MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 6

Problem 1 (10pts)

Let X be the vector field on $\mathbb{R}^n$ given by $X=\sum_{i=1}^{i=n} x_i \frac{\partial}{\partial x_i}$.

(a) *Determine the time t-flow $X_t\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ of X (give a formula).*

(b) *Use (a) to show directly from the definition of the Lie derivative L_X that the homomorphism defined by*

$$R_k : E^k(\mathbb{R}^n) \longrightarrow E^k(\mathbb{R}^n), \qquad f\mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k} \longrightarrow \left(\int_0^1 s^{k-1}f(sx)\mathrm{d}s\right)\mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k}$$

is a left inverse for L_X if $k\ge 1$ (this is used in the proof of the Poincare Lemma).

(c) *Is R_k also a right inverse for L_X for $k\ge1$? What happens for $k=0$?*

(a) For each $p=(p_1,\ldots,p_n)\in\mathbb{R}^n$, we need to solve the initial-value problem

$$x_i'(t)=x_i(t) \quad i=1,\ldots,n, \qquad x_i(0)=p_i \quad i=1,\ldots,n.$$

The solution is $x_i(t)=p_i\mathrm{e}^t$ for all $i=1,\ldots,n$; so

$$X_t\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n, \qquad X_t(p)=p\mathrm{e}^t.$$

(b) We need to show that $(R_kL_X(f\mathrm{d}x_I))_p=f(p)\mathrm{d}_px_I$ for all $f\in C^\infty(\mathbb{R}^n)$, $\mathrm{d}x_I=\mathrm{d}x_{i_1}\wedge\ldots\wedge\mathrm{d}x_{i_k}$ with $k\ge1$, and $p\in\mathbb{R}^n$. By definition,

$$\big(R_kL_X(f\mathrm{d}x_I)\big)_p=\left(R_k\left(\frac{\mathrm{d}}{\mathrm{d}t}X_t^*(f\mathrm{d}x_I)\Big|_{t=0}\right)\right)_p.$$

Since $x_i\circ X_t=\mathrm{e}^tx_i$, for all $q\in\mathbb{R}^n$

$$\begin{aligned}\big(X_t^*(f\mathrm{d}x_I)\big)_q&=f\circ X_t(q)\mathrm{d}_q(x_{i_1}\circ X_t)\wedge\ldots\wedge\mathrm{d}_q(x_{i_k}\circ X_t)=f(\mathrm{e}^tq)\mathrm{d}_q(\mathrm{e}^tx_{i_1})\wedge\ldots\wedge\mathrm{d}_q(\mathrm{e}^tx_{i_k})\\&=\mathrm{e}^{kt}f(\mathrm{e}^tq)\mathrm{d}_qx_{i_1}\wedge\ldots\wedge\mathrm{d}_qx_{i_k}=\mathrm{e}^{kt}f(\mathrm{e}^tq)\mathrm{d}_qx_I.\end{aligned}$$

By the above two equations and the definition of R_k,

$$\begin{aligned}\big(R_kL_X(f\mathrm{d}x_I)\big)_p&=\left(\int_0^1 s^{k-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{kt}f(\mathrm{e}^tsp)\Big|_{t=0}\mathrm{d}s\right)\mathrm{d}_px_I=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 s^{k-1}\mathrm{e}^{kt}f(\mathrm{e}^tsp)\mathrm{d}s\right)\Big|_{t=0}\mathrm{d}_px_I\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{\mathrm{e}^t}s^{k-1}f(sp)\mathrm{d}s\right)\Big|_{t=0}\mathrm{d}_px_I=\big(f(\mathrm{e}^0p)\cdot\mathrm{e}^0\big)\mathrm{d}_px_I=f(p)\mathrm{d}_px_I\,,\end{aligned}$$

as needed.

(c) If f is continuous function on $\mathbb{R}^n$ which is not everywhere zero, then for any $k\ge1$ there exists $x\in\mathbb{R}^n$ such that

$$\int_0^1 s^{k-1}f(sx)\mathrm{d}s\neq0.$$

Thus, R_k is injective. Since $R_k L_X = \mathrm{id}$ on $E^k(\mathbb{R}^n)$, it follows that R_k is an isomorphism and thus $L_X R_k = \mathrm{id}$. In the $k=0$ case, $R_k f$ is not defined for $f \in E^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ such that $f(\mathbf{0}) \neq 0$ because the function $1/s$ is not integrable near $s=0$. The map R_0 is defined on the subspace

$$C^\infty_{\mathbf{0}}(\mathbb{R}^n) \equiv \big\{ f \in C^\infty(\mathbb{R}^n) \colon \; f(\mathbf{0}) = 0 \big\} \subset E^0(\mathbb{R}^n),$$

mapping it injectively to itself. The image of the homomorphism L_X on $E^0(\mathbb{R}^n)$ is also contained in $C^\infty_{\mathbf{0}}(\mathbb{R}^n)$. Since the argument of part (b) applies when restricted to $C^\infty_{\mathbf{0}}(\mathbb{R}^n)$, R_0 is a left inverse of L_X on $C^\infty_{\mathbf{0}}(\mathbb{R}^n)$ and by injectivity also the right inverse.

Problem 2: Chapter 4, #19 (10pts)

Show that if $f, g \colon M \longrightarrow N$ are smooth maps that are smoothly homotopic, then

$$f^* = g^* \colon H^*_{\mathrm{de\,R}}(N) \longrightarrow H^*_{\mathrm{de\,R}}(M).$$

Let $I = [0,1]$. For $k = 0, 1$, define

$$i_k \colon M \longrightarrow I \times M \qquad \text{by} \qquad i_k(p) = (k, p).$$

In the next paragraph we will construct a homomorphism

$$h \colon E^*(I \times M) \longrightarrow E^{*-1}(M) \qquad \text{s.t.} \qquad i_1^* - i_0^* = h \circ \mathrm{d} + \mathrm{d} \circ h \colon E^*(I \times M) \longrightarrow E^*(M).$$

$$\begin{tikzcd}
\vdots \arrow[dr, "h"] & \vdots \\
E^{k+1}(I\times M) \arrow[u, "d"] \arrow[r, shift left, "i_0^*"] \arrow[r, shift right, "i_1^*"'] \arrow[dr, "h"] & E^{k+1}(M) \arrow[u, "d"'] \\
E^{k}(I\times M) \arrow[u, "d"] \arrow[r, shift left, "i_0^*"] \arrow[r, shift right, "i_1^*"'] \arrow[dr, "h"] & E^{k}(M) \arrow[u, "d"'] \\
E^{k-1}(I\times M) \arrow[u, "d"] \arrow[r, shift left, "i_0^*"] \arrow[r, shift right, "i_1^*"'] \arrow[dr, "h"] & E^{k-1}(M) \arrow[u, "d"'] \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "d"']
\end{tikzcd}$$

If $\alpha \in E^*(I \times M)$ is closed, then

$$\begin{aligned} i_1^*\alpha - i_0^*\alpha = h \circ \mathrm{d}\alpha + \mathrm{d} \circ h\alpha = \mathrm{d}(h\alpha) \qquad &\Longrightarrow \qquad [i_1^*\alpha] = [i_0^*\alpha] \in H^*_{\mathrm{de\,R}}(M) \\ &\Longrightarrow \qquad i_0^* = i_1^* \colon H^*_{\mathrm{de\,R}}(I \times M) \longrightarrow H^*_{\mathrm{de\,R}}(M). \end{aligned}$$

Suppose $F \colon I \times M \longrightarrow N$ is a smooth homotopy from f to g, i.e.

$$\begin{gathered} F(0,p) = f(p), \quad F(1,p) = g(p) \quad \forall\, p \in M \qquad \Longrightarrow \qquad f = F \circ i_0, \quad g = F \circ i_1 \\ \Longrightarrow \qquad f^* = (F \circ i_0)^* = i_0^* \circ F^* = i_1^* \circ F^* = (F \circ i_1)^* = g^* \colon H^*_{\mathrm{de\,R}}(N) \longrightarrow H^*_{\mathrm{de\,R}}(M), \end{gathered}$$

as needed.

If α is a differential form on $I\times M$, then $\alpha=\beta+dt\wedge\gamma$ for some $\beta,\gamma\in\Gamma\big(I\times M;\pi_2^*\Lambda^*T^*M\big)$. Define

$$h\colon E^*(I\times M)\longrightarrow E^{*-1}(M)\quad\text{by}\quad h(\beta)=0,\quad \big\{h(\mathrm{d}t\wedge\gamma)\big\}_p(X_1,\ldots,X_k)=\int_0^1\gamma_{(t,p)}(X_1,\ldots,X_k)\,\mathrm{d}t$$
$$\text{if}\qquad \beta,\gamma\in\Gamma\big(I\times M;\pi_2^*\Lambda^*T^*M\big),\qquad X_1,\ldots,X_k\in T_pM.$$

Suppose $p\in M$ and $\mathbf{x}=(x_1,\ldots,x_n)\colon U\longrightarrow\mathbb{R}^n$ is a smooth chart near p. If α is a k-form on $I\times M$,

$$\alpha|_U=\sum_I a_I\mathrm{d}x_I+\sum_J a_J\mathrm{d}t\wedge\mathrm{d}x_J\qquad\text{for some}\qquad a_I,a_J\in C^\infty(U),$$

where the first sum is taken over all increasing k-tuples I and the second over all increasing $(k-1)$-tuples J. Then,

$$h(\alpha)\big|_U=\begin{cases}0, & \text{if } \alpha|_U=a_I\mathrm{d}x_J;\\ \big(\int_0^1 a_J(t,\mathbf{x})\,\mathrm{d}t\big)\mathrm{d}x_J, & \text{if } \alpha|_U=a_J\mathrm{d}t\wedge\mathrm{d}x_J.\end{cases}$$

Thus,

$$\begin{aligned}
\{h\circ\mathrm{d}+\mathrm{d}\circ h\}(a_I\mathrm{d}x_I)&=h\bigg(\frac{\partial a_I}{\partial t}\mathrm{d}t\wedge\mathrm{d}x_I+\sum_{i=1}^{i=n}\frac{\partial a_I}{\partial x_i}\mathrm{d}x_i\wedge\mathrm{d}x_I\bigg)\\
&=\bigg(\int_0^1\frac{\partial a_I}{\partial t}(t,\mathbf{x})\,\mathrm{d}t\bigg)\mathrm{d}x_I=\big(a_I(1,\mathbf{x})-a_I(0,\mathbf{x})\big)\mathrm{d}x_I=\{i_1^*-i_0^*\}(a_I\mathrm{d}x_I);\\
\{h\circ\mathrm{d}+\mathrm{d}\circ h\}\big(a_J\mathrm{d}t\wedge\mathrm{d}x_J\big)&=h\bigg(-\sum_{i=1}^{i=n}\frac{\partial a_J}{\partial x_i}\mathrm{d}t\wedge\mathrm{d}x_i\wedge\mathrm{d}x_J\bigg)+\mathrm{d}\bigg(\int_0^1 a_J(t,\mathbf{x})\,\mathrm{d}t\big)\mathrm{d}x_J\bigg)\\
&=-\sum_{i=1}^{i=n}\bigg(\int_0^1\frac{\partial a_J}{\partial t}(t,\mathbf{x})\,\mathrm{d}t\bigg)\mathrm{d}x_i\wedge\mathrm{d}x_J+\sum_{i=1}^{i=n}\bigg(\int_0^1\frac{\partial a_J}{\partial x_i}(t,\mathbf{x})\,\mathrm{d}t\bigg)\mathrm{d}x_i\wedge\mathrm{d}x_J\\
&=0=\{i_1^*-i_0^*\}(a_J\mathrm{d}t\wedge\mathrm{d}x_J).
\end{aligned}$$

The last equality holds because $i_k^*\mathrm{d}t=0$.

Problem 3 (5pts)

Show that a one-form α on S^1 is exact if and only if

$$\int_{[0,1]}f^*\alpha=0$$

for every smooth function $f\colon[0,1]\longrightarrow S^1$ such that $f(0)=f(1)$.

Suppose α is an exact one-form on S^1, i.e. $\alpha=\mathrm{d}g$ for some $g\in C^\infty(S^1)$. If $f\colon[0,1]\longrightarrow S^1$ is a smooth function such that $f(0)=f(1)$, then

$$\begin{aligned}
\int_{[0,1]}f^*\alpha=\int_{[0,1]}f^*\mathrm{d}g=\int_{[0,1]}\mathrm{d}(f^*g)=\int_{[0,1]}\mathrm{d}(g\circ f)&\\
=\int_0^1\frac{\mathrm{d}(g\circ f)}{\mathrm{d}t}\,\mathrm{d}t=g\circ f\Big|_{t=0}^{t=1}&=g(f(1))-g(f(0))=0.
\end{aligned}$$

Conversely, let $q\colon \mathbb{R}\longrightarrow S^1$, $t\longrightarrow \mathrm{e}^{2\pi \mathrm{i}t}$, be the standard covering map and suppose

$$\int_{[0,1]} q^*\alpha = 0.$$

Define

$$\tilde{g}\colon \mathbb{R}\longrightarrow \mathbb{R} \qquad \text{by} \qquad \tilde{g}(t) = \int_0^t q^*\alpha.$$

Since $q^*\alpha$ is a smooth one-form on $\mathbb{R}$, $\tilde{g}$ is a smooth function on $\mathbb{R}$. Furthermore, for all $t\in\mathbb{R}$,

$$\tilde{g}(t+1) - \tilde{g}(t) = \int_t^{t+1} q^*\alpha = \int_0^1 q_t^*\alpha = \int_0^1 q^*\alpha = 0, \qquad \text{where} \qquad q_t(s) = q(t+s).$$

Thus, $\tilde{g}$ is constant along the fibers of the quotient projection map q and descends to a continuous map g from the quotient:

$$\begin{tikzcd} \mathbb{R} \arrow[d, "q"'] \arrow[dr, "\tilde{g}"] & \\ S^1 \arrow[r, dashed, "g"] & \mathbb{R} \end{tikzcd}$$

Since q is a local diffeomorphism and $\tilde{g}$ is smooth, so is g. Furthermore,

$$\begin{aligned} (q^*\mathrm{d}g)_t &= \mathrm{d}_t(q^*g) = \mathrm{d}_t(g\circ q) = \mathrm{d}_t\tilde{g} = \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t q^*\alpha\bigg)\mathrm{d}t \\ &= \bigg(\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \{q^*\alpha\}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)\,\mathrm{d}s\bigg)\mathrm{d}t = \{q^*\alpha\}\Big(\frac{\mathrm{d}}{\mathrm{d}s}\Big)\Big|_{s=t}\mathrm{d}t = q^*\alpha|_t. \end{aligned}$$

Since q is a local diffeomorphism, it follows that $\mathrm{d}g=\alpha$, i.e. α is an exact one-form.

Problem 4 (5pts)

(a) *Suppose $\varphi : M \longrightarrow \mathbb{R}^N$ is an immersion. Show that M is orientable if and only if the normal bundle to the immersion φ is orientable.*
(b: Chapter 4, #1) *Suppose $\varphi\colon M^d\longrightarrow\mathbb{R}^{d+1}$ is an immersion. Show that M is orientable if and only if there exists a nowhere-vanishing normal vector field along (M,φ).*

(a) By Section 10 in *Lecture Notes*, the normal bundle $\mathcal{N}_\varphi$ is given by

$$\begin{aligned} &\mathcal{N}_\varphi = \varphi^*T\mathbb{R}^N/\mathrm{Im}\,\mathrm{d}\varphi \qquad \Longrightarrow \qquad \varphi^*T\mathbb{R}^N \approx \mathrm{Im}\,\mathrm{d}\varphi\oplus\mathcal{N}_\varphi \approx TM\oplus\mathcal{N}_\varphi \\ \Longrightarrow \qquad & M\times\mathbb{R} = \varphi^*(\mathbb{R}^N\times\mathbb{R}) \approx \varphi^*\Lambda^{\mathrm{top}}(T\mathbb{R}^N) \approx \Lambda^{\mathrm{top}}\varphi^*(T\mathbb{R}^N) \approx \Lambda^{\mathrm{top}}(TM)\otimes\Lambda^{\mathrm{top}}\mathcal{N}_\varphi\,. \end{aligned}$$

The vector bundles $\Lambda^{\mathrm{top}}(TM)$ and $\Lambda^{\mathrm{top}}\mathcal{N}_\varphi$ are line bundles. By (4) of Lemma 12.1 in *Lecture Notes*, a line bundle is orientable if and only if it is trivial. Since the tensor product of any line bundle L with the trivial line bundle is L again, by the above if $\Lambda^{\mathrm{top}}(TM)$ is trivial, then so is $\Lambda^{\mathrm{top}}\mathcal{N}_\varphi$, and vice versa. Thus, the line bundle $\Lambda^{\mathrm{top}}\mathcal{N}_\varphi$ is orientable if and only if the line bundle $\Lambda^{\mathrm{top}}(TM)$ is orientable. On the other hand, by (3) of Lemma 12.1, a vector bundle $V\longrightarrow M$ is orientable if and only if the line bundle $\Lambda^{\mathrm{top}}V$ is orientable. We conclude that the vector bundle $\mathcal{N}_\varphi$ is orientable if and only if the vector bundle TM is orientable, i.e. M is orientable.

(b) By part (a), M is orientable if and only the normal bundle $\mathcal{N}_\varphi$ is orientable. Since $\mathcal{N}_\varphi$ is a line bundle in this case, by (4) of Lemma 12.1 and by Lemma 8.5 in *Lecture Notes* $\mathcal{N}_\varphi$ is orientable if and only if $\mathcal{N}_\varphi$ admits a nowhere-vanishing section. Since $T\mathbb{R}^n$ has a natural metric, such a section corresponds to a vector field Y along (M,φ) which is everywhere normal to $\mathrm{Im}\,\mathrm{d}\varphi$, i.e.

$$Y(p)\in T_{\varphi(p)}\mathbb{R}^{d+1} \quad \text{and} \quad \big\langle Y(p),\mathrm{d}_p\varphi(X)\big\rangle=0 \qquad \forall\ X\in T_pM,\ p\in M.$$

Thus, M is orientable if and only if there exists a nowhere-vanishing normal vector field along (M,φ).

Problem 5 (10pts)

Let M be a smooth manifold.
(a) *Show that every real vector bundle $V\longrightarrow M$ admits a Riemannian metric and every complex vector bundle admits a hermitian metric.*
(b) *Show that if M is connected and there exists a non-orientable vector bundle $V\longrightarrow M$, then M admits a connected double-cover (2:1 covering map).*
(c) *Show that if the order of $\pi_1(M)$ is finite and odd, then M is orientable.*

(a) Let $V\longrightarrow M$ be a real vector bundle of rank k. Choose a locally finite open cover of M by trivializations $\{(U_\alpha,h_\alpha)\}_{\alpha\in\mathcal{A}}$ of V, i.e.

$$h_\alpha\colon V|_{U_\alpha}\longrightarrow U_\alpha\times\mathbb{R}^k$$

is a diffeomorphism commuting with the projections maps to U_α which is linear on each fiber. Such a cover exists because M is paracompact. Let $\{\varphi_\alpha\}_{\alpha\in\mathcal{A}}$ be a partition of unity on M subordinate to $\{U_\alpha\}_{\alpha\in\mathcal{A}}$, i.e.

$$\varphi_\alpha\in C^\infty(M), \qquad \varphi_\alpha(M)\subset[0,1], \qquad \sum_{\alpha\in\mathcal{A}}\varphi_\alpha(p)=1\ \ \forall\,p\in M,$$
$$\text{and} \qquad \operatorname{supp}\varphi_\alpha\equiv\overline{\varphi_\alpha^{-1}(\mathbb{R}-0)}\subset U_\alpha\ \ \forall\,\alpha\in\mathcal{A}.$$

For each $\alpha\in\mathcal{A}$, define a symmetric bilinear form $\langle,\rangle_\alpha$ on V by

$$\langle X_1,X_2\rangle_\alpha=\begin{cases}\varphi_\alpha(p)\langle h_\alpha(X_1),h_\alpha(X_2)\rangle, & \text{if } X_1,X_2\in V_p,\ p\in U_\alpha;\\ 0, & \text{if } X_1,X_2\in V_p,\ p\in M-\operatorname{supp}\varphi_\alpha,\end{cases}$$

where $\langle h_\alpha(X_1),h_\alpha(X_2)\rangle$ denotes the standard inner-product on $\mathbb{R}^k$. Since $\langle,\rangle_\alpha$ is smooth over the open sets U_α and $M-\operatorname{supp}\varphi_\alpha$ and agrees on the overlap, $\langle,\rangle_\alpha$ is a well-defined smooth bilinear symmetric form on all of M. We define a symmetric bilinear form $\langle,\rangle$ on M by

$$\langle X_1,X_2\rangle=\sum_{\alpha\in\mathcal{A}}\langle X_1,X_2\rangle_\alpha \qquad \forall\ X_1,X_2\in V_p,\ p\in M.$$

Since for every $p\in M$ there exists a neighborhood U of p that intersects only finitely many of the open sets U_α, the above sum is a finite sum of smooth bilinear forms, and therefore is smooth. Furthermore,

$$\langle X,X\rangle=\sum_{\alpha\in\mathcal{A}}\langle X,X\rangle_\alpha \qquad \forall\,X\in V_p,\ p\in M.$$

By construction, $\langle X,X\rangle_\alpha \geq 0$ for all $\alpha\in\mathcal{A}$. Since for every $p\in M$ there exists $\beta\in\mathcal{A}$ such that $\varphi_\beta(p)>0$,

$$\langle X,X\rangle_\beta > 0 \quad \forall\, X\in V_p-0 \qquad \Longrightarrow \qquad \sum_{\alpha\in\mathcal{A}}\langle X,X\rangle_\alpha \geq \langle X,X\rangle_\beta > 0 \quad \forall\, X\in V_p-0,$$

i.e. $\langle,\rangle$ is nondegenerate. Thus, $\langle,\rangle$ is a Riemannian metric in V. The construction in the complex case is analogous: simply replace $\mathbb{R}^k$ with its standard inner-product by $\mathbb{C}^k$ with its standard hermitian inner-product.

(b) Suppose M is connected and $V\longrightarrow M$ is a non-orientable vector bundle. By (3) of Lemma 12.1 in *Lecture Notes*, the line bundle $\Lambda^{\mathrm{top}}V\longrightarrow M$ is non-orientable. Choose a Riemannian metric in $\Lambda^{\mathrm{top}}V$. By (5) of Lemma 12.1, $\pi\colon S(\Lambda^{\mathrm{top}}V)\longrightarrow V$ is connected. It is a 2:1 covering map.

(c) We assume that M is connected (otherwise, $\pi_1(M)$ depends on the choice of component). If M is non-orientable, the vector bundle $TM\longrightarrow M$ is not orientable. By part (b), there exists a connected 2:1-covering map $\pi\colon \tilde{M}\longrightarrow M$. By Theorem 54.6 in Munkres,

$$\pi_*\big(\pi_1(\tilde{M})\big)\subset \pi_1(M)$$

is a subgroup of index two, i.e. the corresponding set of cosets consists of two elements. Since all cosets have the same cardinality, the index of every subgroup must divide the order of the group (if it is finite). Thus, if $\pi_1(M)$ is finite and odd, M does not admit a connected 2:1-covering map and must then be orientable.

Problem 6 (15pts)

(a) *Show that the antipodal map on $S^n\subset\mathbb{R}^{n+1}$ (i.e. $x\longrightarrow -x$) is orientation-preserving if n is odd and orientation-reversing if n is even.*
(b) *Show that $\mathbb{R}P^n$ is orientable if and only if n is odd.*
(c) *Describe the orientable double cover of $\mathbb{R}P^n\times\mathbb{R}P^n$ with n even.*

(a) Denote by $\iota\colon S^n\longrightarrow\mathbb{R}^{n+1}$ the inclusion map, by $\tilde{a}\colon\mathbb{R}^{n+1}\longrightarrow\mathbb{R}^{n+1}$ the antipodal map, and by $a\colon S^n\longrightarrow S^n$ its restriction to S^n:

$$\begin{tikzcd} S^n \arrow[r, "a"] \arrow[d, "\iota"'] & S^n \arrow[d, "\iota"'] \\ \mathbb{R}^{n+1} \arrow[r, "\tilde{a}"] & \mathbb{R}^{n+1} \end{tikzcd}$$

Let $\Omega=\mathrm{d}x_1\wedge\ldots\mathrm{d}x_{n+1}$ be the standard volume form on $\mathbb{R}^{n+1}$ and

$$X=\sum_{i=1}^{n+1}x_i\frac{\partial}{\partial x_i}\in\Gamma(\mathbb{R}^{n+1};T\mathbb{R}^{n+1}).$$

Since the function

$$f\colon\mathbb{R}^{n+1}\longrightarrow\mathbb{R},\qquad (x_1,\ldots,x_{n+1})\longrightarrow\sum_{i=1}^{n+1}x_i^2,$$

is constant on $S^n\subset\mathbb{R}^{n+1}$, d_pf vanishes on $T_pS^n\subset\mathbb{R}^{n+1}$. Since

$$X(f)=\sum_{i=1}^{n+1}x_i\frac{\partial}{\partial x_i}(f)=2\sum_{i=1}^{n+1}x_i^2,$$

$X(f)_p \neq 0$ for all $p \in S^n$ and thus $X_p \notin T_pS^n$. Since Ω is a volume form on $\mathbb{R}^{n+1}$, it follows that

$$\alpha \equiv \iota^*(i_X\Omega) = \big(i_X\Omega\big)|_{TS^n}$$

is a volume form on S^n (Ω_p is nonzero on any set of $n+1$ linearly independent vectors in $\mathbb{R}^{n+1}$, in particular if the first one of them is X_p and the remaining are n linearly independent vectors in $T_pS^n \subset \mathbb{R}^{n+1}$). Since $\tilde{a}^*\Omega = (-1)^{n+1}\Omega$ and $\mathrm{d}\tilde{a}(X) = X$ on $\mathbb{R}^{n+1}$,

$$a^*\alpha = a^*\iota^*(i_X\Omega) = \iota^*\tilde{a}^*\big(i_{\mathrm{d}\tilde{a}(X)}\Omega\big) = \iota^*\big(i_X(\tilde{a}^*\Omega)\big) = (-1)^{n+1}\iota^*(i_X\Omega) = (-1)^{n+1}\alpha.$$

Thus, $a\colon S^n \longrightarrow S^n$ is orientation-preserving if and only if $n+1$ is even.

(b) We first make the following general observation. Suppose $\tilde{M}$ is a smooth manifold and G is a group that acts on M by diffeomorphisms and properly discontinuously; see Section 81 in Munkres. By Problem 2 on PS1, $M \equiv \tilde{M}/G$ is a smooth manifold, with smooth structure induced from that of $\tilde{M}$ via the quotient projection map $q\colon \tilde{M} \longrightarrow M$. We claim that

$$\big\{q^*\alpha\colon \alpha \in E^*(M)\big\} = E^*(\tilde{M})^G \equiv \big\{\tilde{\alpha} \in E^*(\tilde{M})\colon g^*\tilde{\alpha} = \tilde{\alpha}\ \forall\, g \in G\big\}. \tag{1}$$

Since $q \circ g = q$, for all $\alpha \in E^*(M)$

$$q^*\alpha = (q \circ g)^*\alpha = g^*(q^*\alpha) \qquad \Longrightarrow \qquad \big\{q^*\alpha\colon \alpha \in E^*(M)\big\} \subset E^*(\tilde{M})^G.$$

Conversely, suppose $\tilde{\alpha} \in E^*(\tilde{M})^G$; define $\alpha \in E^*(M)$ as follows. If $p \in M$, choose $\tilde{p} \in q^{-1}(p)$ and neighborhoods U and $\tilde{U}$ of p and $\tilde{p}$ in M and $\tilde{M}$, respectively, such that $q\colon \tilde{U} \longrightarrow U$ is a diffeomorphism. Define

$$\alpha_p \in \Lambda^* T_p^*M \qquad \text{by} \qquad \{q|_{\tilde{U}}\}^*(\alpha_p) = \tilde{\alpha}_{\tilde{p}}.$$

If $\tilde{p}' \in q^{-1}(p)$ is another point, there exists $g \in G$ such that $g\tilde{p}' = \tilde{p}$. We can then take $\tilde{U}' = g^{-1}(\tilde{U})$. Since $q \circ g = q$,

$$\{q|_{\tilde{U}'}\}^*\big(\alpha_p\big) = \{q|_{\tilde{U}} \circ g\}^*\big(\alpha_p\big) = g^*\big(q|_{\tilde{U}}^*(\alpha_p)\big) = g^*(\tilde{\alpha}_{\tilde{p}}) = \tilde{\alpha}_{\tilde{p}'},$$

i.e. α is well-defined. Since $q|_{\tilde{U}}$ is a diffeomorphism, α is smooth. We have now proved (1). One consequence of (1) is that a volume form (and thus an orientation) on M corresponds to a volume form for $\tilde{M}$ which is preserved by G.

Let α be the volume form on S^n defined in (a) and $a\colon S^n \longrightarrow S^n$ the antipodal map. By definition,

$$\mathbb{R}P^n = S^n/\mathbb{Z}_2, \qquad \text{where} \qquad (-1)x = a(x)\ \ \forall\, x \in S^n.$$

By part (a), $a^*\alpha = (-1)^{n+1}\alpha$. Thus, if n is odd, then $\alpha \in E^*(S^n)^{\mathbb{Z}_2}$ and defines an orientation on $\mathbb{R}P^n$; so $\mathbb{R}P^n$ is orientable in this case. On the other hand, if n is even there exists no non-vanishing $\beta \in E^n(S^n)^{\mathbb{Z}_2}$ and thus $\mathbb{R}P^n$ is not orientable in this case by the previous paragraph. For if $\beta \in E^n(S^n)^{\mathbb{Z}_2}$, then $\beta = f\alpha$ for some $f \in C^\infty(S^n)$ and therefore

$$f\alpha = \beta = a^*\beta = (f \circ a)a^*\alpha = -(f \circ a)\,\alpha \qquad \Longrightarrow \qquad f \circ a = -f.$$

Thus, f and β must vanish somewhere on S^n.

(c) By Theorem 60.1 in Munkres,

$$\pi_1(\mathbb{R}P^n \times \mathbb{R}P^n) \approx \pi_1(\mathbb{R}P^n) \times \pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

The universal cover of $\mathbb{R}P^n\times\mathbb{R}P^n$ is $S^n\times S^n$ and

$$\mathbb{R}P^n\times\mathbb{R}P^n = (S^n\times S^n)/\pi_1(\mathbb{R}P^n\times\mathbb{R}P^n) = (S^n\times S^n)/\mathbb{Z}_2\oplus\mathbb{Z}_2;$$

see Section 81 in Munkres. The group $\mathbb{Z}_2\oplus\mathbb{Z}_2\approx\{\pm1\}\times\{\pm1\}$ acts on $(S^n\times S^n)$ by

$$\big((-1)^p\times(-1)^q\big)\cdot(x\times y) = a^p(x)\times a^q(y).$$

By Section 82 in Munkres, the connected double covers of $\mathbb{R}P^n\times\mathbb{R}P^n$ correspond to (the conjugacy classes) of the subgroups of $\pi_1(\mathbb{R}P^n\times\mathbb{R}P^n)$ of index two. The cover corresponding to a subgroup G of $\mathbb{Z}_2\oplus\mathbb{Z}_2$ is

$$(S^n\times S^n)/G \longrightarrow (S^n\times S^n)/\mathbb{Z}_2\oplus\mathbb{Z}_2.$$

There are three index-two subgroups of $\mathbb{Z}_2\oplus\mathbb{Z}_2$: those generated by $(-1,1)$, $(1,-1)$, and $(-1,-1)$. The covering spaces corresponding to the first two groups are $\mathbb{R}P^n\times S^n$ and $S^n\times\mathbb{R}P^n$; neither is orientable. Thus, the covering space corresponding to the third subgroup is

$$(S^n\times S^n)/G = (S^n\times S^n)/\mathbb{Z}_2, \qquad (-1)\cdot(x\times y) = d_a(x,y) \equiv (-x)\times(-y),$$

must be orientable (this is called the diagonal $\mathbb{Z}_2$-action on $S^n\times S^n$). This can also be seen directly. If α is the volume form on S^n as above, then $\beta\equiv\pi_1^*\alpha\wedge\pi_2^*\alpha$ is a volume form on $S^n\times S^n$. Furthermore,

$$\begin{aligned} d_a^*\beta = d_a^*\pi_1^*\alpha\wedge d_a^*\pi_2^*\alpha &= (\pi_1\circ d_a)^*\alpha\wedge(\pi_2\circ d_a)^*\alpha = (a\circ\pi_1)^*\alpha\wedge(a\circ\pi_2)^*\alpha \\ &= \pi_1^*a^*\alpha\wedge\pi_2^*a^*\alpha = \pi_1^*\big((-1)^{n+1}\alpha\big)\wedge\pi_2^*\big((-1)^{n+1}\alpha\big) = \pi_1^*\alpha\wedge\pi_2^*\alpha = \beta. \end{aligned}$$

Thus, $\beta\in E^{\mathrm{top}}(S^n\times S^n)^{\mathbb{Z}_2}$ is a volume form on $S^n\times S^n$ preserved by the diagonal $\mathbb{Z}_2$-action and therefore induces a volume/orientation form on the corresponding quotient.

Problem 7 (10pts)

(a) *Show that every diffeomorphism $f\colon S^n\longrightarrow S^n$ that has no fixed points is smoothly homotopic to the antipodal map (x is a fixed point of f if $f(x)=x$).*
(b) *Show that if $\pi\colon S^n\longrightarrow M$ is a covering projection onto a smooth manifold M and $|\pi_1(M)|\neq2$, then M is orientable.*

(a) Define

$$F\colon I\times S^n\longrightarrow S^n \qquad \text{by} \qquad F(t,x) = \frac{(1-t)f(x)+t(-x)}{|(1-t)f(x)+t(-x)|}.$$

This map is well-defined, since

$$\begin{aligned} |(1-t)f(x)+t(-x)|=0 \quad&\Longrightarrow\quad (1-t)f(x)=tx \quad\Longrightarrow\quad |1-t|=|t| \\ &\Longrightarrow\quad t=1/2 \quad\Longrightarrow\quad f(x)=x. \end{aligned}$$

However, f has no fixed points. The map F is smooth, since it is smooth as a map into $\mathbb{R}^{n+1}$ and its image lies in S^n, which is an embedded submanifold of $\mathbb{R}^{n+1}$.

(b) The group $\pi_1(M)$ acts on S^n properly discontinuously by diffeomorphisms and $M=S^n/\pi_1(M)$. Since S^n is compact, $\pi_1(M)$ is a finite group. Let

$$\beta = \frac{1}{|\pi_1(M)|}\sum_{g\in\pi_1(M)} g^*\alpha \in E^n(S^n)^{\pi_1(M)},$$

where α is the standard volume form on S^n, as in Problem 6. If $g \in \pi_1(M)$ is orientation-preserving, then $g^*\alpha = f_g\alpha$ for a smooth positive-valued function f_g on S^n. Thus, if $g\colon S^n \longrightarrow S^n$ is orientation-preserving for all g, then β is a nowhere-vanishing top form on S^n preserved by $\pi_1(M)$ and thus induces an orientation on M; see Problem 6b.

By the previous paragraph, it is sufficient to show that every element of $\pi_1(M)$ is orientation-preserving if $|\pi_1(M)| \neq 2$. If $g \in \pi_1(M)$ is not the identity, g has no fixed points and thus homotopic to the antipodal map by part (a). Then,

$$g^* = a^* = (-1)^{n+1}\colon H^n_{\mathrm{de\,R}}(S^n) \longrightarrow H^n_{\mathrm{de\,R}}(S^n)$$

by Problems 2 and 6. In particular, if n is odd, then all elements of $\pi_1(M)$ act by orientation-preserving diffeomorphisms (no matter what $\pi_1(M)$ is). Suppose n is even. We will show that $\pi_1(M)$ contains at most 2 elements. Suppose $g_1, g_2 \in \pi_1(M)$ are different from the identity. Since $\pi_1(M)$ acts without fixed points,

$$\begin{aligned} g_k^* = a^* = (-1)^{n+1} = -1\colon H^n_{\mathrm{de\,R}}(S^n) \longrightarrow H^n_{\mathrm{de\,R}}(S^n) \qquad k=1,2; \\ \Longrightarrow \qquad g_1g_2 = 1 \neq a^*\colon H^n_{\mathrm{de\,R}}(S^n) \longrightarrow H^n_{\mathrm{de\,R}}(S^n). \end{aligned}$$

Thus, g_1g_2 is not homotopic to a and must then have a fixed point by part (a). Since $\pi_1(M)$ acts without fixed points, it follows that $g_1g_2 = \mathrm{id}$. Since this holds for any pair of elements of $\pi_1(M) - \mathrm{id}$, it follows that $\pi_1(M)$ contains at most 2 elements.

Problem 8 (10pts)

(a) *Show that if X is a smooth nowhere-vanishing vector field on a compact manifold M, then the flow $X_t\colon M \longrightarrow M$ of X has no fixed points for some $t \in \mathbb{R}$.*
(b) *Show that S^n admits a nowhere vanishing vector field if and only if n is odd.*
(c) *Show that the tangent bundle of S^n is not trivial if $n \geq 1$ is even.*
Note: In fact, TS^n is trivial if and only if $n = 1, 3, 7$.

(a) Suppose not, i.e. there exists a sequence $t_k \in \mathbb{R}^*$ converging to 0 and a sequence $p_k \in M$ such that $X_{t_k}(p_k) = p_k$ for all $k \in \mathbb{Z}^+$. Since M is compact, after passing to a subsequence we can assume that p_k converges to some $p^* \in M$. If $X_{p^*} \neq 0$, there exists $t^* \in \mathbb{R}^*$ such that $X_{t^*}(p^*) \neq p^*$. Since M is Hausdorff, there exist disjoint open neighborhoods U and V of p^* and $X_{t^*}(p^*)$, respectively. By (d) and (h) of Theorem 1.48, there exist a neighborhood U' of p^* in U and $\epsilon > 0$ such that

$$X_t(p) \in V \qquad \forall\, p \in U',\ t \in (t^* - \epsilon, t^* + \epsilon).$$

Since (t_k, p_k) converges to $(0, p^*)$, there exist $k, N \in \mathbb{Z}^+$ such that

$$p_k \in U' \quad \text{and} \quad Nt_k \in (t^* - \epsilon, t^* + \epsilon) \quad \Longrightarrow \quad X_{Nt_k}(p_k) \in V \quad \Longrightarrow \quad X_{Nt_k}(p_k) \neq p_k \in U' \subset U.$$

However, this is impossible, since $X_{Nt_k}(p_k) = X_{t_k}^N(p_k) = p_k$ because $X_{t_k}(p_k) = p_k$.

(b) If X is a nowhere-vanishing vector field on S^n, by part (a) there exists $t \in \mathbb{R}$ such that the diffeomorphism $X_t\colon S^n \longrightarrow S^n$ has no fixed points. Thus, by part (a) of Problem 7, X_t is homotopic to the antipodal map a and

$$X_t^* = a^* = (-1)^{n+1}\colon H^n_{\mathrm{de\,R}}(S^n) \longrightarrow H^n_{\mathrm{de\,R}}(S^n)$$

by Problems 2 and 6. On the other hand, $s \longrightarrow X_{st}$ is a homotopy from $X_0 = \mathrm{id}$ to X_t. Thus,

$$X_t^* = \mathrm{id}^* = \mathrm{id}\colon H^n_{\mathrm{de\,R}}(S^n) \longrightarrow H^n_{\mathrm{de\,R}}(S^n).$$

It follows that $1 = (-1)^{n+1}$, i.e. n is odd.

On the other hand, if $n = 2k+1$ is odd, let

$$X\big(x_1, x_2, \ldots, x_{2k+1}, x_{2k+2}\big) = \sum_{i=1}^{k+1} \Big(-x_{2i}\frac{\partial}{\partial x_{2i-1}} + x_{2i-1}\frac{\partial}{\partial x_{2i}} \Big).$$

This is a vector field on $\mathbb{R}^{2k+2}$ (it corresponds to rotations in $k+1$ coordinate 2-planes) that does not vanish on $\mathbb{R}^{2k+2} - 0 \supset S^{2k+1}$. We show

$$X|_{S^{2k+1}} \in \Gamma\big(S^{2k+1}; TS^{2k+1}\big) \subset \Gamma\big(S^{2k+1}; T\mathbb{R}^{2k+2}|_{S^{2k+1}}\big).$$

Since S^{2k+1} is defined by $f(\mathbf{x}) \equiv |\mathbf{x}|^2 = 1$ and $\mathrm{d}_{\mathbf{x}} f \neq 0$ for all $\mathbf{x} \in S^{2k+1}$, it is sufficient to show that X lies in the kernel of $\mathrm{d}_{\mathbf{x}} f$:

$$\mathrm{d}f = 2\sum_{i=1}^{2k+2} x_i dx_i \qquad \Longrightarrow \qquad \mathrm{d}f(X) = 2\sum_{i=1}^{k+1} \big(-x_{2i}x_{2i-1} + x_{2i-1}x_{2i} \big) = 0.$$

Thus, $X|_{S^{2k+1}}$ is a nowhere-vanishing vector field on S^{2k+1}.

(c) If TS^n is isomorphic to $S^n \times \mathbb{R}^n$, S^n admits a nowhere-vanishing vector field X (in fact, n vector fields that are linearly independent at every point of S^n). By part (b), if this is the case, then n is odd. In other words, if $n \geq 1$ is even, the vector bundle $TS^n \longrightarrow S^n$ is not trivial.

Problem 9 (5pts)

Suppose M is a compact oriented 3-manifold with boundary and $\partial M = T^2 = S^1 \times S^1$. Let

$$\pi_1, \pi_2 \colon T^2 \longrightarrow S^1$$

be the two projection maps. Show that it is impossible to extend both (as opposed to at least one of) $\alpha_1 \equiv \pi_1^* \mathrm{d}\theta$ and $\alpha_2 \equiv \pi_2^* \mathrm{d}\theta$ to closed forms on M.

Suppose β_1 and β_2 are closed one-forms on M such that $\alpha_1 = \beta_1|_{\partial M}$ and $\alpha_2 = \beta_2|_{\partial M}$. Then,

$$\mathrm{d}(\beta_1 \wedge \beta_2) = \mathrm{d}\beta_1 \wedge \beta + (-1)^1 \beta_1 \wedge \mathrm{d}\beta_2 = 0 + 0 = 0.$$

Thus, by the second version of Stokes' Theorem, Theorem 4.9,

$$\int_{\partial M} \beta_1 \wedge \beta_2 = \int_M \mathrm{d}(\beta_1 \wedge \beta_2) = \int_M 0 = 0.$$

On the other hand,

$$\int_{\partial M} \beta_1 \wedge \beta_2 = \int_{\partial M} \alpha_1 \wedge \alpha_2 = \int_{S^1 \times S^1} \pi_1^* \mathrm{d}\theta \wedge \pi_2^* \mathrm{d}\theta = \int_{S^1} \mathrm{d}\theta \cdot \int_{S^1} \mathrm{d}\theta = 2\pi \cdot 2\pi \neq 0.$$

This is a contradiction.