MAT 542: Algebraic Topology, Fall 2016

Suggested Problems for Week 8

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 31.1, 32.1-3, 33.2-5, 34.1

Problem L

Let X, Z be smooth manifolds, $f : Z \longrightarrow X$ be a smooth map, and $W \subset X$ be a neighborhood of $f(Z) \subset X$. Show that there exists a neighborhood $U \subset W$ of $f(Z) \subset X$ such that

$$H_p(U; R) = 0 \qquad \forall \, p > \dim Z \,.$$

Hint. By Sard's theorem, there exists a smooth triangulation of X such that $\operatorname{Int} \sigma$ is transverse to f for every simplex σ.

Note. Let $N \in \mathbb{Z}^+$ with $N \geq 3$. Let

$$X = \bigcup_{i=1}^{\infty} \left\{ x \in \mathbb{R}^N : \left| x - (1/i, 0, \ldots, 0) \right| = 1/i \right\} \,.$$

Thus, X is a wedge of countably many $(N{-}1)$-spheres with one point (the origin) in common and with collapsing radii. It is the image of a smooth map

$$f : \mathbb{Z}^+ \times S^{N-1} \longrightarrow \mathbb{R}^N \,.$$

It is shown in the 1962 paper of Barratt-Milnor that $H_p(X) \neq 0$ for infinitely many values of p. If you can find a direct proof of this, you can submit it as Problem L$'$.

Problem M

Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit sphere and define

$$r_1 : S^n \longrightarrow S^n, \qquad r_1(x_1, \ldots, x_{n+1}) = \big(-x_1, x_2, \ldots, x_{n+1} \big) \,.$$

Use Mayer-Vietoris to show that there is a commutative diagram

$$
\begin{array}{ccc}
\widetilde{H}_p(S^n) & \xrightarrow{\;\; r_{1*} \;\;} & \widetilde{H}_p(S^n) \\
\approx \downarrow & & \downarrow \approx \\
\widetilde{H}_{p-1}(S^{n-1}) & \xrightarrow[\;\; r_{1*} \;\;]{} & \widetilde{H}_{p-1}(S^{n-1})
\end{array}
$$

Conclude that the degree of r_1 is -1 and the degree of the antipodal map on S^n is $(-1)^{n+1}$.