MAT 531: Topology&Geometry, II
Spring 2011

Problem Set 10
Due on Thursday, 5/5, in class

1. Chapter 6, #6 (p252)

2. Suppose M is a compact Riemannian manifold. If M is not orientable, the Hodge $*$-operator is defined only up to sign. However, as the definition of δ in Section 6.1 involves two $*$'s, the linear operator δ is well-defined. Show that δ is the adjoint of d whether or not M is oriented.
Hint: It is sufficient to assume that M is connected and non-orientable. Deduce the claim from the orientable case proved in the book. The proof is a couple of lines.

3. Suppose M is a compact connected *non*-orientable n-manifold. Show that $H^n_{\mathrm{de\,R}}(M)=0$.
Remark: In fact, if M is a *non*-compact connected n-manifold, orientable or not, $H^n_{\mathrm{de\,R}}(M)=0$.
Hint: Deduce from the orientable case. The proof is a couple of lines.

4. Chapter 6, #16 (p254); you may want to do (d) before (b) and (c).
Remark: In (e)-(h), it suffices to write up only proofs of claims omitted in the statement.

5. Suppose M and N are smooth compact Riemannian manifolds. A differential form γ on $M\times N$ is called `decomposable` if

$$\gamma = \pi_M^*\alpha \wedge \pi_N^*\beta \qquad \text{for some} \quad \alpha \in E^*(M),\ \beta \in E^*(N).$$

(a) Show that

$$\Delta_{M\times N}\big(\pi_M^*\alpha\wedge\pi_N^*\beta\big) = \pi_M^*\Delta_M\alpha \wedge \pi_N^*\beta + \pi_M^*\alpha \wedge \pi_N^*\Delta_N\beta \qquad \forall\ \alpha \in E^*(M),\ \beta \in E^*(N).$$

(b) Show that the $\mathbb{R}$-span of the decomposable forms on $M\times N$ is L^2-dense in $E^*(M\times N)$.
(c) Conclude that

$$\mathcal{H}^*(M\times N) \approx \mathcal{H}^*(M)\otimes\mathcal{H}^*(N), \qquad \pi_M^*\alpha\wedge\pi_N^*\beta \longleftrightarrow \alpha\otimes\beta.$$

(d) Conclude that

$$H^p_{\mathrm{de\,R}}(M\times N) \approx \bigoplus_{q+r=p} H^q_{\mathrm{de\,R}}(M)\otimes H^r_{\mathrm{de\,R}}(N), \qquad \pi_M^*\alpha\wedge\pi_N^*\beta \longleftrightarrow \alpha\otimes\beta.$$

This is the *Kunneth Formula* for compact manifolds.