1)$ and $C_2 = (z_2)$. The holomorphic functions $w_1 \equiv z_1 \circ \pi, w_2 \equiv z_2 \circ \pi \colon \pi^{-1}(U) \longrightarrow \mathbb{C}$ then satisfy $(w_i) = E_p + \overline{C_i}$.

More generally, blowing up S at p reduces the order of contact of two curves at p and the extent of the singularity of a curve at p. For example, if C_1 and C_2 have contact of order 2 at p, then $\overline{C_1}$ and $\overline{C_2}$ intersect transversely at the point $T_pC_1 = T_pC_2$ of $E_p = \mathbb{P}(T_pS)$:



Side Note 3. The Castelnuovo-Enriques Criterion holds in the complex category.
The key needed statement is the following.

Prp. If $\widetilde{S}$ is a cmpt $\mathbb{C}$-surface, $E \subset \widetilde{S}$ is a smooth curve s.t. $E \approx \mathbb{P}^1$ and $E \cdot E = -1$, and $\widetilde{p} \in E$, then there exists a holomorphic function $w\colon W \longrightarrow \mathbb{C}$ on a neighborhood of E in $\widetilde{S}$ so that $w|_E \equiv 0$ and $(\mathrm{d}w\colon \mathcal{N}_{\widetilde{S}} E \longrightarrow \mathbb{C}) = (\widetilde{p})$.

A pair of such functions $(w_1, w_2)\colon \widetilde{U} \longrightarrow U \subset \mathbb{C}^2$, with $\widetilde{p}_1 \neq \widetilde{p}_2$, defines a contraction of E.

More details in Chapter III of Barth-Hulek-Peters-van de Ven and references cited there, or just try to sort this out yourself.