MAT 531: Topology&Geometry, II
Spring 2006

Final Exam

Instructions

- Give concise proofs, quoting established facts as appropriate; no treatises.
- The problems are worth 20 points each, but are not necessarily of the same difficulty. Parts of a problem may not carry equal weight.
- Your final exam score will be based on your work on 5 problems: 2 (highest-scoring) from Part I, 2 from Part II, and 1 from Part III. The points earned on the bonus problem will be added to your final-exam score, whether or not the total exceeds 100. *NO extra credit will be awarded for solutions to any of the other three problems.*
- Please start each problem on a new sheet of paper. When you are finished, please assemble your solutions in order and attach to the cover sheet. Do *not* attach this problem sheet.

Part I (*choose 2 problems from 1,2, and 3*)

1. Suppose M is a compact manifold and α is a nowhere-zero closed one-form. Show that

$$[\alpha] \neq 0 \in H^1_{\mathrm{deR}}(M).$$

2. Let Y and Z be the vector fields on $\mathbb{R}^3$ given by

$$Y(x_1,x_2,x_3) = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_3} \qquad \text{and} \qquad Z(x_1,x_2,x_3) = a\frac{\partial}{\partial x_1} + b\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3},$$

where a, b, and c are constants.
(a) Compute the Lie bracket $[Y,Z]$.
(b) Describe the flows φ_s of Y and ψ_t of Z.
(c) For what constants a, b, and c do these two flows commute (i.e. $\varphi_s \circ \psi_t = \psi_t \circ \varphi_s$)?

3. Describe explicitly trivializations and transition data for the vector bundle $TS^2 \longrightarrow S^2$.

Part II (*choose 2 problems from 4,5, and 6*)

4. Compute the singular homology of a point *directly from the definition* (over $\mathbb{R}$ if you like).

5. (a) Show that $\mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4$ is not orientable.
(b) Describe the orientable double cover of $\mathbb{R}P^2 \times \mathbb{R}P^3 \times \mathbb{R}P^4$.

6. (a) Determine the de Rham cohomology of $\mathbb{R}P^2$.
(b) Determine the de Rham cohomology of $\mathbb{R}P^2 \# \mathbb{R}P^2$.

Part III (*choose 1 problem from 7 and 8*)

7. (a) Show that the normal bundle of S^n in $\mathbb{R}^{n+1}$ is trivial.
(b) Show that $S^3 \times S^4$ can be embedded into $\mathbb{R}^8$.
(c) Let $n_1, \ldots, n_k$ be nonnegative integers and N their sum. Show that

$$S^{n_1} \times S^{n_2} \times \ldots \times S^{n_k}$$

can be embedded into $\mathbb{R}^{N+1}$.

8. Let M be a smooth Riemannian manifold.
(a) What is the symbol of the differential,

$$d_p \colon E^p(M) \longrightarrow E^{p+1}(M)?$$

Under what conditions is this operator elliptic?
(b) What is the symbol of the (formal) adjoint of the differential,

$$\delta_p \colon E^p(M) \longrightarrow E^{p-1}(M)?$$

Under what conditions is this operator elliptic?
(c) What is the symbol of the operator

$$d+\delta \colon E^*(M) \longrightarrow E^*(M)?$$

Under what conditions is this operator elliptic?
Note: $E^*(M)$ denotes the direct sum of all spaces $E^p(M)$.

Bonus Problem

Determine the cohomology ring of $\mathbb{C}P^2$.
Note: no credit will be awarded for answer only (as opposed to justification).