MAT 531: Topology&Geometry, II
Spring 2011

Solutions to Problem Set 3

Problem 1: Chapter 1, #5 (10pts)

If $(M,\mathcal{F})$ is a smooth n-manifold, let

$$TM = \bigsqcup_{m\in M} T_mM \qquad \textit{and} \qquad \pi\colon TM \longrightarrow M, \quad \pi(v)=m \;\; \textit{if} \;\; v\in T_mM.$$

If $(\mathcal{U},\varphi)\in\mathcal{F}$ and $\varphi=(x_1,\ldots,x_n)$, define

$$\tilde\varphi\colon \pi^{-1}(\mathcal{U}) \longrightarrow \mathbb{R}^{2n} \qquad \textit{by} \quad \tilde\varphi(v) = \big(\varphi(\pi(v)), v(x_1),\ldots,v(x_n)\big).$$

Show that

(a) *for all $(\mathcal{U},\varphi),(V,\psi)\in\mathcal{F}$, $\tilde\psi\circ\tilde\varphi^{-1}$ is smooth;*

(b) *the collection*
$$\mathcal{B} \equiv \big\{\tilde\varphi^{-1}(W)\colon W\subset\mathbb{R}^{2n} \text{ open}, \; (\mathcal{U},\varphi)\in\mathcal{F}\big\}$$
is basis for a topology on TM in which TM is a topological $2n$-manifold;

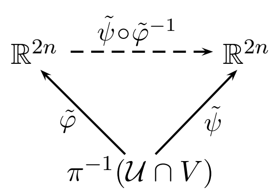
(c) *the collection*
$$\tilde{\mathcal{F}}_0 = \big\{(\pi^{-1}(\mathcal{U}),\tilde\varphi)\colon (\mathcal{U},\varphi)\in\mathcal{F}\big\}$$
induces a differentiable structure $\tilde{\mathcal{F}}$ on TM.

(a) Suppose $\varphi=(x_1,\ldots,x_n)$ and $\psi=(y_1,\ldots,y_n)$. By 1.20b, with i replaced by j,

$$v(y_i) = \bigg\{\sum_{j=1}^n v(x_j)\frac{\partial}{\partial x_j}\Big|_m\bigg\}(y_i) = \sum_{j=1}^n \Big(\frac{\partial y_i}{\partial x_j}\Big)v(x_j).$$

Thus, the overlap map

$$\tilde\psi\circ\tilde\varphi^{-1}\colon \tilde\varphi\big(\pi^{-1}(\mathcal{U})\cap\pi^{-1}(V)\big) \longrightarrow \tilde\psi\big(\pi^{-1}(\mathcal{U})\cap\pi^{-1}(V)\big)$$



is given by

$$\tilde\psi\circ\tilde\varphi^{-1}\colon \varphi\big(\mathcal{U}\cap V\big)\times\mathbb{R}^n \longrightarrow \psi\big(\mathcal{U}\cap V\big)\times\mathbb{R}^n, \qquad \tilde\psi\circ\tilde\varphi^{-1}(p,w) = \big(\psi\circ\varphi^{-1}(p), \mathcal{J}(\psi\circ\varphi^{-1})|_p w\big),$$

where

$$\mathcal{J}(\psi\circ\varphi^{-1})|_p = \left(\frac{\partial y_i}{\partial x_j}\right)_{i,j=1,\ldots,n}\Big|_p$$

is the Jacobian (the matrix of partial derivatives) of $\psi\circ\varphi^{-1}$ at p. Since $(\mathcal{U},\varphi),(V,\psi)\in\mathcal{F}$, the maps

$$\psi\circ\varphi^{-1}\colon \varphi\big(\mathcal{U}\cap V\big)\longrightarrow\mathbb{R}^n \qquad\text{and}\qquad \mathcal{J}(\psi\circ\varphi^{-1})\colon \varphi\big(\mathcal{U}\cap V\big)\longrightarrow \mathrm{Mat}_n\mathbb{R}=\mathbb{R}^{n^2}$$

are smooth. Since the multiplication map is smooth, so is the map

$$\varphi\big(\mathcal{U}\cap V\big)\times\mathbb{R}^n\longrightarrow\mathbb{R}^n,\quad (p,w)\longrightarrow \mathcal{J}(\psi\circ\varphi^{-1})|_p w.$$

Since both "coordinate" functions of the overlap map $\tilde\psi\circ\tilde\varphi^{-1}$ are smooth, the map $\tilde\psi\circ\tilde\varphi^{-1}$ itself is smooth as well.

(b-i) We begin by showing that $\mathcal{B}$ covers TM. Since $\mathcal{F}$ is a differentiable structure on M,

$$TM=\pi^{-1}(M)=\pi^{-1}\bigg(\bigcup_{(\mathcal{U},\varphi)\in\mathcal{F}}\mathcal{U}\bigg)=\bigcup_{(\mathcal{U},\varphi)\in\mathcal{F}}\pi^{-1}(\mathcal{U})=\bigcup_{(\mathcal{U},\varphi)}\tilde\varphi^{-1}(\mathbb{R}^{2n})\subset\bigcup_{V\in\mathcal{B}}V.$$

Furthermore, if $(\mathcal{U},\varphi),(V,\psi)\in\mathcal{F}$ and W,W' are open subsets of $\mathbb{R}^{2n}$, then

$$\begin{aligned}\tilde\varphi^{-1}(W)\cap\tilde\psi^{-1}(W')&=\tilde\varphi^{-1}(W)\cap\tilde\varphi^{-1}(\mathbb{R}^{2n})\cap\tilde\psi^{-1}(W')=\tilde\varphi^{-1}(W)\cap\tilde\varphi^{-1}\big(\tilde\varphi(\tilde\varphi^{-1}(\mathbb{R}^{2n})\cap\tilde\psi^{-1}(W'))\big)\\&=\tilde\varphi^{-1}\big(W\cap\tilde\varphi(\tilde\varphi^{-1}(\mathbb{R}^{2n})\cap\tilde\psi^{-1}(W'))\big)=\tilde\varphi^{-1}\big(W\cap\tilde\varphi(\tilde\psi^{-1}(W')\big)\in\mathcal{B},\end{aligned}$$

since $\tilde\varphi(\tilde\psi^{-1}(W'))\subset\mathbb{R}^{2n}$ is open (because by part (a), $\tilde\psi\circ\tilde\varphi^{-1}$ is smooth and thus continuous). We conclude that $\mathcal{B}$ is a basis for a topology on TM.

(b-ii) We next show that TM is Hausdorff in this topology. Suppose $v,w\in TM$ and $v\neq w$. If $\pi(v)=\pi(w)$, choose $(\mathcal{U},\varphi)\in\mathcal{F}$ such that $\pi(v)\in\mathcal{U}$. Since the map $\tilde\varphi$ is injective, $\tilde\varphi(v)\neq\tilde\varphi(w)\in\mathbb{R}^{2n}$. If V and W are disjoint open subsets of $\mathbb{R}^{2n}$ (which is Hausdorff) containing $\tilde\varphi(v)$ and $\tilde\varphi(w)$, respectively, then

$$\tilde\varphi^{-1}(V),\tilde\varphi^{-1}(W)\in\mathcal{B}$$

are disjoint open subsets of TM containing v and w, respectively. On the other hand, suppose $\pi(v)\neq\pi(w)$. Since M is Hausdorff, there exist disjoint open subsets V and W of M containing $\pi(v)$ and $\pi(w)$. Note that since $\mathcal{F}$ is maximal with respect to the smooth-overlap condition, if $(\mathcal{U},\varphi)\in\mathcal{F}$, then $(\mathcal{U}',\varphi|_{\mathcal{U}'})\in\mathcal{F}$ for every open subset $\mathcal{U}'\subset\mathcal{U}$. Thus, there exist $(V',\varphi),(W',\psi)\in\mathcal{F}$ such that

$$\pi(v)\in V'\subset V,\quad \pi(w)\in W'\subset W\quad\Longrightarrow\quad v\in\pi^{-1}(V')=\tilde\varphi^{-1}(\mathbb{R}^{2n}),\quad w\in\pi^{-1}(W')=\tilde\psi^{-1}(\mathbb{R}^{2n}).$$

Thus, $\tilde\varphi^{-1}(\mathbb{R}^{2n}),\tilde\psi^{-1}(\mathbb{R}^{2n})\in\mathcal{B}$ are are disjoint open subsets of TM containing v and w, respectively.

(b-iii) If $(\mathcal{U},\varphi)\in\mathcal{F}$, the map

$$\tilde\varphi\colon\pi^{-1}(\mathcal{U})=\tilde\varphi^{-1}(\mathbb{R}^{2n})\longrightarrow\varphi(\mathcal{U})\times\mathbb{R}^n$$

is continuous, as $\tilde\varphi^{-1}(W)\in\mathcal{B}$ for all $W\subset\mathbb{R}^n$ open. Furthermore, if $W\subset\mathbb{R}^{2n}$ is open, then

$$\tilde\varphi\big(\tilde\varphi^{-1}(W)\big)=W\cap(\varphi(\mathcal{U})\times\mathbb{R}^n)$$

is open in $\mathbb{R}^{2n}$. Combining this with (b-i), it follows that $\tilde{\varphi}$ takes basis elements for the topology on $\pi^{-1}(\mathcal{U})\subset TM$ to open subsets of $\mathbb{R}^{2n}$. Thus, the map

$$\tilde{\varphi}\colon \pi^{-1}(\mathcal{U}) \longrightarrow \varphi(\mathcal{U})\times\mathbb{R}^n$$

is continuous, open, and bijective, i.e. a homeomorphism. Since $\{\pi^{-1}(\mathcal{U})\}_{(\mathcal{U},\varphi)\in\mathcal{F}}$ is a cover of TM, it follows that $\{(\mathcal{U},\tilde{\varphi})\}_{(\mathcal{U},\varphi)\in\mathcal{F}}$ is a collection of charts covering TM and TM is locally Euclidean of dimension $2n$.

(b-iv) It remains to show that the topology on TM has a countable basis. Since M has a countable basis, there exists a countable subcollection $\mathcal{F}_0=\{(\mathcal{U}_i,\varphi_i)\}_{i\in\mathbb{Z}}$ of $\mathcal{F}$ such that the collection $\{\mathcal{U}_i\}_{i\in\mathbb{Z}}$ covers M. Then, the collection $\{\pi^{-1}(\mathcal{U}_i)\}_{i\in\mathbb{Z}}$ is a countable open cover of TM and $\pi^{-1}(\mathcal{U}_i)$ is second-countable (being homeomorphic to subset of $\mathbb{R}^{2n}$). Thus, TM is second-countable as well.

(c) We need to show that the collection

$$\tilde{\mathcal{F}}_0 = \big\{(\pi^{-1}(\mathcal{U}),\tilde{\varphi})\colon (\mathcal{U},\varphi)\in\mathcal{F}\big\}$$

is a collection of charts on TM covering TM and the overlap maps are smooth. Each of the maps

$$\tilde{\varphi}\colon \pi^{-1}(\mathcal{U})=\tilde{\varphi}^{-1}(\mathbb{R}^{2n}) \longrightarrow \pi^{-1}(\mathcal{U})\times\mathbb{R}^n$$

is a chart on M by (b-iii). The overlap maps, $\tilde{\psi}^{-1}\circ\tilde{\varphi}$, are smooth by part (a). Finally, $\{\pi^{-1}(\mathcal{U})\}_{(\mathcal{U},\varphi)\in\mathcal{F}}$ is a cover of TM, since $\{\mathcal{U}\}_{(\mathcal{U},\varphi)\in\mathcal{F}}$ is a cover of M.

Problem 2 (5pts)

Show that the tangent bundle TM of a smooth n-manifold is a real vector bundle of rank n over M. What is its transition data?

Let $\{(\mathcal{U}_\alpha,\varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ be the smooth structure on M. By Problem 1, TM is a smooth manifold with smooth structure given by the collection $\{(\pi^{-1}(\mathcal{U}_\alpha),\tilde{\varphi}_\alpha)\}_{\alpha\in\mathcal{A}}$. If $\alpha\in\mathcal{A}$ and $\varphi_\alpha=(x_1,\ldots,x_n)$, we define a trivialization of TM over $\mathcal{U}_\alpha$ by

$$h_\alpha\colon TM|_{\mathcal{U}_\alpha}\equiv\pi^{-1}(\mathcal{U}_\alpha)\longrightarrow\mathcal{U}_\alpha\times\mathbb{R}^n, \qquad h_\alpha(v)=\big(\pi(v),v(x_1),\ldots,v(x_n)\big).$$

This map is smooth, since the induced map between the charts

$$\{\varphi_\alpha\times\mathrm{id}\}\circ h_\alpha\circ\tilde{\varphi}_\alpha^{-1}\colon \varphi_\alpha(\mathcal{U}_\alpha)\times\mathbb{R}^n\longrightarrow\varphi_\alpha(\mathcal{U}_\alpha)\times\mathbb{R}^n$$

is the identity (and thus smooth). Furthermore, $\pi_1\circ h_\alpha=\pi|_{TM|\mathcal{U}_\alpha}$ and the restriction of h_α to each fiber of π is an isomorphism of vector spaces. Thus, $\pi: TM\longrightarrow M$ is a real vector bundle of rank n, with trivializations $\{(\pi^{-1}(\mathcal{U}_\alpha),\tilde{\varphi}_\alpha)\}_{\alpha\in\mathcal{A}}$. By Problem 2(a), the corresponding overlap maps are given by

$$h_\alpha\circ h_\beta^{-1}(m,v)=\big(m,g_{\alpha\beta}(m)v\big),$$

where the transition map

$$g_{\alpha\beta}\colon\mathcal{U}_\alpha\cap\mathcal{U}_\beta\longrightarrow\mathrm{GL}_n\mathbb{R}$$

is given by

$$g_{\alpha\beta}(m) = \mathcal{J}(\varphi_\alpha \circ \varphi_\beta^{-1})|_{\varphi_\beta(m)}.$$

Problem 3 (5pts)

Show that the tangent bundle TS^1 of S^1, defined as in 1.25 (p19), is isomorphic to the trivial real line bundle over S^1.

By Lemma 8.5 in *Lecture Notes*, it is sufficient to show that the vector bundle $\pi : TS^1 \longrightarrow S^1$ admits a nowhere-zero section s or *vector field*, i.e. a smooth family of choices of $v_m \in T_m S^1$ for each $m \in M$. Such a section is given by

$$s(m) = \frac{\partial}{\partial\theta}\Big|_m,$$

where θ is the angle "coordinate". Formally, let

$$\psi \colon \mathbb{R} \longrightarrow S^1 \subset \mathbb{C}, \qquad \psi(\theta) = e^{i\theta},$$

be the standard covering projection. Then, we define the section s of TS^1 by

$$s\big(\psi(\theta)\big) = d\psi\big|_\theta \frac{\partial}{\partial\theta}.$$

This section is well-defined, i.e. depends only on $\psi(\theta)$ and not θ. To see this, define

$$h \colon \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{by} \qquad h(\theta) = \theta + 2\pi.$$

If $\psi(\theta) = \psi(\theta')$, then $\theta' = h^m(\theta)$ for some $m \in \mathbb{Z}$. On the other hand,

$$dh|_\theta \frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta} \qquad \Longrightarrow \qquad dh^m|_\theta \frac{\partial}{\partial\theta} = \frac{\partial}{\partial\theta}$$
$$\Longrightarrow \qquad d\psi\big|_\theta \frac{\partial}{\partial\theta} = d(\psi \circ h^m)\big|_\theta \frac{\partial}{\partial\theta} = d\psi\big|_{h^m(\theta)} \circ dh^m|_\theta \frac{\partial}{\partial\theta} = d\psi\big|_{\theta'} \frac{\partial}{\partial\theta}.$$

Furthermore, the restriction of ψ to each interval $(\theta - \pi, \theta + \pi)$ is the inverse map for a coordinate patch $(\mathcal{U}, \varphi)$. Then,

$$\begin{aligned}\tilde{\varphi}\big(s(\psi(\theta))\big) \equiv \big(\varphi\big(\pi\big(s(\psi(\theta))\big)\big), \big\{s(\psi(\theta))\big\}(\varphi)\big) &= \Big(\varphi(\psi(\theta)), \Big\{d\psi\big|_\theta \frac{\partial}{\partial\theta}\Big\}(\varphi)\Big) \\ &= \Big(\theta, \frac{\partial}{\partial\theta}(\varphi \circ \psi)\Big) = \Big(\theta, \frac{\partial}{\partial\theta}(\mathrm{id})\Big) = (\theta, 1).\end{aligned}$$

Thus, the section s is smooth and never zero.

Remark: If we view S^1 as the circle of radius 1 in $\mathbb{R}^2$, $s(m)$ is the unit vector tangent to S^1 at m and pointing counterclockwise.

Problem 4 (5pts)

Suppose that $f\colon X \longrightarrow M$ is a smooth map and $\pi\colon V \longrightarrow M$ is a smooth vector bundle. The pullback of V by f, $\pi_1\colon f^*V \longrightarrow X$, is the vector bundle defined by taking

$$f^*V = \big\{(x,v)\in X\times V : f(x)=\pi(v)\big\} \subset X\times V.$$

In particular, f^*V is supposed to be a smooth manifold. Show that f^*V is in fact a smooth submanifold of $X\times V$.

Apply PS2-5 with $(Y,g)=(V,\pi)$. The condition on the differentials there holds because

$$\mathrm{d}_v\pi\colon T_vV \longrightarrow T_{\pi(v)}V$$

is surjective for all $v\in V$, since on a trivialization of V the map π is the projection on the first component.

Problem 5 (10pts)

Show that the tautological line bundle $\gamma_n \longrightarrow \mathbb{C}P^n$ is indeed a complex line bundle (describe its trivializations). What is its transition data? Why is it non-trivial for $n\geq 1$? (not isomorphic to $\mathbb{C}P^n\times\mathbb{C}\longrightarrow\mathbb{C}P^n$ as line bundle over $\mathbb{C}P^n$)

(a) The topological space γ_n is Hausdorff and second-countable, being a subspace of such a space. The vector space structures in the fibers of the projection map

$$\pi\colon \gamma_n \longrightarrow \mathbb{C}P^n, \qquad \pi(\ell,v)=\ell,$$

are induced from the vector space structures on the fibers of

$$\pi_1\colon \mathbb{C}P^n\times\mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n.$$

Below we construct a differentiable structure on γ_n along with trivializations of γ_n over each of the open sets

$$\mathcal{U}_i = \big\{[X_0,\ldots,X_n]\in\mathbb{C}P^n : X_i\neq 0\big\} \tag{1}$$

with $i=0,1,\ldots,n$.

For each $i=0,1,\ldots,n$, define

$$h_i\colon \gamma_n\big|_{\mathcal{U}_i}\equiv\pi^{-1}(\mathcal{U}_i) \longrightarrow \mathcal{U}_i\times\mathbb{C} \qquad \text{by} \qquad h_i\big(\ell,(c_0,\ldots,c_n)\big) = \big(\ell,c_i\big).$$

This map is continuous, being a projection map. The inverse map, which is given by

$$h_i^{-1}\big([X_0,\ldots,X_n],c\big) = \big([X_0,\ldots,X_n],(cX_0/X_i,\ldots,cX_n/X_i)\big)\in\gamma_n,$$

is also continuous; it is well-defined because $X_i\neq 0$ on $\mathcal{U}_i$. Thus, h_i is a homeomorphism. Furthermore, $\pi_1\circ h_i=\pi|_{\gamma_n|\mathcal{U}_i}$ and the restriction of h_i to each fiber of π is a vector-space isomorphism. If $i,j=0,1,\ldots,n$, the corresponding overlap map

$$h_i\circ h_j^{-1}\colon (\mathcal{U}_i\cap\mathcal{U}_j)\times\mathbb{C} \longrightarrow (\mathcal{U}_i\cap\mathcal{U}_j)\times\mathbb{C}$$

is given by

$$\begin{aligned} h_i \circ h_j^{-1}\big([X_0,\ldots,X_n],c\big) &= h_i\big([X_0,\ldots,X_n],(cX_0/X_j,\ldots,cX_n/X_j)\big) \\ &= \big([X_0,\ldots,X_n],cX_i/X_j\big) = \big([X_0,\ldots,X_n],(X_i/X_j)c\big). \end{aligned} \tag{2}$$

Since all overlap maps are smooth, the collection $\{(\gamma_n|_{\mathcal{U}_i},h_i)\}_{i=0,1,\ldots,n}$ induces a differentiable structure on γ_n such that the projection map π is smooth, as this is the case on the trivializations over $\mathcal{U}_i$. We conclude that

$$\pi\colon \gamma_n \longrightarrow \mathbb{C}P^n$$

is a smooth complex line bundle, with trivializations $\{(\mathcal{U}_i,h_i)\}_{i=0,1,\ldots,n}$. By (2), the corresponding overlap maps are given by

$$h_i \circ h_j^{-1}\big(\ell,c\big) = \big(\ell, g_{ij}([X_0,\ldots,X_n])c\big),$$

where the transition map

$$g_{ij}\colon \mathcal{U}_i\cap\mathcal{U}_j \longrightarrow \mathrm{GL}_1\mathbb{C} = \mathbb{C}^*, \qquad g_{ij}\big([X_0,\ldots,X_n]\big) = X_i/X_j.$$

(b) In order to show that $\gamma_n \longrightarrow \mathbb{C}P^n$ is not isomorphic to the trivial line bundle over $\mathbb{C}P^n$, it is sufficient to show that the complements of the zero sections in the total spaces of the two line bundles are not homotopy-equivalent. In the case of the trivial line bundle, the complement is $\mathbb{C}P^n\times\mathbb{C}^*$; it is homotopy-equivalent to $\mathbb{C}P^n\times S^1$. Since

$$\pi_1\big(\mathbb{C}P^n\times S^1\big) = \pi_1(\mathbb{C}P^n)\times\pi_1(S^1) = \pi_1(\mathbb{C}P^n)\times\mathbb{Z},$$

$\mathbb{C}P^n\times S^1$ is not simply connected (you can in fact show that $\pi_1(\mathbb{C}P^n)=0$, but this is irrelevant here). On the other hand,

$$\gamma_n - s_0(\mathbb{C}P^n) = \big\{(\ell,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1}\colon\ v\in\ell-0\big\}$$

is homotopy-equivalent to the sphere (circle) bundle of γ_n,

$$S(\gamma_n) = \big\{(\ell,v)\in\mathbb{C}P^n\times\mathbb{C}^{n+1}\colon\ v\in\ell,\ |v|=1\big\} = \big\{(\ell,v)\in\mathbb{C}P^n\times S^{2n+1}\colon\ \ell=q(v)\big\},$$

where $q\colon S^{2n+1}\longrightarrow\mathbb{C}P^n$ is the restriction of the quotient projection map $\mathbb{C}^{n+1}-0\longrightarrow\mathbb{C}P^n$. Since $S(\gamma_n)$ is a compact space, S^{2n+1} is Hausdorff, and the projection $\pi_2\colon S(\gamma_n)\longrightarrow S^{2n+1}$ is a continuous bijection, it follows that π_2 is a homeomorphism. Thus, $\gamma_n - s_0(\mathbb{C}P^n)$ is simply connected (since S^{2n+1} is for $n\geq1$).

Problem 6 (10pts)

Suppose $k<n$. Show that the map

$$\iota\colon \mathbb{C}P^k \longrightarrow \mathbb{C}P^n, \qquad [X_0,\ldots,X_k] \longrightarrow [X_0,\ldots,X_k,\underbrace{0,\ldots,0}_{n-k}],$$

is a complex embedding (i.e. a smooth embedding that induces holomorphic maps between the charts that determine the complex structures on $\mathbb{C}P^k$ and $\mathbb{C}P^n$). Show that the normal bundle to the immersion, $\mathcal{N}_\iota$, is isomorphic to

$$(n-k)\gamma_k^* \equiv \underbrace{\gamma_k^* \oplus \ldots \oplus \gamma_k^*}_{n-k},$$

where $\gamma_k \longrightarrow \mathbb{C}P^k$ is the tautological line bundle (isomorphic as complex line bundles).

For each $i=0,1,\ldots,n$ and $j=0,1,\ldots,k$, let

$$\mathcal{U}_i = \big\{[X_0,\ldots,X_n]\in\mathbb{C}P^n\colon X_i\neq 0\big\}, \qquad \mathcal{U}'_j = \big\{[X_0,\ldots,X_k]\in\mathbb{C}P^k\colon X_j\neq 0\big\},$$
$$\varphi_i\colon\mathcal{U}_i \longrightarrow \mathbb{C}^n, \quad \varphi_i\big([X_0,\ldots,X_n]\big) = \big(X_0/X_i,\ldots,X_{i-1}/X_i,X_{i+1}/X_i,\ldots,X_n/X_i\big),$$
$$\varphi'_j\colon\mathcal{U}'_j \longrightarrow \mathbb{C}^k, \quad \varphi'_j\big([X_0,\ldots,X_k]\big) = \big(X_0/X_j,\ldots,X_{j-1}/X_j,X_{j+1}/X_j,\ldots,X_k/X_j\big).$$

The collections

$$\mathcal{F}_0 \equiv \big\{(\mathcal{U}_i,\varphi_i)\big\}_{i=0,1,\ldots,n} \qquad \text{and} \qquad \mathcal{F}'_0 \equiv \big\{(\mathcal{U}'_j,\varphi'_j)\big\}_{j=0,1,\ldots,k}$$

of charts determine smooth and complex structures on $\mathbb{C}P^n$ and $\mathbb{C}P^k$; see Problem 3 on PS1. We will show that the maps on the charts induced by ι,

$$\varphi_i\circ\iota\circ\varphi_j'^{-1}\colon \varphi'_j\big(\mathcal{U}'_j\cap\iota^{-1}(\mathcal{U}_i)\big) \longrightarrow \varphi_i(\mathcal{U}_i)\subset\mathbb{C}^n,$$

are smooth embeddings that are holomorphic. Since $\iota(\mathcal{U}'_j)\subset\mathcal{U}_j$, it is sufficient to consider the case $i=j$. In this case,

$$\varphi_j\circ\iota\circ\varphi_j'^{-1}\colon\mathbb{C}^k\longrightarrow\mathbb{C}^n, \qquad (z_1,\ldots,z_k)\longrightarrow(z_1,\ldots,z_k,0,\ldots,0).$$

Thus, $\varphi_j\circ\iota\circ\varphi_j'^{-1}$ is the holomorphic embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ as $\mathbb{C}^k\times 0$.

It is immediate that the map ι is injective. Since the maps $\varphi_j\circ\iota\circ\varphi_j'^{-1}$ are holomorphic embeddings, it follows that so is ι. We can thus identify $\mathbb{C}P^k$ with its image in $\mathbb{C}P^n$ under ι. Then,

$$\mathcal{N}_\iota = T\mathbb{C}P^n|_{\mathbb{C}P^k}/T\mathbb{C}P^k.$$

We will show that $\mathcal{N}_\iota$ is isomorphic to $(n-k)\gamma_k^*$ as complex vector bundles over $\mathbb{C}P^k$ in four different ways.

(i: *use exact sequence*) We begin by showing that there exists a short exact sequence of vector bundles

$$0\longrightarrow\mathbb{C}P^n\times\mathbb{C}\stackrel{f}{\longrightarrow}(n+1)\gamma_n^*\stackrel{h}{\longrightarrow}T\mathbb{C}P^n\longrightarrow 0. \tag{3}$$

First, we construct the bundle map $f=(f_0,\ldots,f_n)$. If $\ell\in\mathbb{C}P^n$ and $\lambda\in\mathbb{C}$, we define

$$f_i(\ell,\lambda)\in\gamma_n^* \qquad \text{by} \qquad \big\{f_i(\ell,\lambda)\big\}(c_0,\ldots,c_n) = \lambda c_i \quad \forall\,(c_0,\ldots,c_n)\in\gamma_n|_\ell. \tag{4}$$

It is immediate that the map f_i is linear and f is injective, i.e. the sequence (3) is exact at the first (nonzero) position. Each map f_i is smooth because the map

$$\mathbb{C}P^n\times\mathbb{C}\oplus\gamma_n = \mathbb{C}\times\gamma_n\longrightarrow\mathbb{C}, \qquad (\lambda,c)\longrightarrow\big\{f_i(\ell,\lambda)\big\}(c) = \lambda c_i,$$

is smooth (it is the restriction of a smooth map on $\mathbb{C}\times\mathbb{C}P^n\times\mathbb{C}^{n+1}$ to the submanifold $\mathbb{C}\times\gamma_n$).

In order to construct the bundle homomorphism h, it is convenient to introduce the functions

$$z_{i,j}=\frac{X_j}{X_i}, \qquad j\in\{0,\ldots,n\},$$

on $\mathcal{U}_i$. Then, the coordinates of φ_i are $z_{i,j}$ with $j\neq i$ and

$$(\mathrm{id},\mathbf{z}_i)\equiv\big(\mathrm{id},z_{i,0},\ldots,z_{i,n}\big)\colon\mathcal{U}_i\longrightarrow\gamma_n\big|_{\mathcal{U}_i}\subset\mathcal{U}_i\times\mathbb{C}^{n+1} \tag{5}$$

is a bundle section of γ_n over $\mathcal{U}_i$. We can thus define

$$h\colon(n{+}1)\gamma^*\longrightarrow T\mathbb{C}P^n \qquad \text{by}$$

$$(p_0,\ldots,p_n)\longrightarrow\sum_{j\neq i}\big(p_j(\ell,\mathbf{z}_i(\ell))-z_{i,j}p_i(\ell,\mathbf{z}_i(\ell))\big)\frac{\partial}{\partial z_{i,j}} \qquad \forall\, p_l\in\gamma_\ell^*,\ \ell\in\mathcal{U}_i. \tag{6}$$

This is a smooth map over $\mathcal{U}_i$ since the coefficients of the basis vectors in (6) are smooth functions on $\mathcal{U}_i$ whenever each p_i is a smooth section of γ_n^*, by definition of the smooth structure in γ_n^* and because the section of γ_n given by (5) is smooth. Suppose $i'\neq i$. Then,

$$z_{i',j'}=z_{i,i'}^{-1}z_{i,j'}\Longrightarrow\frac{\partial}{\partial z_{i,j}}=\sum_{j'\neq i'}\frac{\partial z_{i',j'}}{\partial z_{i,j}}\frac{\partial}{\partial z_{i',j'}}=\begin{cases}z_{i,i'}^{-1}\frac{\partial}{\partial z_{i',j}}, & \text{if } j\neq i';\\ -z_{i,i'}^{-2}\Big(\frac{\partial}{\partial z_{i',i}}+\sum\limits_{j'\neq i,i'}z_{i,j'}\frac{\partial}{\partial z_{i',j'}}\Big), & \text{if } j=i';\end{cases} \tag{7}$$

see Warner 1.20(c). Since each p_l is a linear functional, for $j\neq i,i'$ the j-th summand in (6) can be written as

$$\begin{aligned}\big(z_{i',i}^{-1}p_j(\ell,\mathbf{z}_{i'}(\ell))-z_{i',i}^{-2}z_{i',j}p_i(\ell,\mathbf{z}_{i'}(\ell))\big)z_{i,i'}^{-1}\frac{\partial}{\partial z_{i',j}}&\\ =\big(p_j(\ell,\mathbf{z}_{i'}(\ell))-z_{i,j}p_i(\mathbf{z}_{i'}(\ell))\big)\frac{\partial}{\partial z_{i',j}}&.\end{aligned} \tag{8}$$

The remaining, $j=i'$, summand in (6) is equal to

$$\begin{aligned}\big(z_{i',i}^{-1}p_{i'}(\ell,\mathbf{z}_{i'}(\ell))-z_{i',i}^{-2}p_i(\ell,\mathbf{z}_{i'}(\ell))\big)(-z_{i,i'}^{-2})\Big(\frac{\partial}{\partial z_{i',i}}+\sum_{j\neq i,i'}z_{i,j}\frac{\partial}{\partial z_{i',j}}\Big)&\\ =\big(p_i(\ell,\mathbf{z}_{i'}(\ell))-z_{i',i}p_{i'}(\ell,\mathbf{z}_{i'}(\ell))\big)\Big(\frac{\partial}{\partial z_{i',i}}+\sum_{j\neq i,i'}z_{i,j}\frac{\partial}{\partial z_{i',j}}\Big)&.\end{aligned} \tag{9}$$

Since $z_{i',i}z_{i,j}=z_{i',j}$, collecting similar terms in (8) and (9), we obtain equation (6) with i replaced by i'. Thus, h is a bundle homomorphism defined everywhere over $\mathbb{C}P^n$. It is immediate from (6) that this homomorphism is surjective and its composition with f is zero. Since the kernel of h must be one-dimensional, it must then equal the image of f. Thus, the sequence (3) of vector bundles is indeed exact.

If $k\leq n$, $\gamma_n^*|_{\mathbb{C}P^k}=\gamma_k^*$ under the above embedding $\iota\colon\mathbb{C}P^k\longrightarrow\mathbb{C}P^n$. Let

$$T\colon(k{+}1)\gamma_k^*\longrightarrow(n{+}1)\gamma_k^*$$

be the bundle homomorphism over $\mathbb{C}P^k$ including the domain as the first $k+1$ components of the image. By (4) and (6), the top two squares in the diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}P^k \times \mathbb{C} & \xrightarrow{f} & (k+1)\gamma_k^* & \xrightarrow{h} & T\mathbb{C}P^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle \mathrm{d}\iota} & & \\
0 & \longrightarrow & \mathbb{C}P^k \times \mathbb{C} & \xrightarrow{f} & (n+1)\gamma_k^* & \xrightarrow{h} & T\mathbb{C}P^n\big|_{\mathbb{C}P^k} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & (n-k)\gamma_k^* & \longrightarrow & \mathcal{N}_\iota & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

then commute; the rows and columns in this diagram are short exact sequences. By the exactness of the last row in this diagram, $\mathcal{N}_\iota \approx (n-k)\gamma_k^*$.

(ii: *construct a vector bundle isomorphism h between the two bundles)* First, for each $j=k+1,\ldots,n$, we define a vector bundle homomorphism

$$\tilde{h}_j : T\mathbb{C}P^n|_{\mathbb{C}P^k} \longrightarrow \gamma_k^*$$

as follows. Suppose $\ell \in \mathcal{U}_i'$, with $i=0,1,\ldots,k$, and $v \in T_\ell \mathbb{C}P^n$. Then, v acts on smooth functions defined on $\mathcal{U}_i$ (on complex-valued functions by linear extension). Thus, we can define

$$\tilde{h}_j(v) \in \gamma_k^*|_\ell \qquad \text{by} \qquad \big\{\tilde{h}_j(v)\big\}(\ell, c_0, \ldots, c_k) = c_i \cdot v(z_{i,j}) \in \mathbb{C}.$$

where $z_{i,j}=X_j/X_i$ as in (i) above. The maps

$$\tilde{h}_j(v) : \gamma_k|_\ell \longrightarrow \mathbb{C} \qquad \text{and} \qquad T_\ell \mathbb{C}P^n|_\ell \longrightarrow \gamma_k^*|_\ell, \quad v \longrightarrow \tilde{h}_j(v),$$

are linear (over $\mathbb{C}$), since for all $\lambda \in \mathbb{C}$

$$\big\{\tilde{h}_j(v)\big\}\big(\lambda \cdot (\ell, c_0, \ldots, c_k)\big) = \big\{\tilde{h}_j(v)\big\}(\ell, \lambda c_0, \ldots, \lambda c_k) = \lambda c_i \cdot v(z_{i,j});$$
$$\big\{\tilde{h}_j(\lambda v)\big\}(\ell, c_0, \ldots, c_k) = c_i \cdot \{\lambda v\}(z_{i,j}) = c_i \cdot \lambda \cdot v(z_{i,j}).$$

If $\ell \in \mathcal{U}_i' \cap \mathcal{U}_{i'}'$ and $(c_0, \ldots, c_k) \in \ell \subset \mathbb{C}^{k+1}$, then $c_{i'} = z_{i,i'}(\ell) c_i$, $z_{i,j} = z_{i,i'} z_{i',j}$, and

$$c_i \cdot v(z_{i,j}) = c_i \cdot v(z_{i,i'} z_{i',j}) = c_i \cdot \big(z_{i,i'}(\ell) \cdot v(z_{i',j}) + z_{i',j}(\ell) \cdot v(z_{i,i'})\big) = c_{i'} \cdot v(z_{i',j}),$$

since v is a derivation with respect to the evaluation at ℓ and $z_{i',j}(\ell)=0$ for all $\ell \in U_{i'}' \subset \mathbb{C}P^k$ and $j>k$. Thus, the element $\tilde{h}_j(v) \in \gamma_k^*|_\ell$ does not depend on i and the bundle homomorphism

$$\tilde{h}_j : T\mathbb{C}P^n|_{\mathbb{C}P^k} \longrightarrow \gamma_k^*$$

is well-defined over the entire space $\mathbb{C}P^k$. If $i=0,1,\ldots,k$ and $l\neq 0,1,\ldots,n$ with $j\neq i$,

$$\big\{\tilde{h}_j(\partial/\partial z_{i,l})_\ell\big\}(c_0,\ldots,c_k) = c_i\cdot\frac{\partial}{\partial z_{i,l}}\bigg|_\ell (z_{i,j}) = c_i\delta_{jl} \qquad \forall\,\ell\in\mathcal{U}_i. \tag{10}$$

Thus, $\tilde{h}_j$ takes smooth sections of $T\mathbb{C}P^n|_{\mathcal{U}_i'}$ to smooth sections of γ_k^* a smooth bundle map. Define the map

$$\tilde{h}\colon T\mathbb{C}P^n|_{\mathbb{C}P^k} \longrightarrow (n-k)\gamma_k^* \qquad \text{by} \qquad \tilde{h}(v) = \big(\tilde{h}_{k+1}(v),\ldots,\tilde{h}_n(v)\big).$$

By the above, $\tilde{h}$ is a smooth bundle homomorphism. By (10), $\tilde{h}$ is surjective on every fiber and vanishes on $T\mathbb{C}P^k$ (which is spanned by the first k coordinate vectors). Thus, $\tilde{h}$ induces a vector bundle isomorphism

$$h\colon \mathcal{N}_\iota = T\mathbb{C}P^n|_{\mathbb{C}P^k}/T\mathbb{C}P^k \longrightarrow (n-k)\gamma_k^*.$$

So, the two vector bundles are isomorphic.

(iii: *compare transition data)* We compare the transition data for the two vector bundles, corresponding to trivializations over $\mathcal{U}_i'=\mathbb{C}P^k\cap\mathcal{U}_i$. By Problem 5, the transition data for the line bundle $\gamma_k\longrightarrow\mathbb{C}P^k$ is given by

$$g_{ij}\colon \mathcal{U}_i'\cap\mathcal{U}_j' \longrightarrow \mathrm{GL}_1\mathbb{C}=\mathbb{C}^*, \qquad [X_0,\ldots,X_k]\longrightarrow X_i/X_j\,.$$

Thus, the transition data for the dual line bundle $\gamma_k^*\longrightarrow\mathbb{C}P^k$ is given by

$$(g_{ij}^*)^{-1}\colon \mathcal{U}_i'\cap\mathcal{U}_j' \longrightarrow \mathrm{GL}_1\mathbb{C}=\mathbb{C}^*, \qquad [X_0,\ldots,X_k]\longrightarrow X_j/X_i,$$

and for the vector bundle $(n-k)\gamma_k^*\longrightarrow\mathbb{C}P^k$ by

$$G_{ij}=(g_{ij}^*)^{-1}\mathbb{I}_{n-k}\colon \mathcal{U}_i'\cap\mathcal{U}_j' \longrightarrow \mathrm{GL}_{n-k}\mathbb{C}, \qquad [X_0,\ldots,X_k]\longrightarrow (X_j/X_i)\mathbb{I}_{n-k}, \tag{11}$$

where $\mathbb{I}_{n-k}\in\mathrm{GL}_{n-k}\mathbb{C}$ is the identity matrix; see Section 10 in *Lecture Notes*. By PS1-3(b), the overlap map between the charts $(\mathcal{U}_i,\varphi_i)$ and $(\mathcal{U}_j,\varphi_j)$ on $\mathbb{C}P^n$ is

$$\varphi_i\circ\varphi_j^{-1}\colon \varphi_j(\mathcal{U}_i\cap\mathcal{U}_j)\longrightarrow\varphi_i(\mathcal{U}_i\cap\mathcal{U}_j), \qquad (\varphi_i\circ\varphi_j^{-1})_l = z_{i,l} = z_{j,l}z_{i,j}\begin{cases} z_l/z_i, & \text{if } l>i>j;\\ z_l/z_{i+1}, & \text{if } l>j>i.\end{cases} \tag{12}$$

By the complex analogue of Problem 2, the transition data for the vector bundle $T\mathbb{C}P^n$ is given by

$$h_{ij}\colon \mathcal{U}_i\cap\mathcal{U}_j\longrightarrow\mathrm{GL}_n\mathbb{C}, \qquad h_{ij}(\ell)=\mathcal{J}\big(\varphi_i\circ\varphi_j^{-1}\big)_{\varphi_j(\ell)}.$$

By (12), the (l,m)-entry of h_{ij} with $i,j\leq k$ and $l,m>k$ is

$$\big(h_{ij}([X_0,\ldots,X_n])\big)_{lm} = \delta_{lm}z_{i,j}. \tag{13}$$

The local charts $(\mathcal{U}_i',\varphi_i')$ for $\mathbb{C}P^k$ are are given by

$$\mathcal{U}_i' = \mathcal{U}_i\cap\mathbb{C}P^k \qquad \text{and} \qquad \varphi_i' = \pi\circ\varphi_i|_{\mathcal{U}_i'}\,,$$

where $\pi:\mathbb{C}^n\longrightarrow\mathbb{C}^k$ is the projection on the first k components. Thus, the upper-left $k\times k$ submatrix of $h_{ij}|_{\mathcal{U}_i'\cap\mathcal{U}_j'}$ gives transition data for the subbundle $T\mathbb{C}P^k$ of $T\mathbb{C}P^n|_{\mathbb{C}P^k}$ and the lower-right $(n-k)\times(n-k)$ submatrix of $h_{ij}|_{\mathcal{U}_i'\cap\mathcal{U}_j'}$ gives transition data for the quotient vector bundle

$$\mathcal{N}_\iota = T\mathbb{C}P^n|_{\mathbb{C}P^k}/T\mathbb{C}P^k;$$

see Section 10 in *Lecture Notes*. By (13), the data for $\mathcal{N}_\iota$ is then

$$H_{ij}:\mathcal{U}_i'\cap\mathcal{U}_j'\longrightarrow \mathrm{GL}_{n-k}\mathbb{C}, \qquad \ell\longrightarrow z_{i,j}(\ell)\mathbb{I}_{n-k}.$$

Along with (11), this implies that the bundle $\mathcal{N}_\iota$ and $(n-k)\gamma_k^*$ are isomorphic.

(iv: *identify neighborhoods)* We will construct a biholomorphism

$$f\colon (n-k)\gamma_k^*\longrightarrow W$$

onto a neighborhood W of $\mathbb{C}P^k$ in $\mathbb{C}P^n$ such that

$$f(0_\ell)=\ell \qquad \forall\,\ell\in\mathbb{C}P^k,$$

where $0_\ell\in(n-k)\gamma_k^*|_\ell$ is the zero vector. By the general lemma stated below, the existence of such a diffeomorphism implies that $(n-k)\gamma_k^*$ and $\mathcal{N}_\iota$ are isomorphic as complex vector bundles. Define

$$f\colon (n-k)\gamma_k^*\longrightarrow \bigcup_{i=0}^{i=k}\mathcal{U}_i\subset\mathbb{C}P^n \qquad \text{by}$$

$$f\big([X_0,\ldots,X_k],\alpha_{k+1},\ldots,\alpha_n\big)=\big[X_0,\ldots,X_k,\alpha_{k+1}(X_0,\ldots,X_k),\ldots,\alpha_n(X_0,\ldots,X_k)\big].$$

Since $(X_0,\ldots,X_k)$ is defined up to multiplication by $\mathbb{C}^*$, the map f is well-defined. It is immediate that f is bijective and takes $(n-k)\gamma_k^*|_{\mathcal{U}_i'}$ to $\mathcal{U}_i$. Holomorphic charts on $(n-k)\gamma_k^*|_{\mathcal{U}_i'}$ are induced by the charts on $\gamma_k|_{\mathcal{U}_i'}$ of Problem 5 and are given by

$$\tilde\varphi_i\colon (n-k)\gamma_k^*|_{\mathcal{U}_i}\longrightarrow \mathbb{C}^k\times\mathbb{C}^{n-k},$$

$$\big(\tilde\varphi_i(\ell,\alpha_{k+1},\ldots,\alpha_n)\big)_l=\begin{cases}\big(\varphi_i(\ell)\big)_l, & \text{if } l\le k;\\ \alpha_l(\ell,z_{i,0}(\ell),\ldots,z_{i,n}(\ell)), & \text{if } l>k,\end{cases}$$

where $z_{i,j}=X_j/X_i$ as in (i) above. The map between these charts induced by f is

$$\varphi_i\circ f\circ\tilde\varphi_i^{-1}\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n$$

is the identity and thus biholomorphic.

Lemma *Suppose M is an embedded submanifold of N and $V\longrightarrow M$ is a vector bundle. If there exists a diffeomorphism between neighborhoods W and W' of M in V and in N, respectively,*

$$f\colon W\longrightarrow W' \qquad s.t. \qquad f(p)=p \quad \forall\,p\in M,$$

then V is isomorphic to the normal bundle $\mathcal{N}$ of M in N. If in addition, N is a complex manifold, M is a complex submanifold, $V\longrightarrow M$ is a complex vector bundle, and the linear map

$$\mathrm{d}_pf\colon T_pV/T_pM\longrightarrow T_pN/T_pM$$

is $\mathbb{C}$-linear for all $p\in M$, then V and $\mathcal{N}$ are isomorphic as complex vector bundles.

Remark: Recall from Section 8 in *Lecture Notes* that M can be viewed as the zero section in V.

Proof: Let $V \longrightarrow M$ be a (complex) vector bundle. If $v \in V$, let $\alpha_v : I \longrightarrow V$ be the curve $\alpha_v(t) = tv \in V$. Then, the map

$$V \longrightarrow TV|_M/TM, \qquad v \longrightarrow \alpha_v'(0) + T_{\pi(v)}M,$$

is an isomorphism of (complex) vector bundles, as this is the case in any trivialization of V. On the other hand, if f is a diffeomorphism that maps the submanifold M of V to the submanifold M of N, then the differential

$$\mathrm{d}f|_M : TV|_M \longrightarrow TN|_M$$

is an isomorphism that restricts to the identity on TM. Thus, $\mathrm{d}f|_M$ induces an isomorphism

$$TV|_M/TM \longrightarrow TN|_M/TM = \mathcal{N} \tag{14}$$

of vector bundles over M. If V, TN, and TM are complex bundles and $\mathrm{d}f|_M$ is $\mathbb{C}$-linear (as is the case if f is a holomorphic map between complex manifolds), then the bundle isomorphism between the quotient bundles above is also $\mathbb{C}$-linear. Combining (14) with the first isomorphism, we obtain the lemma.

Problem 7 (10pts)

Let $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ be the top exterior power of the vector bundle $T\mathbb{C}P^n$ taken over $\mathbb{C}$. Show that $\Lambda^n_{\mathbb{C}} T\mathbb{C}P^n$ is isomorphic to the line bundle

$$\gamma_n^{*\otimes(n+1)} \equiv \underbrace{\gamma_n^* \otimes \ldots \otimes \gamma_n^*}_{n+1},$$

where $\gamma_n \longrightarrow \mathbb{C}P^n$ is the tautological line bundle (isomorphic as complex line bundles).

We will show that this is the case in three different ways.

(i: *use exact sequence*) By the short exact sequence (3),

$$\begin{aligned}\gamma_n^{*\otimes(n+1)} &= \Lambda^{n+1}_{\mathbb{C}}\big((n+1)\gamma_n^*\big) = \Lambda^{\mathrm{top}}_{\mathbb{C}}\big((n+1)\gamma_n^*\big) \approx \Lambda^{\mathrm{top}}_{\mathbb{C}}\big(\mathbb{C}P^n \times \mathbb{C}\big) \otimes \Lambda^{\mathrm{top}}_{\mathbb{C}} T\mathbb{C}P^n \\ &= \Lambda^1_{\mathbb{C}}\big(\mathbb{C}P^n \times \mathbb{C}\big) \otimes \Lambda^n_{\mathbb{C}} T\mathbb{C}P^n \approx \Lambda^n_{\mathbb{C}} T\mathbb{C}P^n,\end{aligned}$$

as claimed.

For approaches (ii) and (iii), let

$$\mathcal{U}_i = \big\{[X_0,\ldots,X_n] \in \mathbb{C}P^n : X_i \neq 0\big\}, \qquad \varphi_i, z_{i,j} : \mathcal{U}_i \longrightarrow \mathbb{C}^n,$$
$$z_{i,j}\big([X_0,\ldots,X_n]\big) = \frac{X_j}{X_i}, \qquad \varphi_i(\ell) = \big(z_{i,0}(\ell),\ldots,z_{i,i-1}(\ell),z_{i,i+1}(\ell),\ldots,z_{i,n}(\ell)\big),$$

as before. The collection $\mathcal{F}_0 \equiv \{(\mathcal{U}_i, \varphi_i)\}_{i=0,1,\ldots,n}$ of charts determines smooth and complex structures $\mathbb{C}P^n$.

(ii: *construct a vector bundle isomorphism h between the two bundles)* We begin by constructing an alternating linear map

$$\tilde{h}\colon n\,T\mathbb{C}P^n \longrightarrow \gamma_n^{*\otimes(n+1)}$$

as follows. Suppose $\ell \in \mathcal{U}_i$, with $i=0,1,\ldots,n$, and $v=(v_1,\ldots,v_n) \in nT_\ell\mathbb{C}P^n$. Then, each component v_m of v acts on functions defined on $\mathcal{U}_i$ (on complex functions by linear extension). Thus, we can define

$$\tilde{h}(v) \in \gamma_n^{*\otimes(n+1)}|_\ell \qquad \text{by} \qquad \{\tilde{h}(v)\}(\ell, c_0,\ldots,c_n)^{\otimes(n+1)} = (-1)^i c_i^{n+1}\det\big(A_i(v)\big) \in \mathbb{C},$$
$$\text{where} \qquad \big(A_i(v)\big)_{jm} = \begin{cases} v_m(z_{i,j-1}), & \text{if } j \le i;\\ v_m(z_{i,j}), & \text{if } j > i.\end{cases}$$

The map

$$\tilde{h}(v)\colon \gamma_n^{\otimes(n+1)}|_\ell \longrightarrow \mathbb{C}$$

is linear (over $\mathbb{C}$), since for all $\lambda \in \mathbb{C}$

$$\begin{aligned}\{\tilde{h}(v)\}\big(\lambda\cdot(\ell,c_0,\ldots,c_n)\big)^{\otimes(n+1)} &= \{\tilde{h}(v)\}(\ell,\lambda c_0,\ldots,\lambda c_n)^{\otimes(n+1)}\\ &= (\lambda c_i)^{n+1}\cdot\det\big(A_i(v)\big) = \lambda^{n+1}\cdot\{\tilde{h}(v)\}(\ell,c_0,\ldots,c_n)^{\otimes(n+1)}.\end{aligned}$$

Since the map $v \longrightarrow A_i(v)$ is linear in each component of v (the determinant of a matrix is linear in each column), the map

$$n\,T_\ell\mathbb{C}P^n|_\ell \longrightarrow \gamma_n^{*\otimes(n+1)}|_\ell, \qquad (v_1,\ldots,v_n) \longrightarrow \tilde{h}(v_1,\ldots,v_n),$$

is multilinear. If $\ell \in \mathcal{U}_i \cap \mathcal{U}_{i'}$, then

$$\begin{aligned} v_m(z_{i,i'}) &= v_m(z_{i',i}^{-1}) = -\big(z_{i,i'}(\ell)\big)^2 v_m(z_{i',i}),\\ v_m(z_{i,j}) &= v_m(z_{i,i'}z_{i',j}) = z_{i,i'}(\ell)\cdot v_m(z_{i',j}) + z_{i',j}(\ell)\cdot v_m(z_{i,i'}).\end{aligned}$$

Thus, if $i' < i$, then

$$\begin{aligned} A_i(v)_{(i'+1)m} &= -\big(z_{i,i'}(\ell)\big)^2 A_{i'}(v)_{im};\\ A_i(v)_{jm} &= z_{i,i'}(\ell)\cdot\begin{cases} A_{i'}(v)_{jm}, & \text{if } j\le i' \text{ or } j>i\\ A_{i'}(v)_{(j-1)m}, & \text{if } i'+2 \le j \le i\end{cases} + A_i(v)_{(i'+1)m}\begin{cases} z_{i',j-1}(\ell), & \text{if } j \le i;\\ z_{i',j}(\ell), & \text{if } j>i.\end{cases}\end{aligned} \tag{15}$$

Since adding a multiple of a row (row #$(i'+1)$ in this case) to another row does not change the determinant, the last term in (15) has no effect on $\det(A_i(v))$. Since moving row #i (in the matrix $A_{i'}(v)$) "up" to make it row #$(i'+1)$ and shifting rows #$(i'+1)$ through #$(i-1)$ "down" by 1 (increasing their row number by 1) multiples the determinant by $(-1)^{i-(i'+1)}$, by (15)

$$\det\big(A_i(v)\big) = (-1)^{i+i'}\big(z_{i,i'}(\ell)\big)^{n+1}\det\big(A_{i'}(v)\big).$$

If in addition $(c_0,\ldots,c_n) \in \ell$, then $c_{i'} = z_{i,i'}(\ell)c_i$. Therefore,

$$(-1)^i c_i^{n+1}\det\big(A_i(v)\big) = (-1)^i c_i^{n+1}\cdot(-1)^{i+i'}\big(z_{i,i'}(\ell)\big)^{n+1}\det\big(A_{i'}(v)\big) = (-1)^{i'}c_{i'}^{n+1}\det\big(A_{i'}(v)\big).$$

Thus, the bundle homomorphism

$$\tilde{h}\colon n\,T\mathbb{C}P^n \longrightarrow \gamma_n^{*\otimes(n+1)}$$

is well-defined over the entire space $\mathbb{C}P^n$. If $i=0,1,\ldots,n$ and $\ell\in\mathcal{U}_i$,

$$A_i\bigg(\frac{\partial}{\partial z_{i,0}}\bigg|_\ell,\ldots,\frac{\partial}{\partial z_{i,i-1}}\bigg|_\ell,\frac{\partial}{\partial z_{i,i+1}}\bigg|_\ell,\ldots,\frac{\partial}{\partial z_{i,n}}\bigg|_\ell\bigg)=\det\mathbb{I}=1.$$

Thus,

$$\big\{\tilde{h}\big((\partial/\partial z_{i,0})_\ell,\ldots,(\partial/\partial z_{i,i-1})_\ell,(\partial/\partial z_{i,i+1})_\ell,\ldots(\partial/\partial z_{i,n})_\ell\big)\big\}(\ell,c_0,\ldots,c_n)=(-1)^i c_i^{n+1}. \tag{16}$$

A permutation of the coordinate vectors on LHS above would change RHS by the sign of the permutation; if any two of the inputs on LHS were the same, RHS would be 0. Thus, $\tilde{h}$ takes smooth sections of $nT\mathbb{C}P^n$ to smooth secrions of $\gamma_n^{*\otimes(n+1)}$, and so is smooth. By (16), the restriction of $\tilde{h}$ to every fiber is nonzero and thus surjective (because the range is one-dimensional). Since the determinant is an alternating function of the columns, $\tilde{h}$ is an alternating multi-linear map between vector bundles. It follows that $\tilde{h}$ descends to a surjective bundle homomorphism

$$h\colon \Lambda^n_{\mathbb{C}}T\mathbb{C}P^n \longrightarrow \gamma_n^{*\otimes(n+1)}.$$

Since the domain of h is a line bundle, h must then be a vector-bundle isomorphism. This is precisely the isomorphism of (i).

(iii: *compare transition data)* We compare the transition data for the two vector bundles, corresponding to trivializations over $\mathcal{U}_i$. Using the trivializations $\{h_i\}$ for $\gamma_n\longrightarrow\mathbb{C}P^n$ of Problem 5, the transition data for γ_n is given by

$$g_{ij}=z_{j,i}:\mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathrm{GL}_1\mathbb{C}=\mathbb{C}^*,\qquad [X_0,\ldots,X_n]\longrightarrow X_i/X_j\,.$$

Thus, the transition data for the dual line bundle $\gamma_n^*\longrightarrow\mathbb{C}P^n$ is given by

$$(g_{ij}^*)^{-1}=z_{i,j}:\mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathrm{GL}_1\mathbb{C}=\mathbb{C}^*,\qquad [X_0,\ldots,X_n]\longrightarrow X_j/X_i,$$

and for the line bundle $\gamma_n^{*\otimes(n+1)}\longrightarrow\mathbb{C}P^n$ by

$$\big((g_{ij}^*)^{-1}\big)^{n+1}=z_{i,j}^{n+1}:\mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathbb{C}^*,\qquad [X_0,\ldots,X_n]\longrightarrow (X_j/X_i)^{n+1};$$

see Section 10 in *Lecture Notes*. These transition data correspond to the trivializations $\big((h_i^*)^{-1}\big)^{\otimes(n+1)}$ induced by h_i. However, for the present purposes it is convenient to compose each of these trivialization with multiplication by $(-1)^i$ (in light of (16), these transition maps correspond to the standard transition maps for $T\mathbb{C}P^n$ via the isomorphism of (i) and (ii) above). Then, the transition maps are modified by $(-1)^{i+j}$ and become

$$G_{ij}=(-1)^{i+j}z_{i,j}^{n+1}:\mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathbb{C}^*,\qquad [X_0,\ldots,X_n]\longrightarrow (-1)^{i+j}(X_j/X_i)^{n+1}. \tag{17}$$

By PS1-3(b), if $j<i$ the overlap map between coordinate charts $(\mathcal{U}_i,\varphi_i)$ and $(\mathcal{U}_j,\varphi_j)$ on $\mathbb{C}P^n$ is

$$\varphi_i\circ\varphi_j^{-1}\colon \varphi_j(\mathcal{U}_i\cap\mathcal{U}_j)\longrightarrow\varphi_i(\mathcal{U}_i\cap\mathcal{U}_j), \qquad (\varphi_i\circ\varphi_j^{-1})_l=\begin{cases} z_l/z_i, & \text{if } l\le j \text{ or } l>i;\\ 1/z_i, & \text{if } l=j+1;\\ z_{l-1}/z_i, & \text{if } j+2\le l\le i.\end{cases} \tag{18}$$

By the complex analogue of Problem 3 on PS2, the transition data for the vector bundle $T\mathbb{C}P^n$ is given by

$$h_{ij}\colon \mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathrm{GL}_n\mathbb{C}, \qquad h_{ij}(\ell)=\mathcal{J}\big(\varphi_i\circ\varphi_j^{-1}\big)_{\varphi_j(\ell)}.$$

Since the (complex) rank of the vector bundle $T\mathbb{C}P^n$ is n, the transition data for the line bundle $\Lambda^n_{\mathbb{C}}T\mathbb{C}P^n$ is given by the determinant of the transition data for $T\mathbb{C}P^n$:

$$H_{ij}\colon \mathcal{U}_i\cap\mathcal{U}_j\longrightarrow \mathrm{GL}_1\mathbb{C}=\mathbb{C}^*, \qquad \ell\longrightarrow \det\mathcal{J}\big(\varphi_i\circ\varphi_j^{-1}\big)_{\varphi_j(\ell)};$$

see Section 10 in *Lecture Notes*. By (18), the only entry in the $(j+1)$-st row of $\mathcal{J}\big(\varphi_i\circ\varphi_j^{-1}\big)_{\varphi_j(\ell)}$ is in the i-th column and equals $-1/z_i^2$. Once the $(j+1)$-st row and i-th column are crossed out, we are left with the matrix $(1/z_i)\mathbb{I}_{n-1}$. Thus,

$$\begin{aligned} H_{ij}(\ell)=\det\big(\mathcal{J}\big(\varphi_i\circ\varphi_j^{-1}\big)_{\varphi_j(\ell)}\big) &= (-1)^{i+j+1}(-1/z_i^2)(1/z_i)^{n-1}\\ &= (-1)^{i+j}(1/z_{j,i})^{n+1}=(-1)^{i+j}z_{i,j}^{n+1}. \end{aligned}$$

Since $H_{ij}=H_{ji}^{-1}$, this formula applies to $i<j$ as well. Along with (17), the above expression implies that the line bundles $\Lambda^n_{\mathbb{C}}T\mathbb{C}P^n$ and $\gamma_n^{*\otimes(n+1)}$ are isomorphic.