MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 9
Due on Thursday, 4/28, in class

1. We have defined Čech cohomology for sheaves or presheaves of K-modules. All such objects are abelian. The sets $\check{H}^0$ and $\check{H}^1$ can be defined for sheaves or presheaves of non-abelian groups as well. The main example of interest is the sheaf $\mathcal{S}$ of germs of smooth (or continuous) functions to a Lie group G over a smooth manifold (or topological space) M.[1]

   Let $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$ be an open cover of M. Analogously to the abelian case, the set $\check{C}^k(\underline{U};\mathcal{S})$ of Čech k-cocycles is a group under pointwise multiplication of sections:

$$\cdot : \check{C}^k(\underline{U};\mathcal{S})\times\check{C}^k(\underline{U};\mathcal{S})\longrightarrow\check{C}^k(\underline{U};\mathcal{S}),$$
$$\{f\cdot g\}_{\alpha_0\alpha_1\ldots\alpha_k}(p)=f_{\alpha_0\alpha_1\ldots\alpha_k}(p)\cdot g_{\alpha_0\alpha_1\ldots\alpha_k}(p)\quad\forall\,\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A},\ p\in U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\,,$$

   where $f_{\alpha_0\alpha_1\ldots\alpha_k}, g_{\alpha_0\alpha_1\ldots\alpha_k}\colon U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\longrightarrow G$ are smooth (or continuous) functions (or equivalently sections of $\mathcal{S}$). The identity element $\mathbf{e}\in\check{C}^k(\underline{U};\mathcal{S})$ is given by

$$\mathbf{e}_{\alpha_0\alpha_1\ldots\alpha_k}(p)=\mathrm{id}_G\qquad\forall\,\alpha_0,\alpha_1,\ldots,\alpha_k\in\mathcal{A},\ p\in U_{\alpha_0}\cap U_{\alpha_1}\cap\ldots\cap U_{\alpha_k}\,.$$

   Define the two bottom boundary maps by

$$\mathrm{d}_0\colon\check{C}^0(\underline{U};\mathcal{S})\longrightarrow\check{C}^1(\underline{U};\mathcal{S}),\ (\mathrm{d}_0f)_{\alpha_0\alpha_1}=f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}$$
$$\mathrm{d}_1\colon\check{C}^1(\underline{U};\mathcal{S})\longrightarrow\check{C}^2(\underline{U};\mathcal{S}),\ (\mathrm{d}_1g)_{\alpha_0\alpha_1\alpha_2}=g_{\alpha_1\alpha_2}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_2}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}}\cdot g_{\alpha_0\alpha_1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}\cap U_{\alpha_2}},$$

   for all $\alpha_0,\alpha_1,\alpha_2\in\mathcal{A}$. We also define an action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ by

$$*\colon\check{C}^0(\underline{U};\mathcal{S})\times\check{C}^1(\underline{U};\mathcal{S})\longrightarrow\check{C}^1(\underline{U};\mathcal{S}),\ \ \{f*g\}_{\alpha_0\alpha_1}=f_{\alpha_0}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\cdot g_{\alpha_0\alpha_1}\cdot f_{\alpha_1}^{-1}\big|_{U_{\alpha_0}\cap U_{\alpha_1}}\in\Gamma(U_{\alpha_0}\cap U_{\alpha_1};\mathcal{S}).$$

   Show that

   (a) $\check{H}^0(\underline{U};\mathcal{S})\equiv\ker\mathrm{d}_0\equiv\mathrm{d}_0^{-1}(\mathbf{e})$ is a subgroup of $\check{C}^0(\underline{U};\mathcal{S})$;

   (b) for every Čech 1-cocycle g (i.e. $g\in\ker\mathrm{d}_1$) for an open cover $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$,

$$g_{\alpha\alpha}=\mathbf{e}|_{U_\alpha},\quad g_{\alpha\beta}g_{\beta\alpha}=\mathbf{e}|_{U_\alpha\cap U_\beta},\quad g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=\mathbf{e}|_{U_\alpha\cap U_\beta\cap U_\gamma},\qquad\forall\,\alpha,\beta,\gamma\in\mathcal{A};$$

   (c) $*$ is a left action of $\check{C}^0(\underline{U};\mathcal{S})$ on $\check{C}^1(\underline{U};\mathcal{S})$ that restricts to an action on $\ker\mathrm{d}_1$ and

$$\mathrm{Im}\,\mathrm{d}_0\subset\check{C}^0(\underline{U};\mathcal{S})\mathbf{e}.$$

   By part (c), we can define

$$\check{H}^1(\underline{U};\mathcal{S})=\ker\mathrm{d}_1/\check{C}^0(\underline{U};\mathcal{S})\,;$$

   this is a pointed set (a set with a distinguished element).

[1] A Lie group G is a smooth manifold and a group so that the group operations are smooth. Examples include $O(k)$, $SO(k)$, $U(k)$, $SU(k)$.

If $\underline{U}'=\{U'_\alpha\}_{\alpha\in\mathcal{A}'}$ is a refinement of $\underline{U}=\{U_\alpha\}_{\alpha\in\mathcal{A}}$, any refining map $\mu\colon \mathcal{A}'\longrightarrow\mathcal{A}$ induces group homomorphisms

$$\mu_k^*\colon \check{C}^k(\underline{U};\mathcal{S})\longrightarrow\check{C}^k(\underline{U}';\mathcal{S}),$$

which commute with d_0, d_1, and the action of $\check{C}^0(\cdot;\mathcal{S})$ on $\check{C}^1(\cdot;\mathcal{S})$, similarly to Section 5.33. Thus, μ induces a group homomorphism and a map

$$R^0_{\underline{U}',\underline{U}}\colon \check{H}^0(\underline{U};\mathcal{S})\longrightarrow\check{H}^0(\underline{U}';\mathcal{S}) \qquad\text{and}\qquad R^1_{\underline{U}',\underline{U}}\colon \check{H}^1(\underline{U};\mathcal{S})\longrightarrow\check{H}^1(\underline{U}';\mathcal{S}).$$

(d) Show that these maps are independent of the choice of μ.

Thus, we can again define $\check{H}^0(M;\mathcal{S})$ and $\check{H}^1(M;\mathcal{S})$ by taking the direct limit of all $\check{H}^0(\underline{U};\mathcal{S})$ and $\check{H}^1(\underline{U};\mathcal{S})$ over open covers of M. The first set is a group, while the second need not be (unless $\mathcal{S}$ is a sheaf of abelian groups). These sets will be denoted by $\check{H}^0(M;G)$ and $\check{H}^1(M;G)$ if $\mathcal{S}$ is the sheaf of germs of smooth (or continuous) functions into a Lie group G. As in the abelian case, $\check{H}^0(M;\mathcal{S})$ is the space of global sections of $\mathcal{S}$.

(e) Show that there is a natural correspondence

$$\big\{\text{isomorphism classes of rank } k \text{ real vector bundles over } M\big\}\longleftrightarrow\check{H}^1\big(M;O(k)\big).$$

(f) What are the analogues of these statements for complex vector bundles? (state them and indicate the changes in the argument; do not re-write the entire solution).

Hint: For (e) and (f), you might want to look over Sections 9 and 11 in *Lecture Notes*. Do not forget that $\check{H}^1(M;\mathcal{S})$ is a *direct limit*.

2. (a) Show that the set of isomorphism classes of line bundles on M forms an abelian group under the tensor product (i.e. satisfies 3 properties for a group and another for abelian). Show that in the real case all nontrivial elements are of order two.

   (b) Show that the correspondence

$$\big\{\text{isomorphism classes of real line bundles over } M\big\}\longleftrightarrow\check{H}^1(M;\mathbb{Z}_2)$$

   of the previous problem is a group isomorphism.

   (c) Show that there is a natural group isomorphism

$$\big\{\text{isomorphism classes of complex line bundles over } M\big\}\longleftrightarrow\check{H}^2(M;\mathbb{Z}).$$

   Hint: Snake Lemma.

   Note: The groups $\check{H}^1(M;\mathbb{Z}_2)$ and $\check{H}^2(M;\mathbb{Z})$ are naturally isomorphic to the singular cohomology groups $H^1(M;\mathbb{Z}_2)$ and $H^2(M;\mathbb{Z})$. The image of a real line bundle L, $w_1(L)\in H^1(M;\mathbb{Z}_2)$, is *the first Stiefel-Whitney class of* L; the image of a complex line bundle, $c_1(L)\in H^2(M;\mathbb{Z})$, is *the first Chern class of* L. However, this is not how these *characteristic classes* are normally defined.

3. Chapter 2, #13 (p79)

Exercise (*figure this out, but do not hand it in*): Chapter 5, #20 (p217).