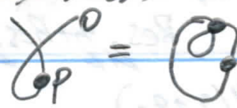
MAT 644: Complex Curves and Surfaces
Notes for 04/01/20

Last time: (1) Construct family of elliptic curves

$$\begin{array}{ccc} \bar{\mathcal{U}} & = & \mathcal{W}\cup_{(\mathcal{W}|_{H'},\,\mathbb{Z}_2\times\mathbb{Z})}(\mathcal{U}',\mathbb{Z}_2) \\ s\uparrow\downarrow\pi & & \downarrow \\ \bar{\mathcal{M}}_{1,1} & = & \mathcal{M}_{1,1}\cup_{(H',\,\mathbb{Z}_2\times\mathbb{Z})}(\mathbb{D},\mathbb{Z}_2) \end{array}$$

$(\pi^{-1}([\tau]), s([\tau])) = (S_\tau \equiv \mathbb{C}/\Lambda_\tau, 0) \quad \forall [\tau]\in\mathcal{M}_{1,1}$

$(\pi^{-1}(\infty), s(\infty)) =$  $\quad Aut(S_\infty, p) = \mathbb{Z}_2 \subset Aut(\mathbb{P}^1)$

(2) Construct weight modular form Δ with $(\Delta)=\{\infty\}\subset\bar{\mathcal{M}}_{1,1}$

$\Rightarrow \lambda_1^{\otimes 12} = \mathcal{O}_{\bar{\mathcal{M}}_{1,1}}(\infty)\to\bar{\mathcal{M}}_{1,1}$

Thm: $(\bar{\mathcal{U}} \overset{\pi}{\underset{s}{\rightleftarrows}} \bar{\mathcal{M}}_{1,1})$ is the universal family over the moduli space of stable nodal genus 1 curves with 1 marked pt:

(a) families $(X \overset{\pi}{\underset{s}{\rightleftarrows}} B)$ of stable nodal genus 1 1-marked curves $/\sim$ correspond to morphisms $\Phi: X\to\bar{\mathcal{M}}_{1,1}$

(b) $X \approx \Phi^*\bar{\mathcal{U}}$ (as equivariant families over universal cover $\tilde{B}\to B$)

$$\begin{array}{ccc} X & \approx & \Phi^*\bar{\mathcal{U}} \\ s\uparrow\searrow & \swarrow & \nearrow\Phi^*s \\ & B & \end{array}$$

Cmpt-ness: if $X \overset{\pi}{\underset{s}{\rightleftarrows}} D^*$ is a family of stable nodal genus 1 1-marked curves, $\exists$ a finite covering $p: D^*\to D^*$, $z\to z^d$, s.t. $p^*X\to D^*$ extends over D as a family of stable nodal genus 1 1-marked curves

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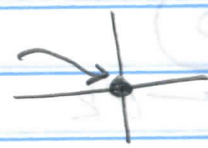
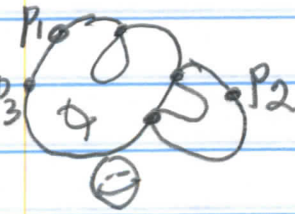
Pre-stable (nodal) k-marked curve: $(C, p_1,\dots,p_k)$

- C = connected $\mathbb{C}$-curve, possibly with nodes  $(xy=0)$
- $p_1,\dots,p_k\in C$ distinct smooth pts

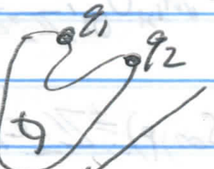
genus of $(C,p_1,\dots,p_k)$ = #holes =

$\equiv \dim \underbrace{H^0(C;K_S)}_{\text{see next}}$



$g=3$
$k=3$

$H^0(C;K_C) = \{$ merom. 1-forms on C with poles only at the nodes, all poles are simple, with residues adding up to 0 at each node $\}$

E.g. $C =$  $\rightsquigarrow \tilde{C}$  $H^0(C;K_C) \equiv \{\eta \in H^0(\tilde{C}; K_{\tilde{C}}(q_1+q_2)) : \mathrm{Res}_{q_1}\eta + \mathrm{Res}_{q_2}\eta = 0\}$

$\dim = 1 - g_{\tilde{C}} + \deg K_{\tilde{C}}(q_1+q_2)$

$= 1 - g_{\tilde{C}} + 2g_{\tilde{C}} - 2 + 2 = g_{\tilde{C}} + 1 = g_C$ ✓

$\mathrm{Aut}(C, p_1, \dots, p_k) = \{\sigma : C \to C$ biholom. s.t. $\sigma(p_i) = p_i \ \forall i = 1, \dots, k\}$

$(C, p_1, \dots, p_k)$ is <u>stable</u> if $|\mathrm{Aut}(C, p_1, \dots, p_k)| < \infty$

<u>Easy:</u> C = smooth, of genus g, $p_1, \dots, p_k \in C$ distinct, then

(i) <u>$g \geq 2$:</u> $\mathrm{Aut}(C, p_1, \dots, p_k) \subset \mathrm{Aut}(C)$ is finite

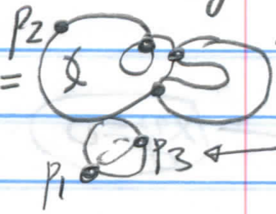
(ii) <u>$g = 1$:</u> $(C, p_1, \dots, p_k)$ is stable iff $k \geq 1$

(iii) <u>$g = 0$:</u> $(C, p_1, \dots, p_k)$ is stable iff $k \geq 3$

$\iff \mathrm{Aut}(C, p_1, \dots, p_k) = \{\mathrm{id}\}$

<u>Crl:</u> pre-stable k-marked curve $(C, p_1, \dots, p_k)$ is stable iff every genus 0 irreducible component of C contains at least 3 nodal or marked pts

and $(g(C), k) \neq (1, 0)$

$C =$  ← not O'K: contains only 2 nodal and marked pts

$\implies (C, p_1, p_2, p_3)$ not stable

← O'K

Pre-stable $(C, p_1, \dots, p_k) \sim (C', p_1', \dots, p_k')$ if $\exists \sigma: C \to C'$ biholom. s.t. $\sigma(p_i) = p_i'$ $\forall i$

A family of stable nodal genus g k-marked curves is

$$\begin{array}{c} X \\ s_1,\dots,s_k \curvearrowright \downarrow \pi \\ \text{sections} \quad B \end{array}$$

$\mathbb{C}$-mflds s.t. $\forall b \in B$, $(X_b \equiv \pi^{-1}(b), s_1(b), \dots, s_k(b))$ is a stable nodal genus g k-marked curves

$(\Longrightarrow s_i(b) \in X_b$ is a smooth pt; $s_i(b) \neq s_j(b)$ $\forall i \neq j)$

Fact: $\forall g, n$ ① $\exists$ universal family over the moduli space of stable nodal genus g k-marked curves: $\overline{\mathcal{U}} \underset{s_1,\dots,s_k}{\overset{\pi}{\rightleftarrows}} \overline{\mathcal{M}}_{g,k}$:

families of stable nodal genus g k-marked curves $(X \underset{s_1,\dots,s_k}{\overset{\pi}{\rightleftarrows}} B)$ correspond to morphisms $\Phi: B \to \overline{\mathcal{M}}_{g,k} \rightsquigarrow X = \Phi^* \overline{\mathcal{U}}$

} in the sense of orbifolds

(2) Cmpt-ness if $X \underset{s_1,\dots,s_k}{\overset{\pi}{\rightleftarrows}} D^*$ is a family of stable nodal genus g k-marked curves, $\exists$ finite covering $p: D^* \to D^*$ s.t. $p^* X \to D^*$ extends over D as a family of stable nodal genus g k-marked curves

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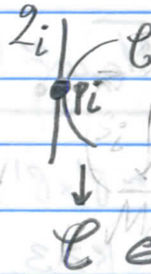
Important Bundles over $\overline{\mathcal{M}}_{g,k}$

(i) universal cotangent line bundle at the i-th marked pt $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,k}$

$$\mathcal{L}_i = s_i^*(T\overline{\mathcal{U}}^v)^*, \quad T\overline{\mathcal{U}}^v = \ker d\pi \to \overline{\mathcal{U}}$$

$$\rightsquigarrow \psi_i \equiv c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}), \quad i = 1, \dots, k$$



$$\begin{array}{c} \overline{\mathcal{U}} \\ s_i \uparrow \downarrow \pi \\ \mathcal{C} \in \overline{\mathcal{M}}_{g,k} \end{array}$$

(ii) Hodge bundle of holomorphic differentials $\mathbb{E} \to \overline{\mathcal{M}}_{g,k}$

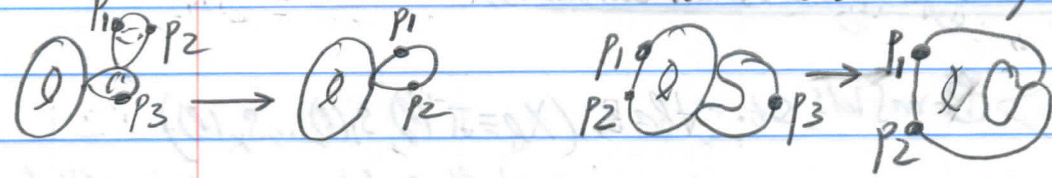
$$\mathbb{E}|_{(C, p_1, \dots, p_k)} = H^0(C; K_C), \quad \operatorname{rk} E = g$$

$$\rightsquigarrow \lambda_j = c_j(\mathbb{E}) \in H^{2j}(\overline{\mathcal{M}}_{g,k}; \mathbb{Q}), \quad j = 1, \dots, g$$

Forgetful morphism $f: \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}$ if $2g+(k-1) \geq 3$

$[C, p_1, \dots, p_k] \to [\overline{C}, p_1, \dots, p_k]$

contract unstable components (if any)



Easy: (i) generic fiber of f is a smooth genus g curve

(ii) $\mathbb{E} = f^*\mathbb{E} \Rightarrow \lambda_j = f^*\lambda_j \quad \forall j=1,\dots,g$

(iii) $L_i = f^*L_i \otimes \mathcal{O}_{\overline{\mathcal{M}}_{g,k}}\big(\underbrace{\{\ \cdots\ \}}_{D_{ik}}\big) \quad \forall i=1,\dots,k-1$

$\Rightarrow \psi_i = f^*\psi_i + D_{ik}$

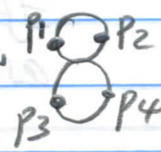
Examples: $g=0 \Rightarrow k \geq 3$ (o/w no stable curves)

$k=3$ (C, p_1, p_2, p_3) stable, $g(C)=0 \Rightarrow (C, p_1, p_2, p_3) \simeq (\mathbb{P}^1, 0, 1, \infty)$

$\Rightarrow \overline{\mathcal{M}}_{0,3} = pt, \quad \mathcal{U} = \mathbb{P}^1 \to pt$

$k=4$ $(C, p_1, p_2, p_3, p_4) \simeq (\mathbb{P}^1, 0, 1, \infty, x)$, $x \in \mathbb{C}^* - \{0,1\}$, or

cross-ratio



3 of these

$\Rightarrow \overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1, \quad \mathcal{U} = \overline{\mathcal{M}}_{0,5} \xrightarrow{f} \overline{\mathcal{M}}_{0,4}$

$k=5$ $\overline{\mathcal{M}}_{0,5} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ blown up at $(0,0), (1,1), (\infty, \infty)$

$\overline{\mathcal{M}}_{0,k}$, $k \geq 3$, a blowup of $(\mathbb{P}^1)^{k-3}$

moduli space exists as a mfld b/c $\mathrm{Aut}(C, p_1, \dots, p_k) = \{id\}$

$\mathbb{E} \to \overline{\mathcal{M}}_{0,k}$ rank 0; (iii) $\Rightarrow \displaystyle\int_{\overline{\mathcal{M}}_{0,k}} \psi_1^{a_1} \cdots \psi_k^{a_k} = \binom{k-3}{a_1, \dots, a_k}$

Previously: constructed $\overline{\mathcal{M}}_{1,1}$

(1) also $\mathbb{L}_1 \to \overline{\mathcal{M}}_{1,1}$; same $\mathbb{L}_1$! HW4 1b: check on the charts and overlap

E.g. $\mathbb{L}_1^{old} = \mathbb{D}\times\mathbb{C} \circlearrowleft (-1)\times$, $\quad \mathbb{L}_1^{old} \to (\mathbb{D},\mathbb{Z}_2)$

$\mathbb{L}_1^{new} = s_1^*(T\mathcal{U}'^{v})^* \to (\mathbb{D},\mathbb{Z}_2) \underset{s_1}{\rightleftarrows} \mathcal{U}' \subset \mathbb{D}\times\mathbb{P}^2 \ni [X,Y,Z] \circlearrowleft (-1)\times$

$s_1(q)=[0,1,0]\in q\times\mathcal{C}_q$, $\{[X,Y,0]\in\mathbb{P}^2\}$ tangent to $\mathcal{C}_q$

$\Longrightarrow \mathbb{L}_1^{new} = \mathbb{D}\times L_\infty \circlearrowleft (-1)\times \Longrightarrow$ same over $\mathbb{D}$

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(2) $\mathbb{E}\approx\mathbb{L}_1\to\overline{\mathcal{M}}_{1,1}$ (only!) Define homom. $\mathbb{E}\otimes\mathbb{L}_1^*\xrightarrow{T}\mathbb{C}$:

$\eta\in\mathbb{E}|_C = H^0(C;\kappa_C)$, $v\in\mathbb{L}_1^* = T_{p_1}C$, $T(\eta\otimes v)=\eta_{p_1}(v)\in\mathbb{C}$

$p_1\in C$ marked pt, smooth: $\eta_{p_1}\neq 0$ if $\eta\neq 0\Longrightarrow T$ is isom.

$\therefore$ On $\overline{\mathcal{M}}_{1,1}$ (only!): $\lambda_1=\psi_1=\frac{1}{12}c_1(\underbrace{\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(\infty)}_{\Delta_0=\{\ p_1\ \}})$

end of last time

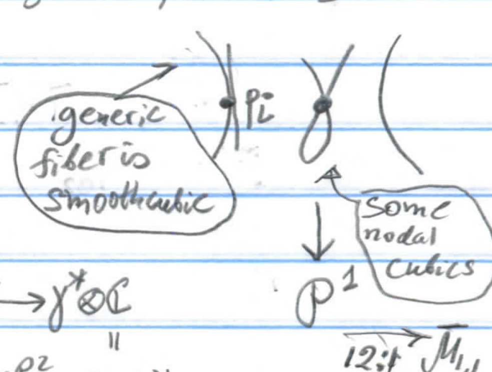
$$\int_{\overline{\mathcal{M}}_{1,1}}\lambda_1=\int_{\overline{\mathcal{M}}_{1,1}}\psi_1=\frac{1}{12}\cdot\frac{1}{|\mathrm{Aut}(\ )|}=\frac{1}{24}$$

HW4 2: Example of non-trivial family of stable nodal genus 1 1-marked curves

$p_1,\dots,p_8\in\mathbb{P}^2$ in general position $\Longrightarrow \underbrace{\mathbb{P}\{f\in \underbrace{H^0(\mathbb{P}^2;\mathcal{O}_{\mathbb{P}^2}(3))}_{\approx\mathbb{C}^{10}}: s(p_i)=0\ \forall i\}}_{\{\text{cubics thr. 8 general pts in }\mathbb{P}^2\}}\approx\mathbb{P}^1$

$X\equiv\{([f],q)\in\mathbb{P}^1\times\mathbb{P}^2: f(q)=0\}$

$X \xrightarrow{} \mathbb{P}^1$, $s:\mathbb{P}^1\to X$, $\quad s([f])=([f],p_i)$ $i=1,\dots,8$ (any)



Implicit FT $\Longrightarrow X\subset\mathbb{P}^1\times\mathbb{P}^2$ smooth submfld

What is $\mathbb{L}_1\to\mathbb{P}_1$? $T_{p_i}X_{[f]}=\mathrm{Ker}\,\nabla f|_p: \gamma^*\otimes\mathbb{C}^2\to\gamma^*\otimes\mathbb{C}$

$\gamma^*\downarrow\mathbb{P}^1$, $\quad \gamma^*\otimes\mathbb{C}^2 = T_{p_i}\mathbb{P}^2$, $\quad \gamma^*\otimes\mathbb{C} = \mathcal{O}_{\mathbb{P}^2}(3)|_{p_i}$

References

"Cmpt-ness" of $\overline{\mathcal{M}}_{1,1}$: Section 5.3 of Hain

Hodge line bundle $\mathbb{E} \to \overline{\mathcal{M}}_{1,1}$: Section 5.4

Pic of $\mathcal{M}_{1,1}$ and $\overline{\mathcal{M}}_{1,1}$: Section 6

Topology of $\overline{\mathcal{M}}_{1,1}$: Section 7

Topology of $\overline{\mathcal{M}}_{0,k}$: Keel'92, MR1034665

Construction of $\overline{\mathcal{M}}_{g,0}$ via AG : Harris-Morrison'98

analysis: Robbin-Salamon'06, MR2262197