MAT 542: Algebraic Topology, Fall 2016
Suggested Problems for Week 11

You may hand in solutions to at *most* 2 problems every 2 weeks and no later than 2 weeks after the necessary material for them is covered in class.

From Munkres: 47.1, 47.3, 47.5, 47.7, 48.1-3, 49.1-3, 49.8, 66.4-6, 57.1

Problem Q

Let R be a commutative ring with 1 and M be a module over R. A **directed set** is a set S with a partial order $\prec$ such that for any pair of elements $i, j \in S$ there exists $k \in S$ with $i, j \preceq k$. A **directed system** of R-modules over a directed set $(S, \prec)$ is an assignment of an R-module M_i for each $i \in S$ and an R-module homomorphism $\rho_{ji} \colon M_i \longrightarrow M_j$ for all $i, j \in S$ with $i \preceq j$ such that

$$\rho_{ii} = \mathrm{id}_{M_i} \quad \forall \, i \in S, \qquad \rho_{ki} = \rho_{kj} \circ \rho_{ji} \quad \forall \, i, j, k \in S, \ i \preceq j \preceq k.$$

An **inverse system** of R-modules over a directed set $(S, \prec)$ is an assignment of an R-module M_i for each $i \in S$ and an R-module homomorphism $\varrho_{ij} \colon M_j \longrightarrow M_i$ for all $i, j \in S$ with $i \preceq j$ such that

$$\varrho_{ii} = \mathrm{id}_{M_i} \quad \forall \, i \in S, \qquad \varrho_{ik} = \varrho_{ij} \circ \varrho_{jk} \quad \forall \, i, j, k \in S, \ i \preceq j \preceq k.$$

The **direct limit** of a directed system and the **inverse limit** of an inverse system are

$$\varinjlim M_i \equiv \left(\bigsqcup_i M_i \right) \Big/ \sim, \quad A_i \ni x_i \sim x_j \in A_j \quad \text{if} \ \exists \, k \in S \ \text{s.t.} \ i, j \preceq k, \ \rho_{ki}(x_i) = \rho_{kj}(x_j),$$

$$\varprojlim M_i \equiv \Big\{ (x_i)_{i \in S} \in \prod_i M_i \colon x_i = \varrho_{ij}(x_j) \ \forall \, i, j \in S, \ i \prec j \Big\},$$

respectively. For example, the collection of compact subsets K of a topological spaces forms a directed set. The associated homology and cohomology groups form a direct system and an inverse system under the homomorphisms

$$\rho_{K'K} \colon H_*(K; R) \longrightarrow H_*(K'; R) \qquad \text{and} \qquad \varrho_{KK'} \colon H^*(K'; M) \longrightarrow H^*(K; M)$$

induced by the inclusions $K \subset K'$.

(a) Show that the direct and inverse limits are R-modules.

(b) Construct isomorphisms

$$\varinjlim H_*(K; R) \longrightarrow H_*(X; R), \qquad H_*(X; R) \longrightarrow \varprojlim H^*(K; R).$$

Problem R

Let R be a commutative ring with 1. Let X, Y be Hausdorff topological spaces and $X \vee Y$ be their wedge product at points that are deformation retracts of neighborhoods in their spaces.

(a) Construct an isomorphism

$$H^p(X \vee Y; R) \longrightarrow H^p(X; R) \oplus H^p(Y; R) \qquad \forall \, p \in \mathbb{Z}^+$$

and show that it respects the cup product.

(b) Use this to establish the claims of Example 4 and Exercise 5 in Section 49 without any simplicial complexes.