MAT 542: Algebraic Topology, Fall 2016

Suggested Problems for Week 2

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 5.4-5.7

The H_* of 5.6 is also known as Borel-Moore homology or homology with closed supports (instead of compact supports). It is useful when working with non-compact manifolds, including in the context of Poincare Duality.

Problem C

Let R be a commutative ring with unity and M be an R-module. If $R=\mathbb{Z}$, then M is an abelian group. One such example is S^1. If X is a CW or simplicial complex, the R-modules $H_p(X;M)$ are defined precisely as in the case $M=R$ in class. Use the CW decomposition of $\mathbb{RP}^n$ you obtained in Problem B to show that

$$H_p(\mathbb{RP}^n;S^1)=\begin{cases} S^1, & \text{if } p=0;\\ S^1, & \text{if } p=n\notin 2\mathbb{Z};\\ \mathbb{Z}_2, & \text{if } 2\le p\le n,\ p\in 2\mathbb{Z};\\ 0, & \text{otherwise.}\end{cases}$$

This could be a good question to discuss on Friday; many thanks to Tyler.

Problem D

For a finite simplicial complex K and each $p\in\mathbb{Z}$, define

$$\alpha_p(K)=\#\text{ of } p\text{-cells in } K,\quad \beta_p(K)=\text{rank of } H_p(K)\,,\quad \chi(K)=\sum_{q=0}^{\infty}(-1)^q\alpha_q(K).$$

The last sum is the Euler characteristic of K. Show that

$$\chi(K)=\sum_{q=0}^{\infty}(-1)^q\beta_q(K)\,.$$

Assuming $H_p(K)$ depends only on the homeomorphism type of $|K|$, this implies that so does $\chi(K)$.

Problem E

Let K be a triangulation of a compact connected surface (without boundary). Show that

$$3\alpha_2(K)=2\alpha_1(K),\quad \alpha_1(K)=3\big(\alpha_0(K)-\chi(K)\big),\quad \alpha_0(K)\ge\frac{1}{2}\big(7+\sqrt{49-24\chi(K)}\big)\,.$$

In particular, a triangulation of the torus has at least 7 vertices, 21 edges, and 14 triangles. Find such a triangulation.

Problems D and E are from a problem set in MIT's 18.905 in Fall 1996 taught by F. Peterson.