MAT 644: Complex Curves and Surfaces

Problem Set 5

Written Solutions (if any) due by Wednesday, 04/29, 10am

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *you are encouraged to write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

Let S be a (compact) complex surface, $\mathcal{C}\subset S$ be a (closed) connected curve, and $h:\tilde{\mathcal{C}}\longrightarrow\mathcal{C}$ a normalization of $\mathcal{C}$. Let $S_0=S,S_1,\ldots$ be a sequence of surfaces and $\mathcal{C}_0=\mathcal{C},\mathcal{C}_1,\ldots$ a sequence of curves, with $\mathcal{C}_k\subset S_k$, such that S_k is the blowup of S_{k-1} at a singular point $p_k\in\mathcal{C}_{k-1}$ and $\mathcal{C}_k$ is the proper transform of $\mathcal{C}_{k-1}$ under the blowup $S_k\longrightarrow S_{k-1}$. It was shown in class that

$$a(\mathcal{C}_k)=a(\mathcal{C})-\sum_{l=1}^{l=k}\binom{m_{p_l}(\mathcal{C}_{l-1})}{2}, \tag{1}$$

where $m_{p_l}(\mathcal{C}_{l-1})$ is the order of the point p_l in the curve $\mathcal{C}_{l-1}$. The above sequences terminate at the k-th stage if and only if $\mathcal{C}_k\subset S_k$ is a smooth curve. If this does happen, $\mathcal{C}_k\approx\tilde{\mathcal{C}}$ and equation (1) implies that

$$g(\tilde{\mathcal{C}})\le a(\mathcal{C})-\sum_{p\in\mathcal{C}}\binom{m_p(\mathcal{C})}{2},\qquad a(\mathcal{C})\in\mathbb{Z}^{\ge0}. \tag{2}$$

In particular, $g(\tilde{\mathcal{C}})\le a(\mathcal{C})$ and the equality holds if and only if $\mathcal{C}\subset S$ is smooth. In class, the first statement in (2) was proved in a different way for S projective; it implies that $a(\mathcal{C})\ge1-N(\mathcal{C})$ for any curve, where $N(\mathcal{C})$ is the number of connected components of $\mathcal{C}$. This was used along with (1) to show that the above sequence does terminate. The estimate $a(\mathcal{C})\ge1-N(\mathcal{C})$ can also be obtained by showing either that $a(\mathcal{C})+N(\mathcal{C})-1$ is the dimension of the space of holomorphic 1-forms on $\mathcal{C}$, properly defined (since $\mathcal{C}$ is singular), or that $a(\mathcal{C})$ is the genus of a smooth curve in the same homology class as $\mathcal{C}$, which exists at least after deforming the complex structure. The purpose of this exercise is to show that the sequences terminate without an *a priori* lower bound on $a(C)$.

(a) Let $B_{i;0}\subset\mathcal{C}$ be a branch of $\mathcal{C}$ at a singular point $p\in\mathcal{C}$ and $B_{i;k}\subset S_k$ be the proper transform of $B_{i;k-1}$ in S_k. Show that $\pi_{i;k}:B_{i;k}\longrightarrow B_{i;0}$ is a bijection, $B_{i;k}$ contains at most 1 singular point, $p_{i;k}\equiv\pi_{i;k}^{-1}(p)$, and $p_k\neq p_{i;k}$ if k is sufficiently large (i.e. $B_{i;k}$ becomes smooth after enough blowups at $p_{i;k}$).

(b) If B_1 and B_2 are distinct smooth branches at $p_k\in\mathcal{C}_{k-1}$, the order of contact of the proper transforms of B_1 and B_2 in S_k is strictly less than $\mathrm{mult}_{p_k}(B_1,B_2)$.

(c) Show that $\mathcal{C}_k$ is a smooth curve if k is sufficiently large.

Problem 2 (10 pts)

(a) If S is a minimal surface and $D \subset S$ is an effective divisor such that $D \cdot K_S < 0$, then

$$P_n(S) \equiv h^0(K_S^n) = 0 \qquad \forall n \in \mathbb{Z}^+ .$$

(b) If S, S' are minimal (projective) surfaces and $f : S \longrightarrow S'$ is a birational map, then either f is a biholomorphic map or $P_n(S) = P_n(S') = 0$ for all $n \in \mathbb{Z}^+$.
Note: This says that a surface S with Kodaira dimension $\kappa(S) \geq 0$ has a unique minimal model.

Problem 3 (10 pts)

Let S be a compact connected surface and D an effective divisor on S.
(a) If $D^2 \equiv D \cdot D < 0$, show that the base locus of the linear system of curves $|D|$ contains a curve. Is the converse true?
(b) If the base locus of $|D|$ does not contain a curve, show that it consists of at most D^2 points.
(c) If D is irreducible and $h^0(D) \geq 2$, then $D^2 \geq 0$. Show that the conclusion need not hold if D is not assumed to be irreducible.
(d) If S is a minimal rational surface and $D^2 \geq 0$, show that $h^0(D) \geq 2$. Show that the conclusion need not hold if S is not assumed to be minimal.
Hint: you can achieve arbitrarily high D^2 with $h^0(D) = 1$ by blowing up $\mathbb{P}^2$ at sufficiently many general points.

Problem 4 (5 pts)

Let $\mathbb{F}_n = S_n$ be the n-th Hirzebruch surface and $D \subset \mathbb{F}_n$ an irreducible curve. Compute $H^q(\mathbb{F}_n, \mathcal{O}(D))$ in terms of the decomposition of D relative to a standard basis for $\mathrm{Pic}(\mathbb{F}_n)$.