MAT 545: Complex Geometry

Problem Set 7

Written Solutions due by Friday, 12/06, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

(a) Let $X_a \subset \mathbb{P}^n$ be a smooth degree a hypersurface with $n \geq 3$ and $a \geq 1$. Show that

$$\dim_{\mathbb{C}} H^0_{\bar\partial}\big(X_a; \mathcal{K}_{X_a}\big) = \begin{cases} 0, & \text{if } a \leq n; \\ \binom{a-1}{n}, & \text{if } a > n; \end{cases} \qquad \chi(X_a) = 1+n+\frac{1}{a}\big((1-a)^{n+1}-1\big).$$

(b) Determine the Hodge diamond for a smooth degree a hypersurface $X_a \subset \mathbb{P}^3$;
(c) Determine the Hodge diamond for a smooth degree a hypersurface $X_a \subset \mathbb{P}^4$.
(d) Determine the Hodge diamond for a smooth degree 2 hypersurface $X_2 \subset \mathbb{P}^5$.
Note: the quartic surface $X_4 \subset \mathbb{P}^3$ is a K3 surface; the quintic $Y_5 \subset \mathbb{P}^4$ is a Calabi-Yau threefold, popular in string theory.

Problem 2 (5 pts)

Let $u\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^n$ be a holomorphic map of degree d (thus, $u_*[\mathbb{P}^1] = d[\mathbb{P}^1] \in H_2(\mathbb{P}^n)$). If $d \leq n$, show that $u(\mathbb{P}^1)$ is contained in some linearly embedded $\mathbb{P}^d$ in $\mathbb{P}^n$.
Note: this is a special case of the Castenuovo bound. It implies for example that every degree 2 (rational) curve in $\mathbb{P}^3$ is in fact contained in some hyperplane $\mathbb{P}^2 \subset \mathbb{P}^3$. This makes it possible to use classical Schubert calculus (homology intersections on $G(k,n)$) to determine the number of such conics in $\mathbb{P}^3$ that pass through a points and $8-2a$ lines in general position.

Problem 3 (5 pts)

Let Σ be a compact connected Riemann surface (complex one-dimensional manifold). Show that Σ can be holomorphically embedded into $\mathbb{P}^N$ for some N.

Problem 4 (5 pts)

Let M be a complex manifold of dimension at least 2 and $x \in M$. Show that the sheaf $\mathfrak{I}_x$ of $\mathcal{O}$-modules is not isomorphic to the sheaf of holomorphic sections of any line bundle $L \longrightarrow M$.
Note: Recall that for any open subset $U \subset M$,

$$\mathfrak{I}_x(U) = \big\{ f \in \mathcal{O}(U) \colon\ f(x)=0 \text{ if } x \in U \big\};$$

this is a module over the ring $\mathcal{O}(U)$.

Problem 5 (10 pts)

Let Γ be a complete lattice in $\mathbb{C}^2$ (i.e. the $\mathbb{Z}$-span of 4 $\mathbb{R}$-linearly independent vectors $v_1, \dots, v_4 \in \mathbb{C}^2$). Thus, $M \equiv \mathbb{C}^2/\Gamma$ is diffeomorphic to $(S^1)^4$.
(a) Show that the Kahler structure (complex structure and symplectic form) on $\mathbb{C}^4$ induce a Kahler structure on M. Describe a basis for $H_2(M;\mathbb{Z})$.
(b) Find a lattice Γ so that $H^{1,1}(M;\mathbb{Z}) = \{0\}$ and thus M is not projective (cannot be embedded into $\mathbb{P}^N$ for any N).
Hint: Find Γ so that $\alpha(H_2(M;\mathbb{Z})) \not\subset \mathbb{Z}$ for every $\alpha \in H^{1,1}(M;\mathbb{C})$.
(c) With M as in (b), find a holomorphic line bundle $L \longrightarrow M$ so that $L \neq [D]$ for any divisor D on M.

Problem 6 (10 pts)

(a) Let $C \subset \mathbb{P}^3$ be a complete intersection of bi-degree (a,b) (so $C = s^{-1}(0)$, where s is a holomorphic section of the bundle $\mathcal{O}(a) \oplus \mathcal{O}(b) \longrightarrow \mathbb{P}^3$ which is transverse to the zero set). Determine the degree of C in $\mathbb{P}^3$ and the genus of C.
(b) If $C \subset \mathbb{P}^3$ is a smooth rational curve of degree 3 (thus, $C \approx \mathbb{P}^1$ and $[C] = 3[\mathbb{P}^1] \in H_2(\mathbb{P}^3)$) and C is not contained in any hyperplane $\mathbb{P}^2$ of $\mathbb{P}^3$, then C is not a complete intersection in $\mathbb{P}^3$. Show that such a curve C actually exists.