MAT 545: Complex Geometry
Notes on Lefschetz Decomposition

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1 Statement

Let (M,J,ω) be a Kähler manifold. Since ω is a closed 2-form, it induces a well-defined homomorphism

$$L\colon H^k(M) \longrightarrow H^{k+2}(M), \qquad L\big([\alpha]\big) = \big[\omega\wedge\alpha\big].$$

Hard Lefschetz Theorem: If (M,J,ω) is a compact Kähler m-manifold, the homomorphism

$$L^r\colon H^{m-r}(M) \longrightarrow H^{m+r}(M) \tag{1}$$

is an isomorphism for all $r\ge 0$.

Since ω is a $(1,1)$-form, by the Hodge decomposition theorem the above claim is equivalent to the statement that each of the homomorphisms

$$L^{m-(p+q)}\colon H^{p,q}(M) \longrightarrow H^{m-q,m-p}(M) \tag{2}$$

with $p+q\le m$ is an isomorphism. In the Hodge diamond, L corresponds to moving up 1 step along the vertical lines. The isomorphism (2) takes the (p,q)-slot to its reflection about the horizontal diagonal. Thus, the Hodge diamond of a compact Kähler manifold is symmetric about the horizontal diagonal. This fact also follows from the Hodge theorem (which implies that the Hodge diamond is symmetric about the vertical diagonal) and the Kodaira-Serre duality (symmetry about the center of the diamond). Furthermore, the Hard Lefschetz Theorem provides a relative restriction on the numbers along each of the vertical lines in the Hodge diamond: these numbers are non-decreasing in the bottom half and non-increasing in the top half (see (3) below).

By the Hard Lefschetz Theorem, $r+1$ is the smallest value of s such that the homomorphism

$$L^s\colon H^{m-r}(M) \longrightarrow M^{m-r+2s}(M)$$

may have a kernel. For each $r=0,1,\ldots,m$, let

$$PH^{m-r}(M) \equiv \big\{\alpha\in H^r(M)\colon\ L^{r+1}\alpha=0\big\}$$

be the primitive cohomology of M. By the Hard Lefschetz Theorem,

$$H^{m-r}(M) = PH^{m-r}(M) \oplus L\big(H^{m-r-2}(M)\big).$$

Thus, the Hard Lefschetz Theorem implies that

$$H^k(M) = \bigoplus_{\substack{s\ge 0\\ k-2s\le m}} L^s\big(PH^{k-2s}(M)\big). \tag{3}$$

On the other hand, (3) and Poincare Duality imply (1).

Let $V \subset M$ be a smooth hypersurface representing the Poincare dial of $[\omega]$. Denote by

$$\cdot_M \colon H_k(M)\otimes H_l(M) \longrightarrow H_{k+l-2m}(M)$$

the homology intersection product on M (the Poincare dual of the cup product). Via Poincare Duality, the Hard Lefschetz Theorem is equivalent to the statement that

$$(V\cdot_M)^r \colon H_{m+r}(M) \longrightarrow H_{m-r}(M), \qquad \eta \longrightarrow \underbrace{H \cdot_M \ldots H \cdot_M}_{r} \eta,$$

is an isomorphism for all $r \geq 0$. Furthermore,

$$\begin{aligned} PD_M\big(L^r(PH^{m-r}(M))\big) &= PD_M\big(\{\alpha \in H^{m+r}(M)\colon\; L(\alpha)=0\}\big) \\ &= \big\{\eta \in M_{m-r}(M) \colon V\cdot_M \eta = 0\big\}. \end{aligned}$$

Along with the long exact sequence

$$\ldots \longrightarrow H_k(M-V) \longrightarrow H_k(M) \xrightarrow{V\cdot_M} H_{k-2}(V) \longrightarrow H_{k-1}(M-V) \longrightarrow \ldots,$$

this implies that the primitive k-th cohomology of M, with $k \leq m$, corresponds to the image of $H_k(M-V)$ and $H_k(M)$, i.e. to the k-cycles in M that are disjoint from V.

The Poincare dual of the Fubini-Study symplectic form ω_{FS} on $\mathbb{P}^n$ is represented by a hyperplane class $V \approx \mathbb{P}^{n-1}$, since

$$\int_{\mathbb{P}^1} \omega_{FS} = 1 = \mathbb{P}^1 \cdot_{\mathbb{P}^n} \mathbb{P}^{n-1}$$

and $H^2(\mathbb{P}^n)$ is one-dimensional. If $M \subset \mathbb{P}^n$ is a compact Kähler submanifold of dimension m,

$$PD_M\big(\omega_{FS}|_M\big) = \big[V \cap M\big].$$

Thus, the Hard Lefschetz Theorem in this case is equivalent to the statement that

$$\mathbb{P}^{n-r} \cap \colon H_{m+r}(M) \longrightarrow H_{m-r}(M)$$

is an isomorphism. The primitive k-th cohomology of M, with $k \leq m$, corresponds to the k-cycles in M that are disjoint from V. Thus, they lie in the affine variety

$$M - V \cap M \subset \mathbb{P}^n - V = \mathbb{C}^n.$$

2 Proof

The Hard Lefschetz Theorem is a consequence of Hodge identities and the Hodge theorem. Let (M, J, ω) be a Kähler m-manifold, $A^k(M)$ be the space of differential k-forms on M (with coefficients in $\mathbb{R}$ or $\mathbb{C}$), and

$$L^* \colon A^k(M) \longrightarrow A^{k-2}(M)$$

be the adjoint of the homomorphism

$$L \colon A^{k-2}(M) \longrightarrow A^k(M), \qquad L(\alpha) = \omega \wedge \alpha,$$

with respect to the inner-product induced by (ω, J). Denote by

$$\Delta \equiv \Delta_{\mathrm{d}} \equiv \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} \colon A^k(M) \longrightarrow A^k(M)$$

the corresponding d-Laplacian and let

$$\mathcal{H}^p \equiv \big\{\alpha \in A^p(M) \colon \Delta\alpha = 0\big\}.$$

Hodge Identities: If (M, J, ω) is a Kähler m-manifold

$$L\Delta = \Delta L, \qquad L^*\Delta = \Delta L^*, \tag{4}$$

$$LL^* - L^*L = (m-k)\,\mathrm{Id} \colon A^k(M) \longrightarrow A^k(M). \tag{5}$$

Hodge Theorem: If (M, J, ω) is a compact Kähler m-manifold, the homomorphism

$$\mathcal{H}^k \longrightarrow H^k(M), \qquad \alpha \longrightarrow [\alpha],$$

is well-defined and is an isomorphism.

The last statement is valid for any Riemannian manifold, while (5) is a pointwise statement and thus follows from a direct check for $\mathbb{C}^n$. The identities (4) imply that L and L^* restrict to homomorphisms

$$L \colon \mathcal{H}^k \longrightarrow \mathcal{H}^{k+2}, \qquad L^* \colon \mathcal{H}^k \longrightarrow \mathcal{H}^{k-2}.$$

By Hodge theorem and Poincare Duality, it is sufficient to show that the homomorphism

$$L^{m-r} \colon \mathcal{H}^r \longrightarrow \mathcal{H}^{m+r},$$

is injective injective for every $r \le m$.s This is the case by the first statement of Corollary 2 below.

From (5), we obtain the following lemma with

$$C_{k,s} = \big(s-(m-k+1)\big)s\,.$$

Lemma 1 *Let $k \in \mathbb{Z}^{\ge 0}$ and $s \in \mathbb{Z}^+$. There exists $C_{k,s} \in \mathbb{Z}$ such that*

$$L^*L^s\alpha = L^{s-1}\big(C_{k,s}\alpha + LL^*\alpha\big) \qquad \forall\, \alpha \in \mathcal{H}^k$$

and $C_{k,s} = 0$ if and only if $s = m-k+1$.

Corollary 2 *Let $k \in \mathbb{Z}^{\ge 0}$ with $k \le m$ and $\alpha \in \mathcal{H}^k$. If $\alpha \ne 0$, then $L^{m-k}\alpha \ne 0$. If $L^{m-k+1}\alpha = 0$, then $L^*\alpha = 0$.*

Proof: It is sufficient to establish both statements under the assumption that $L^{m-k+1}\alpha = 0$. Both of them hold for $k < 0$. Suppose $0 \le k \le m$ and the first statement with k replaced by $k-2$ holds. By Lemma 1,

$$0 = L^*L^{m-k+1}\alpha = L^{m-k}\big(C_{k,m-k+1}\alpha + LL^*\alpha\big) = L^{m-(k-2)-1}(L^*\alpha).$$

Thus, $L^*\alpha = 0$ by the inductive assumption. We now assume that $\alpha \ne 0$ and show that

$$L^s\alpha \ne 0 \qquad \forall\, s = 0, 1, \ldots, m-k.$$

The $s = 0$ case of this statement clearly holds. Suppose $1 \le s \le m-k$ and (ii) holds with s replaced by $s-1$. By Lemma 1 and the just established second statement of Corollary 2,

$$L^*L^s\alpha = L^{s-1}\big(C_{k,s}\alpha + L(L^*\alpha)\big) = C_{k,s}L^{s-1}\alpha \ne 0 \qquad \text{if} \quad \alpha \ne 0.$$

Thus, $L^s\alpha \ne 0$.

3 Applications

The Hodge theorem provides restrictions on topological and complex manifolds that admit a Kähler structure. One of the standard invariants of a topological manifold M is the k-th Betti number,

$$h_k(M) \equiv \dim_{\mathbb{R}} H^k(M;\mathbb{R}) = \dim_{\mathbb{C}} H^k(M;\mathbb{C}) = \dim_{\mathbb{R}} H^k_{\mathrm{deR}}(M;\mathbb{R});$$

the last equality holds if M admits a smooth structure. If M is a compact $2m$-dimensional topological manifold that admits a Kähler structure, then $h_{2r+1}(M)$ is even for all $r\in\mathbb{Z}$ and $h_{2r}(M)>0$ for all $r=0,1,\ldots,m$. Furthermore, $H^2(M;\mathbb{R})$ contains an element α such that $\alpha^m\neq 0$. If (M,J) is a compact complex manifold that admits a compatible Kähler structure, then

$$\begin{gathered} h^{p,q}(M) \equiv \dim_{\mathbb{C}} H^{p,q}_{\mathrm{d}}(M) = \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(M) \quad \forall\, p,q, \\ h^{p,q}(M) = h^{q,p}(M) \quad \forall\, p,q, \qquad h^k(M) = \sum_{p+q=r} h^{p,q}(M) \quad \forall\, r. \end{gathered}$$

Furthermore, the homology class of any analytic subvariety in (M,J) is nonzero in the homology of M, as is every holomorphic p-form in $H^p(M)$.

The Hard Lefschetz Theorem provides additional restrictions. If M is a compact $2m$-dimensional topological manifold that admits a Kähler structure, then $H^2(M;\mathbb{R})$ contains an element α such that the homomorphisms

$$H^{m-r}(M) \longrightarrow H^{m+r}(M), \quad \beta \longrightarrow \alpha^r\wedge\beta, \qquad r\geq 0,$$

are isomorphisms. If (M,J) is a compact complex m-manifold that admits a compatible Kähler structure, then $H^{1,1}_{\mathrm{d}}(M)\cap H^2(M;\mathbb{R})$ contains an element α such that the homomorphisms

$$H^{p,q}_{\mathrm{d}}(M) \longrightarrow H^{m-q,m-p}_{\mathrm{d}}(M), \quad \beta \longrightarrow \alpha^{m-p-q}\wedge\beta, \qquad p+q\leq m,$$

are isomorphisms.

If M is a compact topological oriented $4k$-dimensional manifold, the pairing

$$Q\colon H^{2k}(M;\mathbb{R})\otimes H^{2k}(M;\mathbb{R}) \longrightarrow \mathbb{R}, \qquad Q\big(\alpha\otimes\beta\big) = \big\langle \alpha\cup\beta, [M]\big\rangle\,,$$

is nondegenerate by Poincare duality and is symmetric. Thus, $H^{2k}(M;\mathbb{R})$ admits a basis with respect to which this pairing is diagonal with each of the non-zero entries equal to $+1$ or -1. Let $\lambda_{\pm}(M)=\lambda_{\pm}(Q)$ denote the number of ± 1 entries; this number is determined by the bilinear form Q and thus by the topology and the orientation of M. So, is the number

$$\sigma(M) = \lambda_+(M) - \lambda_-(M),$$

which is known as the signature of M. If in addition J is a complex structure on M, Q restricts to a nondegenerate symmetric pairing $Q^{k,k}$ on $H^{k,k}(M)\cap H^{2k}(M;\mathbb{R})$. Let $\lambda^{p,p}_{\pm}(M)=\lambda_{\pm}(Q^{p,p})$.

Index Theorem for Surfaces: If (M,J) is a compact connected complex surface ($\dim_{\mathbb{C}} M=2$) that admits a compatible Kähler structure, then $\lambda_+^{1,1}(M)=1$.

Proof: Let ω be a symplectic form on M compatible with J and $\mathcal{H}^{1,1}$ be the corresponding space of harmonic $(1,1)$-forms. By the Hard Lefschetz Theorem,

$$\mathcal{H}^{1,1} = \mathbb{C}\omega \oplus \big\{\alpha\in\mathcal{H}^{1,1}\colon\, \omega\wedge\alpha=0\big\} \equiv V_+ \oplus V_0.$$

This decomposition is Q-orthogonal, and Q restricted to $V_+\cap H^2(M;\mathbb{R})$ is positive-definite. Thus, it is sufficient to show that

$$Q(\alpha,\alpha) \le 0 \qquad \forall\, \alpha \in V_0\cap A^2(M;\mathbb{R}).$$

Given $p\in M$, let (z_1,z_2) be holomorphic coordinates around p on M so that

$$\omega|_p = -\frac{1}{2}\mathrm{Im}\big(dz_1\wedge d\bar{z}_1 + dz_2\wedge d\bar{z}_2\big)|_p = dx_1\wedge dy_1 + dx_2\wedge dy_2\,,$$

where $z_j=x_j+\mathfrak{i}y_j$. If $\alpha\in\mathcal{H}^{1,1}\cap A^2(M;\mathbb{R})$,

$$\alpha|_p = \big(A\,dx_1\wedge dy_1 + B\,dx_2\wedge dy_2 + C(dx_1\wedge dx_2 + dy_1\wedge dy_2) + D(dx_1\wedge dy_2 - dy_1\wedge dx_2)\big)|_p$$

for some $A,B,C,D\in\mathbb{R}$. If in addition $\omega\wedge\alpha=0$, then $A=-B$ and

$$\alpha|_p\wedge\alpha|_p = -2(A^2+C^2+D^2)dx_1\wedge dy_1\wedge dx_2\wedge dy_2\big|_p.$$

Thus, $\alpha|_p\wedge\alpha|_p$ is a non-positive multiple of the volume form on M for all $p\in M$ and so $Q(\alpha,\alpha)\le 0$ as needed.

If (M,J) is a compact complex $2k$-manifold that admits a compatible Kähler structure, then

$$\sigma(M) = \sum_{p\equiv q (\mathrm{mod}\, 2)} (-1)^p h^{p,q}(M).$$

This is deduced from the *Hodge-Riemann bilinear relations* (a single formula) in Griffiths&Harris, pp123-126. As explained in the top half of p124, these relations follow from the Hard Lefschetz Theorem and quite a bit of representation theory (a classical subject).

There is an important typo on p123, in the line before the statement of the Hodge-Riemann bilinear relations: "$k=p+q$" should in fact be "$n-k=p+q$".