MAT 545: Complex Geometry

Problem Set 1

Written Solutions due by Tuesday, 9/10, 1pm

Please figure out all of the problems below and discuss them with others.

If you have not passed the orals yet, *please write up concise solutions to problems worth 10 points.*

Problem 1 (10 pts)

Let $U \subset \mathbb{C}^n$ be a connected open subset.
(a) *Unique Continuation:* If $f, g : U \longrightarrow \mathbb{C}$ are holomorphic functions and $V \subset \mathbb{C}^n$ is a nonempty open subset such that $f|_V = g|_V$, then $f = g$.
(b) *Maximum Principle:* If $f : U \longrightarrow \mathbb{C}$ is a holomorphic function and $\max_{z \in U} |f(z)| = |f(z_0)|$ for some $z_0 \in U$, then f is a constant function.
(c) *Elliptic Regularity:* If $f : U \longrightarrow \mathbb{C}$ is a holomorphic function (and thus f is assumed to be C^1), then f is smooth.

Problem 2 (5 pts)

Let $f(z, w) = \sin(w^2) - z$. Find the Weierstrass polynomial

$$g(z, w) = w^d + a_1(z)w^{d-1} + \ldots + a_d(z)$$

such that $f = g \cdot h$ near $(z, w) = (0, 0)$ with $h(0, 0) \neq 0$.

Problem 3 (5 pts)

Find a value of τ such that the tori $\mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ and $\mathbb{C}^*/(z \sim 2z)$ are isomorphic as Riemann surfaces.

Problem 4 (5 pts)

Show the complex projective space $\mathbb{P}^n$ and the total space of *the tautological line bundle*

$$\gamma \equiv \{(\ell, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} : v \in \ell\} \longrightarrow \mathbb{P}^n$$

are complex manifolds. Describe transition maps explicitly.

Problem 5 (10 pts)

Let R be an integral domain, i.e. a commutative ring with identity such that $fg \neq 0$ whenever $f, g \in R - 0$.

- An element $u \in R$ is a unit if u is invertible in R, i.e. $uv = 1$ for some $v \in R$;
- An element $u \in R$ is irreducible if u is not a unit and $u = vw$ for some $v, w \in R$ implies that v or w is a unit;
- An element $u \in R$ is prime if u is not a unit and $uz = vw$ for some $v, w, z \in R$ implies that either $v = z'u$ or $w = z'u$ for some $z' \in R$;
- R is a principal ideal domain (PID) if every ideal is principal, i.e. of the form pR for some $p \in R$;
- R is a unique factorization domain (UFD) if for every $f \in R$ such that f is not a unit there exist irreducible elements $f_1, \ldots, f_k \in R$ such that $f = f_1 \ldots f_k$ and $f_1, \ldots, f_k$ are uniquely determined by f up to a permutation and multiplication by units in R;
- A polynomial $f = a_0 + a_1 x + \ldots \in R[x]$ is primitive if only the units in R divide all the coefficients $a_0, a_1, \ldots$.

Show that:
(a) If R is an integral domain and $p \in R$ is prime, then R/pR is an integral domain.
(b) Any prime element of R is irreducible. If R is UFD, every irreducible element is prime.
(c) If R is UFD and $f, g \in R[x]$ are primitive, then fg is primitive.
(d) If R is UFD, F is the field of fractions of R, and $f \in R[x]$ is irreducible, then f is also irreducible in $F[x]$.
(e) If R is PID, every irreducible element is prime and R is a UFD.
(f) If F is a field, then $F[x]$ is a PID.
(g) If R is UFD, so is $R[x]$.
(h) If R is UFD and $f, g \in R[x]$ are relatively prime (have no common divisors other than units), there exist relatively prime $\alpha, \beta \in R[x]$ and $\gamma \in R - 0$ such that

$$\alpha f + \beta g = \gamma.$$