MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 8
Due on Thursday, 4/14, in class

1. Suppose X is a topological space and $\mathcal{P}=\{S_U;\rho_{U,V}\}$ is a presheaf on X. Let

$$\bar{S}_U=\Big\{(U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\colon U_\alpha\subset U \text{ open},\ U=\bigcup_{\alpha\in\mathcal{A}}U_\alpha;\ f_\alpha\in S_{U_\alpha};$$
$$\forall\,\alpha,\beta\in\mathcal{A},\ p\in U_\alpha\cap U_\beta\ \exists\,W\subset U_\alpha\cap U_\beta \text{ open s.t. } p\in W,\ \rho_{W,U_\alpha}f_\alpha=\rho_{W,U_\beta}f_\beta\Big\}\Big/\sim,$$

where $\quad (U_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\sim(U'_{\alpha'},f'_{\alpha'})_{\alpha'\in\mathcal{A}'}$ if $\ \forall\,\alpha\in\mathcal{A},\ \alpha'\in\mathcal{A}',\ p\in U_\alpha\cap U'_{\alpha'}$
$$\exists\,W\subset U_\alpha\cap U'_{\alpha'} \text{ s.t. } p\in W,\ \rho_{W,U_\alpha}f_\alpha=\rho_{W,U'_{\alpha'}}f'_{\alpha'}.$$

Whenever $U\subset V$ are open subsets of X, the homomorphisms $\rho_{U,V}$ induce homomorphisms

$$\bar{\rho}_{U,V}\colon\bar{S}_V\longrightarrow\bar{S}_U,\qquad \big[(V_\alpha,f_\alpha)_{\alpha\in\mathcal{A}}\big]\longrightarrow\big[(V_\alpha\cap U,\rho_{V_\alpha\cap U,V_\alpha}f_\alpha)_{\alpha\in\mathcal{A}}\big],$$

so that $\bar{\mathcal{P}}\equiv\{\bar{S}_X;\bar{\rho}_{U,V}\}$ is a presheaf on X. Show that

(a) $\bar{\mathcal{P}}=\alpha(\beta(\mathcal{P}))$;

(b) the presheaf homomorphism $\{\varphi_U\}\colon\mathcal{P}\longrightarrow\bar{\mathcal{P}}$

$$\varphi_U\colon S_U\longrightarrow\bar{S}_U,\qquad f\longrightarrow\big[(U,f)\big],$$

is injective (resp. an isomorphism) if and only if $\mathcal{P}$ satisfies 5.7(C_1) (resp. is complete);

(c) if $\mathcal{R}$ is a subsheaf of $\mathcal{S}$, then $\alpha(\mathcal{S}/\mathcal{R})\approx\overline{\alpha(\mathcal{S})/\alpha(\mathcal{R})}$.

Hint: see 5.8 for (b) and Chapter 5 #2,5 (p216) for (c).

2. Chapter 5, #17 (p217); hint: this is barely a two-liner, including justification.

3. Let K be any ring containing 1. For each $i\in\mathbb{Z}^+$, let $V_i=K$; this is a K-module. Whenever $i\le j$, define
$$\rho_{ji}\colon V_i\longrightarrow V_j\qquad\text{by}\qquad\rho_{ji}(v)=2^{j-i}v;$$
this is a homomorphism of K-modules. Since $\rho_{ki}=\rho_{kj}\rho_{ji}$ whenever $i\le j\le k$, we have a directed system and get a direct-limit K-module
$$V_\infty=\varinjlim_{\mathbb{Z}^+}V_i=\lim_{i\longrightarrow\infty}V_i\,.$$

(a) Suppose $2=0\in K$ (e.g. $K=\mathbb{Z}_2$). Show that $V_\infty=\{0\}$.

(b) Suppose 2 is a unit in K (e.g. $K=\mathbb{R}$). Show that $V_\infty\approx K$ as K-modules.

(c) Suppose 2 is not a unit in K, but $2\neq 0\in K$, and K is an integral domain (e.g. $K=\mathbb{Z}$). Show that the K-module V_∞ is not finitely generated.

Note: if you prefer, you can do the *e.g.* cases; this makes no difference in the argument.