MAT 531: Topology&Geometry, II
Spring 2006

Midterm Solutions

Problem 1 (15 pts)

Suppose M is a smooth manifold and X and Y are smooth vector fields on M. Show directly from definitions that

$$[X,Y] = -[Y,X].$$

(You can assume that $[X,Y]$ is whatever object it is supposed to be, but do state what you are taking it to be).

By definition, the Lie bracket $[X,Y]$ of two vector fields X and Y is another vector field on M, i.e. an element of $\Gamma(M;TM)$. In particular,

$$[X,Y],[Y,X]\colon C^\infty(M) \longrightarrow C^\infty(M)$$

are linear maps. They are given by

$$\begin{gathered}
[X,Y]f = X(Yf) - Y(Xf) \quad \forall\, f\in C^\infty(M) \qquad \Longrightarrow \\
[Y,X]f = Y(Xf) - X(Yf) = -\big(X(Yf) - Y(Xf)\big) = -\big([X,Y]f\big) \equiv \big\{-[X,Y]\big\}f \quad \forall\, f\in C^\infty(M) \\
\Longrightarrow \qquad [X,Y] = [Y,X] \in \Gamma(M;TM).
\end{gathered}$$

Problem 2 (20 pts)

Show that the topological subspace

$$\big\{(x,y)\in\mathbb{R}^2 : x^3+xy+y^3=1\big\}$$

of $\mathbb{R}^2$ is a smooth curve (i.e. admits a natural structure of smooth 1-manifold with respect to which it is a submanifold of $\mathbb{R}^2$).

Define

$$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad \text{by} \qquad f(x,y) = x^3+xy+y^3.$$

Then, f is a smooth map and

$$P \equiv \big\{(x,y)\in\mathbb{R}^2 : x^3+xy+y^3=1\big\} = f^{-1}(1).$$

We show below that 1 is regular value for f. By the Implicit Function Theorem, P is then a smooth submanifold of $\mathbb{R}^2$ and

$$\dim P = \dim\mathbb{R}^2 - \dim\mathbb{R} = 1,$$

as needed.

We need to show that $df|_{(x,y)}$ is surjective for all $(x,y)\in f^{-1}(1)$. Since the target space for f is $\mathbb{R}$, we can view $df|_{(x,y)}$ as a linear map into $\mathbb{R}$ (instead of $T_{f(x,y)}\mathbb{R}$). It is given by

$$df|_{(x,y)} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (3x^2{+}y)dx + (x{+}3y^2)dy.$$

Since dx and dy are linearly independent in $T^*_{(x,y)}\mathbb{R}^2$,

$$df|_{(x,y)} = 0 \qquad \Longrightarrow \qquad \begin{cases} 3x^2{+}y = 0 \\ x{+}3y^2 = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} y = -3x^2 \\ x{+}27x^4 = 0 \end{cases} \qquad \Longrightarrow$$
$$(x,y) = (0,0) \;\text{ or }\; (x,y) = (-1/3,-1/3) \quad \Longrightarrow \quad f(x,y) \in \{0, 1/27\} \quad \Longrightarrow \quad (x,y) \notin f^{-1}(1).$$

We conclude that 1 is a regular value for f.

Problem 3 (5+15 pts)

Let X be a non-vanishing vector field on $\mathbb{R}^3$, written in coordinates as

$$X(x,y,z) = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z} \qquad \text{for some} \quad f,g,h \in C^\infty(\mathbb{R}^3).$$

(a) *Find a one-form α on $\mathbb{R}^3$ so that at each point of $\mathbb{R}^3$ the kernel of α is orthogonal to X, with respect to the standard inner-product on $\mathbb{R}^3$.*
(b) *Find a necessary and sufficient condition on X so that for every point $p\in\mathbb{R}^3$ there exists a surface $S\subset\mathbb{R}^3$ passing through p which is everywhere orthogonal to X (i.e. S is a smooth two-dimensional submanifold of $\mathbb{R}^3$ and $T_mS\subset T_m\mathbb{R}^3$ is orthogonal to $X(m)$ for all $m\in S$).*

(a) Every element of $T_{(x,y,z)}\mathbb{R}^3$ can be written as

$$v = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \qquad \text{for some} \qquad a,b,c \in \mathbb{R}$$

and

$$\begin{aligned} \langle X(x,y,z), v\rangle &= f(x,y,z)a + g(x,y,z)b + h(x,y,z)c \\ &= \big\{f(x,y,z)dx{+}g(x,y,z)dy{+}h(x,y,z)dz\big\}v = \alpha|_{(x,y,z)}v, \end{aligned}$$

where α is the one-form on $\mathbb{R}^3$ defined by

$$\alpha|_{(x,y,z)} = f(x,y,z)dx{+}g(x,y,z)dy{+}h(x,y,z)dz.$$

For each $(x,y,z)\in\mathbb{R}^3$, the kernel of α is the subspace of $T_{(x,y,z)}\mathbb{R}^3\approx\mathbb{R}^3$ orthogonal to $X(x,y,z)$.

(b) Let

$$\mathcal{J}(\mathbb{R}\alpha) = \mathcal{J}(\alpha) = \big\{\alpha\wedge\beta\colon \beta\in E^*(\mathbb{R}^3)\big\} \subset E^*(\mathbb{R}^3)$$

be the ideal generated by α. By part (a), we need to find a necessary and sufficient condition on α so that for every point $p\in\mathbb{R}^3$ there exists a surface $S\subset\mathbb{R}^3$ passing through p such that

$$T_{(x,y,z)}S = \ker\alpha|_{(x,y,z)} \subset T_{(x,y,z)}\mathbb{R}^3.$$

Since X is a nowhere-vanishing vector field on $\mathbb{R}^3$, α is a nowhere-vanishing one-form on $\mathbb{R}^3$ and $\mathbb{R}\alpha$ is a vector subbundle of $T^*\mathbb{R}^3$ of rank one. Thus, by the second version of Frobenius Theorem the necessary and sufficient condition is that the ideal $J(\alpha)$ be differential, i.e. closed under d so that

$$d\gamma \in \mathcal{J}(\alpha) \qquad \forall\, \gamma \in \mathcal{J}(\alpha).$$

Since $\mathbb{R}^3$ is a three-dimensional manifold, by Problem 5 on PS5 this is the case if and only if

$$\alpha \wedge d\alpha = 0.$$

Since $\alpha = f dx + g dy + h dz$,

$$\begin{aligned}
d\alpha &= df \wedge dx + dg \wedge dy + dh \wedge dz \\
&= \Big(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\Big)\wedge dx + \Big(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz\Big)\wedge dy + \Big(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz\Big)\wedge dz \\
&= f_y dy\wedge dx + f_z dz\wedge dx + g_x dx\wedge dy + g_z dz\wedge dy + h_x dx\wedge dz + h_y dy\wedge dz \\
&= (g_x - f_y) dx\wedge dy + (h_x - f_z) dx\wedge dz + (h_y - g_z) dy\wedge dz \\
\implies \qquad \alpha\wedge d\alpha &= h(g_x - f_y) dz\wedge dx\wedge dy + g(h_x - f_z) dy\wedge dx\wedge dz + f(h_y - g_z) dx\wedge dy\wedge dz \\
&= \big(f(h_y - g_z) - g(h_x - f_z) + h(g_x - f_y)\big) dx\wedge dy\wedge dz.
\end{aligned}$$

Thus, the necessary and sufficient condition on X is

$$\big\langle X, \vec{\nabla}\times X\big\rangle \equiv \det\begin{pmatrix} f & g & h \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix} \equiv f(h_y - g_z) - g(h_x - f_z) + h(g_x - f_y) = 0.$$

Problem 4 (20 pts)

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth structure and orientation. Find

$$\int_{S^2} \big(x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2\big).$$

With its standard orientation, S^2 is the oriented boundary of the unit ball

$$B^3 \equiv \big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2 + x_2^2 + x_3^3 \le 1\big\}$$

about the origin, with its standard smooth structure and orientation. Thus, by the second version of Stokes' Theorem,

$$\begin{aligned}
\int_{S^2} \big(x_1 dx_2\wedge dx_3 + x_2 dx_1\wedge dx_3 + x_3 dx_1\wedge dx_2\big) &= \int_{\partial B^3} \big(x_1 dx_2\wedge dx_3 + x_2 dx_1\wedge dx_3 + x_3 dx_1\wedge dx_2\big) \\
&= \int_{B^3} d\big(x_1 dx_2\wedge dx_3 + x_2 dx_1\wedge dx_3 + x_3 dx_1\wedge dx_2\big) \\
&= \int_{B^3} \big(dx_1\wedge dx_2\wedge dx_3 + dx_2\wedge dx_1\wedge dx_3 + dx_3\wedge dx_1\wedge dx_2\big) \\
&= \int_{B^3} \big(dx_1\wedge dx_2\wedge dx_3 - dx_1\wedge dx_2\wedge dx_3 + dx_1\wedge dx_2\wedge dx_3\big) = \int_{B^3} 1\, dx_1\wedge dx_2\wedge dx_3.
\end{aligned}$$

Since B^3 with its standard orientation is a regular subspace of $\mathbb{R}^3$,

$$\int_{B^3} 1\,dx_1\wedge dx_2\wedge dx_3 = \int_{B^3} 1\,dx_1 dx_2 dx_3 = \frac{4\pi}{3}.$$

Problem 5 (25 pts)

Suppose M and N are smooth manifolds. Show that
(a) *if M and N are orientable, then $M\times N$ is orientable;*
(b) *if M is orientable and nonempty and N is not orientable, then $M\times N$ is not orientable;*
(c) *if M and N are not orientable, then $M\times N$ is not orientable.*

(a) Since M and N are orientable, the line bundles

$$\Lambda^{\mathrm{top}}TM \longrightarrow M \qquad \text{and} \qquad \Lambda^{\mathrm{top}}TN \longrightarrow N$$

are trivial. On the other hand, let

$$\pi_1, \pi_2 \colon M\times N \longrightarrow M, N$$

be the two projection maps. Then,

$$\begin{aligned} & T(M\times N) = \pi_1^*TM \oplus \pi_2^*TN \longrightarrow M\times N \\ \Longrightarrow \qquad & \Lambda^{\mathrm{top}}\big(T(M\times N)\big) = \Lambda^{\mathrm{top}}\big(\pi_1^*TM\oplus\pi_2^*TN\big) = \Lambda^{\mathrm{top}}(\pi_1^*TM)\otimes\Lambda^{\mathrm{top}}(\pi_2^*TN) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \pi_1^*(\Lambda^{\mathrm{top}}TM)\otimes\pi_2^*(\Lambda^{\mathrm{top}}TN). \end{aligned}$$

Since the line bundles $\Lambda^{\mathrm{top}}TM$ and $\Lambda^{\mathrm{top}}TN$ over M and N, respectively, are trivial, so are their pullbacks to $M\times N$ and their tensor product. Since the line bundle

$$\Lambda^{\mathrm{top}}\big(T(M\times N)\big) = \pi_1^*(\Lambda^{\mathrm{top}}TM)\otimes\pi_2^*(\Lambda^{\mathrm{top}}TN) \longrightarrow M\times N$$

is trivial, the manifold $M\times N$ is orientable.

Alternatively, since M and N are orientable, there exist nowhere-vanishing top forms on M and N:

$$\alpha \in E^{\mathrm{top}}(M) \qquad \text{and} \qquad \beta \in E^{\mathrm{top}}(N).$$

Then, $\pi_1^*\alpha\wedge\pi_2^*\beta$ is a nowhere-vanishing top form on $M\times N$. Since $M\times N$ admits such a form, $M\times N$ is orientable. To see that $\pi_1^*\alpha\wedge\pi_2^*\beta$ does not vanish on $M\times N$, suppose $x\in M$, $y\in N$, and

$$\{v_1,\ldots,v_k\}\subset T_xM \qquad \text{and} \qquad \{w_1,\ldots,w_n\}\subset T_yN$$

are bases for T_xM and T_yN. Since

$$\alpha|_x \in \Lambda^{\mathrm{top}}T_x^*M \qquad \text{and} \qquad \beta|_y \in \Lambda^{\mathrm{top}}T_y^*N$$

are not zero,

$$\alpha|_x(v_1,\ldots,v_k)\neq 0 \qquad \text{and} \qquad \beta|_y(w_1,\ldots,w_n)\neq 0.$$

On the other hand,

$$T_{(x,y)}(M\times N)=T_xM\oplus T_yN \qquad\Longrightarrow\qquad v_1,\ldots,v_k,w_1,\ldots,w_n\in T_{(x,y)}(M\times N).$$

Furthermore,

$$\begin{aligned}\{\pi_1^*\alpha\wedge\pi_2^*\beta\}|_{(x,y)}(v_1,\ldots,v_k,w_1,\ldots,w_n)&=\{\pi_1^*\alpha\}|_{(x,y)}(v_1,\ldots,v_k)\cdot\{\pi_2^*\beta\}|_{(x,y)}(w_1,\ldots,w_n)\\&=\alpha|_x\big(d\pi_1(v_1),\ldots,d\pi_1(v_k)\big)\cdot\beta|_y\big(d\pi_2(w_1),\ldots,d\pi_2(w_n)\big)\\&=\alpha|_x\big(v_1,\ldots,v_k\big)\cdot\beta|_y\big(w_1,\ldots,w_n\big)\neq0,\end{aligned}$$

as claimed. Note that a priori the expression on RHS of the first line above should be a sum over all possible permutations of $\{v_1,\ldots,v_k,w_1,\ldots,w_n\}$. However, since $d\pi_1(w_j)=0$, the only such permutations that yield nonzero terms are the permutations that preserve the subsets $\{v_1,\ldots,v_k\}$ and $\{w_1,\ldots,w_n\}$.

(b) By part (a),

$$\Lambda^{\rm top}\big(T(M\times N)\big)=\pi_1^*(\Lambda^{\rm top}TM)\otimes\pi_2^*(\Lambda^{\rm top}TN)\longrightarrow M\times N.$$

Since M is orientable, the line bundle $\pi_1^*(\Lambda^{\rm top}M)$ is again trivial and

$$\Lambda^{\rm top}\big(T(M\times N)\big)=\pi_1^*(\Lambda^{\rm top}TM)\otimes\pi_2^*(\Lambda^{\rm top}TN)\approx\pi_2^*(\Lambda^{\rm top}TN).$$

Since N is not orientable, the line bundle $\Lambda^{\rm top}TN\longrightarrow N$ is not trivial. This does not mean that its pullback by every map is not trivial, but we will show that its pullback by π_2 is indeed not trivial. In turn, since

$$\Lambda^{\rm top}\big(T(M\times N)\big)\approx\pi_2^*(\Lambda^{\rm top}TN)$$

and $\pi_2^*(\Lambda^{\rm top}TN)$ is not trivial, it follows that $M\times N$ is not orientable. To see that $\pi_2^*(\Lambda^{\rm top}TN)$ is not trivial, pick any $x\in M$ and define

$$\iota_x\colon N\longrightarrow M\times N \qquad\text{by}\qquad \iota_x(y)=(x,y).$$

Since $\pi_2\circ\iota_x=\mathrm{id}_N$,

$$\iota_x^*\big(\pi_2^*(\Lambda^{\rm top}TN)\big)=\iota_x^*\pi_2^*(\Lambda^{\rm top}TN)=\{\pi_2\circ\iota_x\}^*\Lambda^{\rm top}TN=\mathrm{id}_N^*\Lambda^{\rm top}TN=\Lambda^{\rm top}TN.$$

Thus, $\iota_x^*(\pi_2^*(\Lambda^{\rm top}TN))$ is not trivial, which implies that $\pi_2^*(\Lambda^{\rm top}TN)$ is not trivial either.

Here is another approach. Suppose $M\times N$ is orientable, i.e. there exists a nowhere-vanishing $\gamma\in E^{\rm top}(M\times N)$. We will construct a nowhere-vanishing $\beta\in E^{\rm top}(M\times N)$ by restricting γ to the vertical slice $x\times N$ and contracting the M-part of γ. Denote by k and n be the dimensions of M and N. Let

$$v_1,\ldots,v_k\subset T_xM$$

be a basis. For each $i=1,\ldots,k$, let

$$X_i\in\Gamma\big(x\times N;T(M\times N)|_{x\times N}\big)$$

be the (horizontal) vector field along $x\times N$ defined by

$$X_i(x,y)=v_i\in T_{(x,y)}(X\times Y)=T_xX\oplus T_yY.$$

These are smooth vector fields and thus

$$\beta_x \equiv \iota_{X_k} \dots \iota_{X_1} \gamma = \gamma(X_1, \dots, X_k, \cdot, \dots, \cdot) \in \Gamma\big(x \times N; \Lambda^n T(M \times N)|_{x \times N}\big)$$

is a smooth n-form. If $y \in N$ and

$$w_1, \dots, w_n \subset T_y N$$

is a basis for $T_y N$, then

$$v_1, \dots, v_k, w_1, \dots, w_n \subset T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$$

is a basis. Since γ does not vanish at (x, y),

$$\beta_x|_y(w_1, \dots, w_n) = \gamma(v_1, \dots, v_k, w_1, \dots, w_n) \neq 0.$$

Thus, β_x does not vanish along $x \times N$. Furthermore, $\iota_{v_i}\beta_x|_y = 0$ for all i, i.e.

$$\beta_x|_y \in \Lambda^n T_y^* N \subset \Lambda^n_{(x,y)} T^*(M \times N) = \bigoplus_{p+q=n} \Lambda^p T_x^* M \otimes \Lambda^q T_y^* N.$$

This means that

$$\beta \equiv \iota_x^* \beta_x \in \Gamma\big(N; \Lambda^n T^* N\big) = E^n(N)$$

is a nowhere-zero top form on N, i.e N is orientable. Note that this argument also implies part (c), since the only fact we used about M is that it is nonempty.

Here is a third approach. We can assume that N is connected. Let

$$p\colon \tilde{N} \longrightarrow N$$

be the orientable (connected) double cover of N. It can be obtained by choosing a metric on $\Lambda^{\mathrm{top}} TN$ and taking

$$\tilde{N} = S\big(\Lambda^{\mathrm{top}} TN\big);$$

see more below. Let g be the nontrivial deck transformation for p, so that

$$N = \tilde{N}/\{\mathrm{id}, g\}.$$

Since $\tilde{N}$ is orientable and N is not orientable, the diffeomorphism $g\colon \tilde{N} \longrightarrow \tilde{N}$ must be orientation-reversing. By part (a), $M \times \tilde{N}$ is an orientable manifold. Furthermore,

$$M \times N = M \times \tilde{N}/\big\{\mathrm{id}_M \times \mathrm{id}, \mathrm{id}_M \times g\big\}.$$

Since id_M is orientation-preserving and g is orientation-reversing,

$$\mathrm{id}_M \times g\colon M \times \tilde{N} \longrightarrow M \times \tilde{N}$$

is orientation-reversing. Thus, $M \times N$ is not orientable.

We now show that the total space $\tilde{N}$ of the double cover

$$p\colon \tilde{N} \equiv S\big(\Lambda^{\mathrm{top}} TN\big) \longrightarrow N$$

is orientable. Since p is a local diffeomorphism,

$$dp\colon T\tilde{N} \longrightarrow p^*TN$$

is an isomorphism of vector bundles. Thus,

$$\Lambda^{\mathrm{top}}T\tilde{N} \approx \Lambda^{\mathrm{top}}(p^*TN) \approx p^*\big(\Lambda^{\mathrm{top}}TN\big).$$

To show that the line bundle $p^*(\Lambda^{\mathrm{top}}TN)$ is trivial (and thus $\tilde{N}$ is orientable), we construct a nowhere-vanishing section of

$$p^*\big(\Lambda^{\mathrm{top}}TN\big) \equiv \big\{(e,v)\in\tilde{N}\times\Lambda^{\mathrm{top}}TN\colon p(e)=\pi(v)\big\} \longrightarrow \tilde{N},$$

where $\pi\colon \Lambda^{\mathrm{top}}TN \longrightarrow N$ is the bundle projection map. By definition,

$$\tilde{N} = \big\{e\in\Lambda^{\mathrm{top}}TN\colon |e|=1\big\} \qquad \Longrightarrow$$
$$p^*\big(\Lambda^{\mathrm{top}}TN\big) \equiv \big\{(e,v)\in\Lambda^{\mathrm{top}}TN \times\Lambda^{\mathrm{top}}TN\colon \pi(v)=\pi(e)\big\} \longrightarrow \tilde{N} = \big\{e\in\Lambda^{\mathrm{top}}TN\colon |e|=1\big\}.$$

We define a section of $p^*(\Lambda^{\mathrm{top}}TN)$ over $\tilde{N}$ by

$$s(e) = (e,e).$$

This section does not vanish.

(c) We will show that the line bundle

$$\Lambda^{\mathrm{top}}\big(T(M\times N)\big) = \pi_1^*(\Lambda^{\mathrm{top}}TM) \otimes \pi_2^*(\Lambda^{\mathrm{top}}TN) \longrightarrow M\times N$$

is not trivial by showing that its pullback by ι_x is again not trivial. This implies that $M\times N$ is not orientable. Given $x\in M$, let

$$f_x\colon N \longrightarrow M, \qquad f_x(y) = x,$$

be the constant map sending N to x. Since $\pi_1\circ\iota_x = f_x$,

$$\iota_x^*\pi_1^*(\Lambda^{\mathrm{top}}TM) = \big\{\pi_1\circ\iota_x\big\}^*(\Lambda^{\mathrm{top}}TM) = f_x^*(\Lambda^{\mathrm{top}}TM) = N\times(\Lambda^{\mathrm{top}}TM)_x$$

is the trivial line bundle. Thus,

$$\begin{aligned}\iota_x^*\big(\pi_1^*(\Lambda^{\mathrm{top}}TM)\otimes\pi_2^*(\Lambda^{\mathrm{top}}TN)\big) &= \iota_x^*\big(\pi_1^*(\Lambda^{\mathrm{top}}TM)\big)\otimes\iota_x^*\big(\pi_2^*(\Lambda^{\mathrm{top}}TN)\big)\\ &= (N\times\mathbb{R})\otimes\Lambda^{\mathrm{top}}TN \approx \Lambda^{\mathrm{top}}TN.\end{aligned}$$

Since N is not orientable, the line bundle $\Lambda^{\mathrm{top}}TN$ is not trivial as needed.

Alternatively, we can assume that M is connected. Let

$$p\colon \tilde{M} \longrightarrow M$$

be the orientable (connected) double cover of M similarly to part (b). Then,

$$p\times\mathrm{id}\colon \tilde{M}\times N \longrightarrow M\times N$$

is a covering projection. Since $\tilde{M}\times N$ is not orientable by part (b), $M\times N$ is not orientable either (a nowhere-vanishing top form on the base induces a nowhere-vanishing top form on the total space of a covering map (but not conversely)).