MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 5
Due on Thursday, 3/10, in class

Note: This problem set has two pages.

1. Let V be a vector space of dimension n and $\Omega \in \Lambda^n V^*$ a nonzero element. Show that the homomorphism
$$V \longrightarrow \Lambda^{n-1}V^*, \qquad v \longrightarrow i_v\Omega,$$
where i_v is the contraction map, is an isomorphism.

2. Suppose M is a smooth n-manifold.
   (a) Let Ω be a nowhere-zero n-form on M. Show that for every $p \in M$ there exists a chart $(x_1,\ldots,x_n)\colon U \longrightarrow \mathbb{R}^n$ around p such that
$$\Omega|_U = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$
   (b) Let α be a nowhere-zero closed $(n-1)$-form on M. Show that for every $p \in M$ there exists a chart $(x_1,\ldots,x_n)\colon U \longrightarrow \mathbb{R}^n$ around p such that
$$\alpha|_U = \mathrm{d}x_2 \wedge dx_3 \wedge \ldots \wedge \mathrm{d}x_n.$$

3. Let M be a smooth manifold and $X, Y \in \Gamma(M;TM)$ smooth vector fields on M. Show that the Lie derivative satisfies
$$L_{[X,Y]} = [L_X, L_Y] \equiv L_X \circ L_Y - L_Y \circ L_X$$
as homomorphisms on $\Gamma(M;TM)$ and $E^k(M)$. *Hint:* use 1.44,1.45d, 2.25abe.

4. Let α be a k-form on a smooth manifold M and $X_0, X_1, \ldots, X_k$ smooth vector fields on M. Show directly from the definitions that
$$\begin{aligned}\mathrm{d}\alpha(X_0, X_1, \ldots, X_k) = &\sum_{i=0}^{i=k}(-1)^i X_i\big(\alpha(X_0,\ldots,\widehat{X_i},\ldots,X_k)\big)\\ &+\sum_{i<j}(-1)^{i+j}\alpha\big([X_i,X_j],X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k\big).\end{aligned}$$
Hint: first show that the values of LHS and RHS at any $p \in M$ depend only on the values of $X_0, X_1, \ldots, X_k$ at p.

5. Let $V \longrightarrow M$ be a smooth vector bundle of rank k and $W \subset V$ a smooth subbundle of V of rank k'. Show that
$$\mathrm{Ann}(W) \equiv \big\{\alpha \in V_p^*\colon\ \alpha(w)=0\ \forall\, w \in W,\ p \in M\big\}$$
is a smooth subbundle of V^* of rank $k-k'$.

6. Suppose M is a 3-manifold, α is a nowhere-zero one-form on M, and $p\in M$. Show that

   (a) if there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$, then $(\alpha\wedge \mathrm{d}\alpha)|_p=0$.

   (b) if there exists a neighborhood U of p in M such that $(\alpha\wedge \mathrm{d}\alpha)|_U=0$, then there exists an embedded 2-dimensional submanifold $P\subset M$ such that $p\in P$ and $\alpha|_{TP}=0$.

   Note: If the top form $\alpha\wedge \mathrm{d}\alpha$ on M is nowhere-zero, α is called a contact form. In this case, it has no integrable submanifolds at all.

7. A two-form ω on a smooth manifold M is called symplectic if ω is closed (i.e. $\mathrm{d}\omega=0$) and everywhere nondegenerate[1]. Suppose ω is a symplectic form on M.

   (a) Show that the dimension of M is even and the map

$$TM\longrightarrow T^*M, \qquad X\longrightarrow i_X\omega,$$

   is a vector bundle isomorphism (i_X is the contraction w.r.t. X, i.e. the dual of $X\wedge$).

   (b) If $H: M\longrightarrow \mathbb{R}$ is a smooth map, let $X_H\in\Gamma(M;TM)$ be the preimage of $\mathrm{d}H$ under this isomorphism. Assume that X_H is a complete vector field, so that the flow

$$\varphi\colon \mathbb{R}\times M\longrightarrow M, \qquad (t,p)\longrightarrow \varphi_t(p),$$

   is globally defined. Show that for every $t\in\mathbb{R}$, the time-t flow $\varphi_t\colon M\longrightarrow M$ is a symplectomorphism, i.e. $\varphi_t^*\omega=\omega$.

   Note: In such a situation, H is called a Hamiltonian and φ_t a Hamiltonian symplectomorphism.

[1] This means that $\omega_p\in\Lambda^2T_p^*M$ is nondegenerate for every $p\in M$, i.e. for every $v\in T_pM-0$ there exists $v'\in T_pM$ such that $\omega_p(v,v')\neq 0$.