MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 4

Problem 1: Chapter 1, #13ad (10pts)

(a) *Show that* $[X,Y]$ *is a smooth vector field on* M *for any two smooth vector fields* X *and* Y *on* M.
(d) *Show that* $[,]$ *satisfies the Jacobi identity, i.e.*

$$\big[[X,Y],Z\big]+\big[[Y,Z],X\big]+\big[[Z,X],Y\big]=0$$

for all smooth vector fields on X, Y, *and* Z *on* M.

(a) First, we need to see that for every $p\in M$ the map

$$[X,Y]_p\colon C^\infty(M)\longrightarrow\mathbb{R},\qquad [X,Y]_p(f)=X_p(Yf)-Y_p(Xf),$$

is well-defined and is an element of T_pM, i.e. it is bilinear, satisfies the product rule, and its value depends only on the germ of f at p. Since X, Y, and f are smooth, Yf and Xf are smooth functions on M by Proposition 1.43. Since X_p and Y_p are linear functionals on $C^\infty(M)$,

$$X_p(Yf),Y_p(Xf)\in\mathbb{R}\qquad\Longrightarrow\qquad [X,Y]_p(f)\in\mathbb{R},$$

i.e. the map $[X,Y]_p$ is well-defined. Since it is a composition of linear maps, $[X,Y]_p$ is a linear map as well. Furthermore, if $f,g\in C^\infty(M)$,

$$\begin{aligned}
[X,Y]_p(fg)&=X_p\big(Y(fg)\big)-Y_p\big(X(fg)\big)=X_p\big(fY(g)+gY(f)\big)-Y_p\big(fX(g)+gX(f)\big)\\
&=\big(f(p)X_p(Y(g))+Y_p(g)X_p(f)+g(p)X_p(Y(f))+Y_p(f)X_p(g)\big)\\
&\qquad\qquad-\big(f(p)Y_p(X(g))+X_p(g)Y_p(f)+g(p)Y_p(X(f))+X_p(f)Y_p(g)\big)\\
&=f(p)\big(X_p(Y(g))-Y_p(X(g))\big)+g(p)\big(X_p(Y(f))-Y_p(X(f))\big)\\
&=f(p)[X,Y]_p(g)+g(p)[X,Y]_p(g),
\end{aligned}$$

i.e. the linear map $[X,Y]_p$ satisfies the product rule. Finally, if U is a neighborhood of p in M and $f|_U=g|_U$, then

$$(Xf)|_U=(Xg)|_U\qquad\text{and}\qquad (Yf)|_U=(Yg)|_U,$$

since for all $q\in U$ the real numbers X_qf and X_qg depend only on the germs of f and g at q. Since the values of X_p and Y_p on $C^\infty(M)$ depend only on the germs of functions at q, we conclude that

$$Y_p(Xf)=Y_p(Xg)\quad\text{and}\quad X_p(Yf)=X_p(Yg)\qquad\Longrightarrow\qquad [X,Y]_pf=[X,Y]_pg,$$

i.e. the value of $[X,Y]_p$ on $f\in C^\infty(M)$ depends only on the germ of f at p. Thus, $[X,Y]$ is a vector field on M.

If X, Y, and f are smooth, then Xf and Yf are smooth functions on M by Proposition 1.43. By Proposition 1.43 again, $Y(Xf)$ and $X(Yf)$ are also smooth functions on M. It follows that the function $[X,Y]f$ is smooth for every smooth function f on M. Thus, $[X,Y]$ is a smooth vector field on M by Proposition 1.43.

(d) We need to show that LHS of the identity is the zero map, i.e. the function obtained by applying LHS to any smooth function f on M is zero. The first summand gives:

$$\big[[X,Y],Z\big]f = [X,Y](Zf) - Z\big([X,Y]f\big) = \big(X(Y(Zf)) - Y(X(Zf))\big) - Z\big(X(Yf) - Y(Xf)\big)$$
$$= X\big(Y(Zf)\big) - Y\big(X(Zf)\big) - Z\big(X(Yf)\big) + Z\big(Y(Xf)\big).$$

Permuting X, Y, and Z cyclicly, we then obtain

$$\big[[Y,Z],X\big]f = Y\big(Z(Xf)\big) - Z\big(Y(Xf)\big) - X\big(Y(Zf)\big) + X\big(Z(Yf)\big) \qquad \text{and}$$
$$\big[[Z,X],Y\big]f = Z\big(X(Yf)\big) - X\big(Z(Yf)\big) - Y\big(Z(Xf)\big) + Y\big(X(Zf)\big).$$

The three expressions add up to zero.

Problem 2: Chapter 1, #22 (5pts)

Let $\gamma(t)$ be an integral curve for a vector field X on M. Show that if $\gamma'(t)=0$ for some t, then γ is a constant map.

Suppose $\gamma\colon (a,b) \longrightarrow M$, $\gamma'(t_0)=0$ for some $t_0 \in (a,b)$, and $\gamma(t_0)=p$. Since γ is an integral curve for X,

$$X(p) = X\big(\gamma(t_0)\big) = \gamma'(t_0) = 0.$$

Let $\beta\colon (a,b) \longrightarrow M$ be the curve defined by $\beta(t)=p$ for all $t \in (a,b)$. Then,

$$\beta(t_0) = p \qquad \text{and} \qquad \beta'(t) = 0 = X(p) = X\big(\beta(t)\big) \quad \forall\, t \in (a,b).$$

We also have

$$\gamma(t_0) = p \qquad \text{and} \qquad \gamma'(t) = X\big(\gamma(t)\big) \quad \forall\, t \in (a,b).$$

By the uniqueness theorem for first-order ODEs, or Theorem 1.48, $\beta=\gamma$, i.e. γ is a constant map.

Problem 3: Chapter 1, #17 (5pts)

Show that any smooth vector field on a compact manifold is complete.

Suppose M is a compact m-manifold, X is a smooth vector field on M, and $\gamma\colon (a,b) \longrightarrow M$ is a maximal integral curve for X. Thus, $a<0$ and $b>0$. We need to show that $(a,b)=\mathbb{R}$.

Suppose $b \in \mathbb{R}$. Choose a sequence $t_n \in (a,b)$ converging to b. Since M is a compact, a subsequence converges to a point $p \in M$. By (3) of Theorem 1.48, there exists $\epsilon \in (0,|a|)$ and a neighborhood U of p in M such that the flow

$$(-\epsilon,\epsilon)\times U \longrightarrow M, \qquad (t,q) \longrightarrow X_t(q),$$

is well-defined. Choose $t_n \in (a,b)$ such that $b-t_n < \epsilon$ and $\gamma(t_n) \in U$. Let

$$\beta\colon (-\epsilon,\epsilon) \longrightarrow M$$

be the integral curve for X such that $\beta(0)=\gamma(t_n)$. Define

$$\alpha\colon (a,t_n+\epsilon) \longrightarrow M \qquad \text{by} \qquad \alpha(t) = \begin{cases} \gamma(t), & \text{if } t\in(a,b); \\ \beta(t-t_n), & \text{if } t\in(t_n-\epsilon,t_n+\epsilon). \end{cases}$$

Let $\tilde{\gamma}(t)=\gamma(t+t_n)$ for $t\in(-\epsilon,b-t_n)$. Since

$$\tilde{\gamma}(0)=\beta(0), \qquad \tilde{\gamma}'(t)=X(\tilde{\gamma}(t)), \quad \text{and} \quad \beta'(t)=X(\beta(t)),$$

$\tilde{\gamma}=\beta$ on $(-\epsilon,b-t_n)$ by the uniqueness of integral curves. Thus, α is well-defined. Furthermore,

$$\alpha'(t) = \begin{cases} \gamma'(t), & \text{if } t\in(a,b) \\ \beta'(t-t_n), & \text{if } t\in(t_n-\epsilon,t_n+\epsilon) \end{cases} = \begin{cases} X(\gamma(t)), & \text{if } t\in(a,b) \\ X(\beta(t-t_n)), & \text{if } t\in(t_n-\epsilon,t_n+\epsilon) \end{cases} = X(\alpha(t)),$$

i.e. α is an integral curve for X. Since $t_n+\epsilon>b$ and $\alpha|_{(a,b)}=\gamma$, we conclude that α is an integral curve for X extending γ. Thus, γ is not maximal unless $b=\infty$. The proof that $a=-\infty$ is similar (or apply the conclusion to the vector field $-X$).

Problem 4 (5pts)

Let V be the vector field on $\mathbb{R}^3$ given by

$$V(x,y,z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Explicitly describe and sketch the flow of V.

The integral curves for this vector fields are the solutions of the system

$$x'(t)=y(t), \qquad y'(t)=-x(t), \qquad z'(t)=1.$$

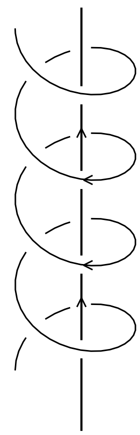
The first pair of equations is independent of the third; its solutions are

$$\big(x(t),y(t)\big) = (x_0+\mathfrak{i}y_0)e^{-\mathfrak{i}t} \in \mathbb{C}.$$

The corresponding curve goes around a circle centered at the origin *clockwise* at the unit angular speed. The solution of the third equation is $z(t)=z_0+t$. Thus, the flow for X is given by

$$X_t\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3=\mathbb{C}\times\mathbb{R}, \qquad X_t\big(x,y,z\big) = \big((x+\mathfrak{i}y)e^{-\mathfrak{i}t}, z+t\big),$$

i.e. the flow rotates clockwise around the vertical axis at the unit angular speed and climbs up at the unit angular speed (the vertical axis itself simply moves up).



Problem 5 (10pts)

Suppose X and Y are smooth vector fields on a manifold M. Show that for every $p \in M$ and $f \in C^\infty(M)$,

$$\lim_{s,t\longrightarrow 0} \frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(p))\big))\big) - f(p)}{s\,t} = [X,Y]_p f \in \mathbb{R}.$$

Do not forget to explain why the limit exists.
Note: This means that the extent to which the flows $\{X_t\}$ of X and $\{Y_s\}$ of Y do not commute (i.e. the rate of change in the "difference" between $Y_s \circ X_t$ and $X_t \circ Y_s$) is measured by $[X,Y]$.

By (d) of Theorem 1.48, we can choose a neighborhood U of p and $\epsilon > 0$ such that the maps

$$(-\epsilon,\epsilon)\times U \longrightarrow M, \quad (z,q) \longrightarrow X_z(q), \qquad (-\epsilon,\epsilon)^2\times U \longrightarrow M, \quad (w,z,q) \longrightarrow Y_w(X_z(q)),$$
$$(-\epsilon,\epsilon)^3\times U \longrightarrow M, \quad (v,w,z,q) \longrightarrow X_v\big(Y_w(X_z(q))\big), \qquad \text{and}$$
$$(-\epsilon,\epsilon)^4\times U \longrightarrow M, \quad (u,v,w,z,q) \longrightarrow Y_u\big(X_v\big(Y_w(X_z(q))\big)\big),$$

are defined and smooth. Define

$$K\colon (-\epsilon,\epsilon)^4 \longrightarrow \mathbb{R} \qquad \text{and} \qquad H\colon (-\epsilon,\epsilon)^2 \longrightarrow \mathbb{R} \qquad \text{by}$$
$$K(u,v,w,z) = f\big(Y_u\big(X_v(Y_w(X_z(p)))\big)\big) - f(p) \qquad \text{and} \qquad H(s,t) = K(-s,-t,s,t).$$

Since K is a composition of smooth functions, K is smooth. Since H is a composition of smooth functions, H is smooth. Furthermore,

$$X_0 = Y_0 = \mathrm{id}_X, \quad Y_{-s}\circ Y_s = \mathrm{id}_{\mathrm{Dom}_s Y}, \quad X_{-t}\circ X_t = \mathrm{id}_{\mathrm{Dom}_t X} \qquad \Longrightarrow \qquad H(s,0) = H(0,t) = 0$$

for all s and t. Thus[1], the limit in the statement of the problem exists and equals to the mixed second partial derivative of H:

$$\lim_{s,t\longrightarrow 0} \frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(p))\big))\big) - f(p)}{s\,t} = \lim_{s,t\longrightarrow 0} \frac{H(s,t)-H(0,0)}{s\,t} = \frac{\partial^2 H}{\partial s\,\partial t}\Big|_{(0,0)}.$$

[1] this follows, for example, from Lemma 3.5 in *Lecture Notes* applied twice

On the other hand, by the Chain Rule,

$$\frac{\partial^2 H}{\partial s\,\partial t}\Big|_{(0,0)} = \frac{\partial^2 K}{\partial u\,\partial v}\Big|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial u\,\partial z}\Big|_{(0,0,0,0)} - \frac{\partial^2 K}{\partial v\,\partial w}\Big|_{(0,0,0,0)} + \frac{\partial^2 K}{\partial w\,\partial z}\Big|_{(0,0,0,0)}.$$

Note that

$$\begin{aligned}\frac{\partial^2 K}{\partial u\,\partial v}\Big|_{(0,0,0,0)} &= \frac{\partial}{\partial v}\bigg(\frac{\partial}{\partial u}f(Y_u(X_v(p)))\Big|_{u=0}\bigg)\Big|_{v=0} = \frac{\partial}{\partial v}\bigg(\mathrm{d}_{X_v(p)}f\Big(\frac{\partial}{\partial u}Y_u(X_v(p))\Big|_{u=0}\Big)\bigg)\Big|_{v=0}\\ &= \frac{\partial}{\partial v}\Big(\mathrm{d}_{X_v(p)}f(Y)\Big)\Big|_{v=0} = \frac{\partial}{\partial v}\Big(\{Yf\}\big(X_v(p)\big)\Big)\Big|_{v=0} = \mathrm{d}_p\{Yf\}(X) = X_p(Yf).\end{aligned}$$

Similarly,

$$\frac{\partial^2 K}{\partial u\,\partial z}\Big|_{(0,0,0,0)} = X_p(Yf), \qquad \frac{\partial^2 K}{\partial v\,\partial w}\Big|_{(0,0,0,0)} = Y_p(Xf), \qquad \frac{\partial^2 K}{\partial w\,\partial z}\Big|_{(0,0,0,0)} = X_p(Yf).$$

Putting these together, we conclude that

$$\lim_{s,t\longrightarrow 0}\frac{f\big(Y_{-s}(X_{-t}\big(Y_s(X_t(p))\big))\big)-f(p)}{s\,t} = \frac{\partial^2 H}{\partial s\,\partial t}\Big|_{(0,0)} = X_p(Yf)-Y_p(Xf) = [X,Y]_p f.$$

Problem 6 (10pts)

Let U and V be the vector fields on $\mathbb{R}^3$ given by

$$U(x,y,z)=\frac{\partial}{\partial x} \qquad \textit{and} \qquad V(x,y,z)=F(x,y,z)\frac{\partial}{\partial y}+G(x,y,z)\frac{\partial}{\partial z},$$

where F and G are smooth functions on $\mathbb{R}^3$. Show that there exists a proper[2] foliation of $\mathbb{R}^3$ by 2-dimensional embedded submanifolds such that the vector fields U and V everywhere span the tangent spaces of these submanifolds if and only if

$$F(x,y,z)=f(y,z)\,e^{h(x,y,z)} \qquad \textit{and} \qquad G(x,y,z)=g(y,z)\,e^{h(x,y,z)}$$

for some $f,g\in C^\infty(\mathbb{R}^2)$ and $h\in C^\infty(\mathbb{R}^3)$ such that (f,g) does not vanish on $\mathbb{R}^2$.

If at every point of $\mathbb{R}^3$ the vector fields U and V span the tangent space of a 2-dimensional submanifold, then their span is two-dimensional, i.e. (F,G) does not vanish. If this is the case, by Frobenius Theorem there exists an integral submanifold for the distribution $\mathcal{D}\subset T\mathbb{R}^3$ spanned by U and V through every point of $\mathbb{R}^3$ if and only if the vector field

$$[U,V]=F_x\frac{\partial}{\partial y}+G_x\frac{\partial}{\partial z}$$

lies in the span of U and V over $C^\infty(\mathbb{R}^3)$. This is the case if and only if there exists $\lambda\in C^\infty(\mathbb{R}^3)$ such that

$$\begin{aligned}[U,V]=\lambda V \qquad &\Longleftrightarrow \qquad F_x=\lambda F, \quad G_x=\lambda G\\ &\Longleftrightarrow \qquad F(x,y,z)=f(y,z)e^{h(x,y,z)}, \quad G(x,y,z)=g(y,z)e^{h(x,y,z)},\end{aligned}$$

[2] in the sense of Definition 10.4 in *Lecture Notes*

where $h\in C^\infty(\mathbb{R}^3)$ is such that $h_x=\lambda$ and $f,g\in C^\infty(\mathbb{R}^2)$ are such that (f,g) does not vanish on $\mathbb{R}^2$ (so that V does not vanish).

If the above is the case, the maximal connected integral submanifolds for the distribution $\mathcal{D}$ spanned by U and V partition $\mathbb{R}^3$. We will show that all such submanifolds are embedded. Since e^h does not vanish, $\mathcal{D}$ is spanned by the vector fields

$$U(x,y,z)=\frac{\partial}{\partial x} \qquad \text{and} \qquad W(x,y,z)=f(y,z)\frac{\partial}{\partial y}+g(y,z)\frac{\partial}{\partial z}.$$

Let $\gamma\colon(a,b)\longrightarrow\mathbb{R}^2$ be a maximal integral curve for the vector field

$$\widetilde{W}(y,z)=f(y,z)\frac{\partial}{\partial y}+g(y,z)\frac{\partial}{\partial z}.$$

Since (f,g) does not vanish on $\mathbb{R}^2$ and $\gamma'(t)=\widetilde{W}(\gamma(t))$ γ is a maximal connected integral submanifold for the distribution $\tilde{\mathcal{D}}$ on $\mathbb{R}^2$ spanned by $\widetilde{W}$. Furthermore,

$$\psi=\mathrm{id}\times\gamma\colon\mathbb{R}\times(a,b)\longrightarrow\mathbb{R}^3=\mathbb{R}\times\mathbb{R}^2$$

is a maximal connected integral submanifold for $\mathcal{D}$ and every maximal connected submanifold for $\mathcal{D}$ has this form. It is an integral submanifold for $\mathcal{D}$ because

$$\begin{aligned} d\psi\big|_{(s,t)}\frac{\partial}{\partial s}&=\frac{\partial}{\partial s}\times 0=U\big(\psi(s,t)\big),\\ d\psi\big|_{(s,t)}\frac{\partial}{\partial t}&=\big(0,\gamma'(t)\big)=\big(0,\widetilde{W}(\gamma(t))\big)=W\big(\psi(s,t)\big),\end{aligned}$$

if (s,t) are the standard coordinates on $\mathbb{R}\times(a,b)$. Since the maximal integral curves γ for W partition $\mathbb{R}^2$, the images of the maps $\mathrm{id}\times\gamma$ partition $\mathbb{R}^3$. Thus, each map $\mathrm{id}\times\gamma$ must be a maximal connected submanifold for $\mathcal{D}$. In the following paragraph, we show that every integral curve γ for $\widetilde{W}$ must be embedded in $\mathbb{R}^2$. This implies that every maximal connected integral submanifold $\mathrm{id}\times\gamma$ for $\mathcal{D}$ is embedded in $\mathbb{R}^3$.

Suppose $\gamma\colon(a,b)\longrightarrow\mathbb{R}^2$ is a maximal integral curve for $\widetilde{W}$ and $t_0\in(a,b)$. By Proposition 1.53, we can choose a coordinate chart

$$\varphi=(x_1,x_2)\colon\big(\mathcal{U},\gamma(t_0)\big)\longrightarrow(\mathbb{R}^2,0)$$

and a neighborhood (c,d) of t_0 in (a,b) such that

$$[-2,2]\times[-2,2]\subset\varphi(\mathcal{U}),\qquad W|_{\mathcal{U}}=\frac{\partial}{\partial x_1}\Big|_{\mathcal{U}},\quad\text{and}\quad \gamma|_{(c,d)}\colon(c,d)\longrightarrow\varphi^{-1}\big(0\times(-2,2)\big)\tag{1}$$

is a diffeomorphism. The middle condition implies that

$$\operatorname{Im}\gamma\cap\varphi^{-1}\big((-2,2)\times(-2,2)\big)$$

is a union of horizontal slices $\varphi^{-1}\big((-2,2)\times y\big)$ with $y\in S_\gamma$, where S_γ is a subset of $(-2,2)$. To show that γ is an embedding, we show that there exists $\epsilon>0$ such that

$$S_\gamma\cap(-\epsilon,\epsilon)=\{0\}.^3$$

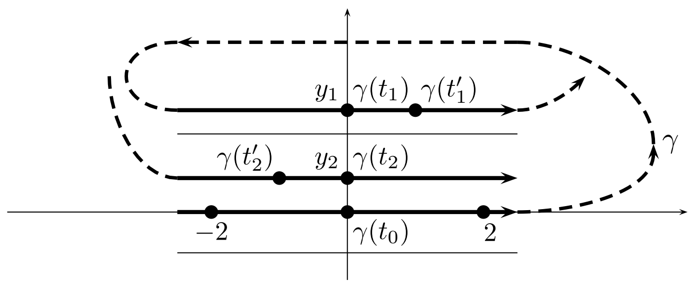
[3] In such a case, if $(c',d')\subset(c,d)$ is a basis element around $t_0\in(a,b)$, then

$$\gamma\big((c',d')\big)=\big(\gamma(c'),\gamma(d')\big)\times 0=\gamma\big((c',d')\big)\cap\big((\gamma(c'),\gamma(d'))\times(-\epsilon,\epsilon)\big),$$

Suppose not, i.e. there exists a sequence $t_k \in (a,b)$ converging to either a or b such that $\gamma(t_k)=0\times y_k$ with $y_k\in S_\gamma$ converging to $0\in\mathbb{R}$. By taking a subsequence and by symmetry, it is sufficient to assume that $t_k\longrightarrow b$ and $y_k\in\mathbb{R}^+$. We can then choose $t_1,t_2\in(t_0,b)$ with $t_1<t_2$ so that

$$\varphi(\gamma(t_1))=0\times y_1,\quad \varphi(\gamma(t_2))=0\times y_2\qquad\text{s.t.}$$
$$0<y_2<y_1<2,\quad (t_0,t_1)\cap\gamma^{-1}\big(\varphi^{-1}(0\times(0,y_1))\big)=\emptyset,\quad (t_1,t_2)\cap\gamma^{-1}\big(\varphi^{-1}(0\times(0,y_1))\big)=\emptyset.$$

In other words, $t_1\in(t_0,b)$ is the smallest number so that $\varphi(\gamma(t_1))\in 0\times(0,2)$ and $t_2\in(t_1,b)$ is the smallest number so that $\varphi(\gamma(t_2))\in 0\times(0,y_1)$.



By the middle condition in eq1,

$$t_1'\equiv\gamma^{-1}\big(\varphi^{-1}(1\times y_1)\big)\in(t_1,t_2),\qquad t_2'\equiv\gamma^{-1}\big(\varphi^{-1}((-1)\times y_2)\big)\in(t_1',t_2),$$

i.e. the flow is to the right in each slice as indicated in the diagram.

Since γ is an integral curve, it has no self-intersections. Thus,

$$\mathcal{C}\equiv\gamma\big((t_0,t_1)\big)\cup\varphi^{-1}\big(0\times[0,y_1]\big)$$

is a simple closed curve in $\mathbb{R}^2$ since

$$(t_0,t_1)\cap\gamma^{-1}\big(\varphi^{-1}(0\times(0,y_1))\big)=\emptyset.$$

Let ℓ be the straight line segment between $\varphi(\gamma(t_1'))$ and $\varphi(\gamma(t_2'))$ in $\mathbb{R}^2$. Since the curve $\varphi^{-1}(\ell)$ intersects the simple closed curve $\mathcal{C}$ exactly once, one of its endpoints, i.e. $\gamma(t_1')$ or $\gamma(t_2')$, must lie inside of $\mathcal{C}$ and the other outside. Thus, the curve $\gamma((t_1',t_2'))$ must intersect $\mathcal{C}$ at least once. Since

$$\gamma\big((t_1',t_2')\big)\cap\varphi^{-1}\big(0\times[0,y_1]\big)\subset\gamma\big((t_1,t_2)\big)\cap\varphi^{-1}\big(0\times[0,y_1]\big)=\emptyset$$

and $\gamma((t_1',t_2'))\cap\mathcal{C}\neq\emptyset$, we conclude that

$$\gamma\big((t_1',t_2')\big)\cap\gamma\big((t_0,t_1)\big)\neq\emptyset.$$

However, this is impossible as well, since $t_1<t_1'$ and an integral curve cannot intersect itself.

Remark: The above argument implies that if $\mathcal{D}$ is any distribution on $\mathbb{R}^2$, every connected integral submanifold for $\mathcal{D}$ is embedded. This is not the case for other manifolds, including $\mathbb{R}^3$ and T^2 (see Chapter 1, #21, p51).

i.e. γ takes open subsets of (a,b) to sets in $\mathrm{Im}\,\gamma$ that are open with respect to the topology $\mathrm{Im}\,\gamma$ inherits as a subspace of $\mathbb{R}^2$. So, γ is an embedding.