

MAT 531: Topology & Geometry, II

Spring 2006

Midterm Solutions

Problem 1 (15 pts)

Suppose M is a smooth manifold and X and Y are smooth vector fields on M . Show directly from definitions that

$$[X, Y] = -[Y, X].$$

(You can assume that $[X, Y]$ is whatever object it is supposed to be, but do state what you are taking it to be).

By definition, the Lie bracket $[X, Y]$ of two vector fields X and Y is another vector field on M , i.e. an element of $\Gamma(M; TM)$. In particular,

$$[X, Y], [Y, X]: C^\infty(M) \longrightarrow C^\infty(M)$$

are linear maps. They are given by

$$\begin{aligned} [X, Y]f &= X(Yf) - Y(Xf) \quad \forall f \in C^\infty(M) && \implies \\ [Y, X]f &= Y(Xf) - X(Yf) = -(X(Yf) - Y(Xf)) = -([X, Y]f) \equiv \{-[X, Y]\}f \quad \forall f \in C^\infty(M) \\ &\implies [X, Y] = -[Y, X] \in \Gamma(M; TM). \end{aligned}$$

Problem 2 (20 pts)

Show that the topological subspace

$$\{(x, y) \in \mathbb{R}^2 : x^3 + xy + y^3 = 1\}$$

of \mathbb{R}^2 is a smooth curve (i.e. admits a natural structure of smooth 1-manifold with respect to which it is a submanifold of \mathbb{R}^2).

Define

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R} \quad \text{by} \quad f(x, y) = x^3 + xy + y^3.$$

Then, f is a smooth map and

$$P \equiv \{(x, y) \in \mathbb{R}^2 : x^3 + xy + y^3 = 1\} = f^{-1}(1).$$

We show below that 1 is a regular value for f . By the Implicit Function Theorem, P is then a smooth submanifold of \mathbb{R}^2 and

$$\dim P = \dim \mathbb{R}^2 - \dim \mathbb{R} = 1,$$

as needed.

We need to show that $df|_{(x,y)}$ is surjective for all $(x,y) \in f^{-1}(1)$. Since the target space for f is \mathbb{R} , we can view $df|_{(x,y)}$ as a linear map into \mathbb{R} (instead of $T_{f(x,y)}\mathbb{R}$). It is given by

$$df|_{(x,y)} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (3x^2 + y)dx + (x + 3y^2)dy.$$

Since dx and dy are linearly independent in $T_{(x,y)}^*\mathbb{R}^2$,

$$\begin{aligned} df|_{(x,y)} = 0 &\implies \begin{cases} 3x^2 + y = 0 \\ x + 3y^2 = 0 \end{cases} \implies \begin{cases} y = -3x^2 \\ x + 27x^4 = 0 \end{cases} \implies \\ (x,y) = (0,0) \text{ or } (x,y) = (-1/3, -1/3) &\implies f(x,y) \in \{0, 1/27\} \implies (x,y) \notin f^{-1}(1). \end{aligned}$$

We conclude that 1 is a regular value for f .

Problem 3 (5+15 pts)

Let X be a non-vanishing vector field on \mathbb{R}^3 , written in coordinates as

$$X(x,y,z) = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} + h \frac{\partial}{\partial z} \quad \text{for some } f, g, h \in C^\infty(\mathbb{R}^3).$$

- (a) Find a one-form α on \mathbb{R}^3 so that at each point of \mathbb{R}^3 the kernel of α is orthogonal to X , with respect to the standard inner-product on \mathbb{R}^3 .
 (b) Find a necessary and sufficient condition on X so that for every point $p \in \mathbb{R}^3$ there exists a surface $S \subset \mathbb{R}^3$ passing through p which is everywhere orthogonal to X (i.e. S is a smooth two-dimensional submanifold of \mathbb{R}^3 and $T_m S \subset T_m \mathbb{R}^3$ is orthogonal to $X(m)$ for all $m \in S$).

(a) Every element of $T_{(x,y,z)}\mathbb{R}^3$ can be written as

$$v = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} \quad \text{for some } a, b, c \in \mathbb{R}$$

and

$$\begin{aligned} \langle X(x,y,z), v \rangle &= f(x,y,z)a + g(x,y,z)b + h(x,y,z)c \\ &= \{f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz\}v = \alpha|_{(x,y,z)}v, \end{aligned}$$

where α is the one-form on \mathbb{R}^3 defined by

$$\alpha|_{(x,y,z)} = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz.$$

For each $(x,y,z) \in \mathbb{R}^3$, the kernel of α is the subspace of $T_{(x,y,z)}\mathbb{R}^3 \approx \mathbb{R}^3$ orthogonal to $X(x,y,z)$.

(b) Let

$$\mathcal{J}(\mathbb{R}\alpha) = \mathcal{J}(\alpha) = \{\alpha \wedge \beta : \beta \in E^*(\mathbb{R}^3)\} \subset E^*(\mathbb{R}^3)$$

be the ideal generated by α . By part (a), we need to find a necessary and sufficient condition on α so that for every point $p \in \mathbb{R}^3$ there exists a surface $S \subset \mathbb{R}^3$ passing through p such that

$$T_{(x,y,z)}S = \ker \alpha|_{(x,y,z)} \subset T_{(x,y,z)}\mathbb{R}^3.$$

Since X is a nowhere-vanishing vector field on \mathbb{R}^3 , α is a nowhere-vanishing one-form on \mathbb{R}^3 and $\mathbb{R}\alpha$ is a vector subbundle of $T^*\mathbb{R}^3$ of rank one. Thus, by the second version of Frobenius Theorem the necessary and sufficient condition is that the ideal $\mathcal{J}(\alpha)$ be differential, i.e. closed under d so that

$$d\gamma \in \mathcal{J}(\alpha) \quad \forall \gamma \in \mathcal{J}(\alpha).$$

Since \mathbb{R}^3 is a three-dimensional manifold, by Problem 5 on PS5 this is the case if and only if

$$\alpha \wedge d\alpha = 0.$$

Since $\alpha = f dx + g dy + h dz$,

$$\begin{aligned} d\alpha &= df \wedge dx + dg \wedge dy + dh \wedge dz \\ &= \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \right) \wedge dx + \left(\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz \right) \wedge dy + \left(\frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial z} dz \right) \wedge dz \\ &= f_y dy \wedge dx + f_z dz \wedge dx + g_x dx \wedge dy + g_z dz \wedge dy + h_x dx \wedge dz + h_y dy \wedge dz \\ &= (g_x - f_y) dx \wedge dy + (h_x - f_z) dx \wedge dz + (h_y - g_z) dy \wedge dz \\ &\implies \alpha \wedge d\alpha = h(g_x - f_y) dz \wedge dx \wedge dy + g(h_x - f_z) dy \wedge dx \wedge dz + f(h_y - g_z) dx \wedge dy \wedge dz \\ &\quad = (f(h_y - g_z) - g(h_x - f_z) + h(g_x - f_y)) dx \wedge dy \wedge dz. \end{aligned}$$

Thus, the necessary and sufficient condition on X is

$$\langle X, \vec{\nabla} \times X \rangle \equiv \det \begin{pmatrix} f & g & h \\ \partial_x & \partial_y & \partial_z \\ f & g & h \end{pmatrix} \equiv f(h_y - g_z) - g(h_x - f_z) + h(g_x - f_y) = 0.$$

Problem 4 (20 pts)

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth structure and orientation. Find

$$\int_{S^2} (x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2).$$

With its standard orientation, S^2 is the oriented boundary of the unit ball

$$B^3 \equiv \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

about the origin, with its standard smooth structure and orientation. Thus, by the second version of Stokes' Theorem,

$$\begin{aligned} \int_{S^2} (x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) &= \int_{\partial B^3} (x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\ &= \int_{B^3} d(x_1 dx_2 \wedge dx_3 + x_2 dx_1 \wedge dx_3 + x_3 dx_1 \wedge dx_2) \\ &= \int_{B^3} (dx_1 \wedge dx_2 \wedge dx_3 + dx_2 \wedge dx_1 \wedge dx_3 + dx_3 \wedge dx_1 \wedge dx_2) \\ &= \int_{B^3} (dx_1 \wedge dx_2 \wedge dx_3 - dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge dx_3) = \int_{B^3} 1 dx_1 \wedge dx_2 \wedge dx_3. \end{aligned}$$

Since B^3 with its standard orientation is a regular subspace of \mathbb{R}^3 ,

$$\int_{B^3} 1 dx_1 \wedge dx_2 \wedge dx_3 = \int_{B^3} 1 dx_1 dx_2 dx_3 = \frac{4\pi}{3}.$$

Problem 5 (25 pts)

Suppose M and N are smooth manifolds. Show that

- (a) if M and N are orientable, then $M \times N$ is orientable;
- (b) if M is orientable and nonempty and N is not orientable, then $M \times N$ is not orientable;
- (c) if M and N are not orientable, then $M \times N$ is not orientable.

(a) Since M and N are orientable, the line bundles

$$\Lambda^{\text{top}}TM \longrightarrow M \quad \text{and} \quad \Lambda^{\text{top}}TN \longrightarrow N$$

are trivial. On the other hand, let

$$\pi_1, \pi_2: M \times N \longrightarrow M, N$$

be the two projection maps. Then,

$$\begin{aligned} T(M \times N) &= \pi_1^*TM \oplus \pi_2^*TN \longrightarrow M \times N \\ \implies \Lambda^{\text{top}}(T(M \times N)) &= \Lambda^{\text{top}}(\pi_1^*TM \oplus \pi_2^*TN) = \Lambda^{\text{top}}(\pi_1^*TM) \otimes \Lambda^{\text{top}}(\pi_2^*TN) \\ &= \pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN). \end{aligned}$$

Since the line bundles $\Lambda^{\text{top}}TM$ and $\Lambda^{\text{top}}TN$ over M and N , respectively, are trivial, so are their pullbacks to $M \times N$ and their tensor product. Since the line bundle

$$\Lambda^{\text{top}}(T(M \times N)) = \pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN) \longrightarrow M \times N$$

is trivial, the manifold $M \times N$ is orientable.

Alternatively, since M and N are orientable, there exist nowhere-vanishing top forms on M and N :

$$\alpha \in E^{\text{top}}(M) \quad \text{and} \quad \beta \in E^{\text{top}}(N).$$

Then, $\pi_1^*\alpha \wedge \pi_2^*\beta$ is a nowhere-vanishing top form on $M \times N$. Since $M \times N$ admits such a form, $M \times N$ is orientable. To see that $\pi_1^*\alpha \wedge \pi_2^*\beta$ does not vanish on $M \times N$, suppose $x \in M$, $y \in N$, and

$$\{v_1, \dots, v_k\} \subset T_xM \quad \text{and} \quad \{w_1, \dots, w_n\} \subset T_yN$$

are bases for T_xM and T_yN . Since

$$\alpha|_x \in \Lambda^{\text{top}}T_x^*M \quad \text{and} \quad \beta|_y \in \Lambda^{\text{top}}T_y^*N$$

are not zero,

$$\alpha|_x(v_1, \dots, v_k) \neq 0 \quad \text{and} \quad \beta|_y(w_1, \dots, w_n) \neq 0.$$

On the other hand,

$$T_{(x,y)}(M \times N) = T_x M \oplus T_y N \quad \implies \quad v_1, \dots, v_k, w_1, \dots, w_n \in T_{(x,y)}(M \times N).$$

Furthermore,

$$\begin{aligned} \{\pi_1^* \alpha \wedge \pi_2^* \beta\}|_{(x,y)}(v_1, \dots, v_k, w_1, \dots, w_n) &= \{\pi_1^* \alpha\}|_{(x,y)}(v_1, \dots, v_k) \cdot \{\pi_2^* \beta\}|_{(x,y)}(w_1, \dots, w_n) \\ &= \alpha|_x(d\pi_1(v_1), \dots, d\pi_1(v_k)) \cdot \beta|_y(d\pi_2(w_1), \dots, d\pi_2(w_n)) \\ &= \alpha|_x(v_1, \dots, v_k) \cdot \beta|_y(w_1, \dots, w_n) \neq 0, \end{aligned}$$

as claimed. Note that a priori the expression on RHS of the first line above should be a sum over all possible permutations of $\{v_1, \dots, v_k, w_1, \dots, w_n\}$. However, since $d\pi_1(w_j) = 0$, the only such permutations that yield nonzero terms are the permutations that preserve the subsets $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_n\}$.

(b) By part (a),

$$\Lambda^{\text{top}}(T(M \times N)) = \pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN) \longrightarrow M \times N.$$

Since M is orientable, the line bundle $\pi_1^*(\Lambda^{\text{top}}TM)$ is again trivial and

$$\Lambda^{\text{top}}(T(M \times N)) = \pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN) \approx \pi_2^*(\Lambda^{\text{top}}TN).$$

Since N is not orientable, the line bundle $\Lambda^{\text{top}}TN \longrightarrow N$ is not trivial. This does not mean that its pullback by every map is not trivial, but we will show that its pullback by π_2 is indeed not trivial. In turn, since

$$\Lambda^{\text{top}}(T(M \times N)) \approx \pi_2^*(\Lambda^{\text{top}}TN)$$

and $\pi_2^*(\Lambda^{\text{top}}TN)$ is not trivial, it follows that $M \times N$ is not orientable. To see that $\pi_2^*(\Lambda^{\text{top}}TN)$ is not trivial, pick any $x \in M$ and define

$$\iota_x: N \longrightarrow M \times N \quad \text{by} \quad \iota_x(y) = (x, y).$$

Since $\pi_2 \circ \iota_x = \text{id}_N$,

$$\iota_x^*(\pi_2^*(\Lambda^{\text{top}}TN)) = \iota_x^* \pi_2^*(\Lambda^{\text{top}}TN) = \{\pi_2 \circ \iota_x\}^* \Lambda^{\text{top}}TN = \text{id}_N^* \Lambda^{\text{top}}TN = \Lambda^{\text{top}}TN.$$

Thus, $\iota_x^*(\pi_2^*(\Lambda^{\text{top}}TN))$ is not trivial, which implies that $\pi_2^*(\Lambda^{\text{top}}TN)$ is not trivial either.

Here is another approach. Suppose $M \times N$ is orientable, i.e. there exists a nowhere-vanishing $\gamma \in E^{\text{top}}(M \times N)$. We will construct a nowhere-vanishing $\beta \in E^{\text{top}}(N)$ by restricting γ to the vertical slice $x \times N$ and contracting the M -part of γ . Denote by k and n be the dimensions of M and N . Let

$$v_1, \dots, v_k \subset T_x M$$

be a basis. For each $i = 1, \dots, k$, let

$$X_i \in \Gamma(x \times N; T(M \times N)|_{x \times N})$$

be the (horizontal) vector field along $x \times N$ defined by

$$X_i(x, y) = v_i \in T_{(x,y)}(M \times N) = T_x M \oplus T_y N.$$

These are smooth vector fields and thus

$$\beta_x \equiv \iota_{X_k} \dots \iota_{X_1} \gamma = \gamma(X_1, \dots, X_k, \cdot, \dots, \cdot) \in \Gamma(x \times N; \Lambda^n T(M \times N)|_{x \times N})$$

is a smooth n -form. If $y \in N$ and

$$w_1, \dots, w_n \subset T_y N$$

is a basis for $T_y N$, then

$$v_1, \dots, v_k, w_1, \dots, w_n \subset T_{(x,y)}(X \times Y) = T_x X \oplus T_y Y$$

is a basis. Since γ does not vanish at (x, y) ,

$$\beta_x|_y(w_1, \dots, w_n) = \gamma(v_1, \dots, v_k, w_1, \dots, w_n) \neq 0.$$

Thus, β_x does not vanish along $x \times N$. Furthermore, $\iota_{v_i} \beta_x|_y = 0$ for all i , i.e.

$$\beta_x|_y \in \Lambda^n T_y^* N \subset \Lambda_{(x,y)}^n T^*(M \times N) = \bigoplus_{p+q=n} \Lambda^p T_x^* M \otimes \Lambda^q T_y^* N.$$

This means that

$$\beta \equiv \iota_x^* \beta_x \in \Gamma(N; \Lambda^n T^* N) = E^n(N)$$

is a nowhere-zero top form on N , i.e N is orientable. Note that this argument also implies part (c), since the only fact we used about M is that it is nonempty.

Here is a third approach. We can assume that N is connected. Let

$$p: \tilde{N} \longrightarrow N$$

be the orientable (connected) double cover of N . It can be obtained by choosing a metric on $\Lambda^{\text{top}} TN$ and taking

$$\tilde{N} = S(\Lambda^{\text{top}} TN);$$

see more below. Let g be the nontrivial deck transformation for p , so that

$$N = \tilde{N} / \{\text{id}, g\}.$$

Since \tilde{N} is orientable and N is not orientable, the diffeomorphism $g: \tilde{N} \longrightarrow \tilde{N}$ must be orientation-reversing. By part (a), $M \times \tilde{N}$ is an orientable manifold. Furthermore,

$$M \times N = M \times \tilde{N} / \{\text{id}_M \times \text{id}, \text{id}_M \times g\}.$$

Since id_M is orientation-preserving and g is orientation-reversing,

$$\text{id}_M \times g: M \times \tilde{N} \longrightarrow M \times \tilde{N}$$

is orientation-reversing. Thus, $M \times N$ is not orientable.

We now show that the total space \tilde{N} of the double cover

$$p: \tilde{N} \equiv S(\Lambda^{\text{top}} TN) \longrightarrow N$$

is orientable. Since p is a local diffeomorphism,

$$dp: T\tilde{N} \longrightarrow p^*TN$$

is an isomorphism of vector bundles. Thus,

$$\Lambda^{\text{top}}T\tilde{N} \approx \Lambda^{\text{top}}(p^*TN) \approx p^*(\Lambda^{\text{top}}TN).$$

To show that the line bundle $p^*(\Lambda^{\text{top}}TN)$ is trivial (and thus \tilde{N} is orientable), we construct a nowhere-vanishing section of

$$p^*(\Lambda^{\text{top}}TN) \equiv \{(e, v) \in \tilde{N} \times \Lambda^{\text{top}}TN : p(e) = \pi(v)\} \longrightarrow \tilde{N},$$

where $\pi: \Lambda^{\text{top}}TN \longrightarrow N$ is the bundle projection map. By definition,

$$\begin{aligned} \tilde{N} &= \{e \in \Lambda^{\text{top}}TN : |e| = 1\} \implies \\ p^*(\Lambda^{\text{top}}TN) &\equiv \{(e, v) \in \Lambda^{\text{top}}TN \times \Lambda^{\text{top}}TN : \pi(v) = \pi(e)\} \longrightarrow \tilde{N} = \{e \in \Lambda^{\text{top}}TN : |e| = 1\}. \end{aligned}$$

We define a section of $p^*(\Lambda^{\text{top}}TN)$ over \tilde{N} by

$$s(e) = (e, e).$$

This section does not vanish.

(c) We will show that the line bundle

$$\Lambda^{\text{top}}(T(M \times N)) = \pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN) \longrightarrow M \times N$$

is not trivial by showing that its pullback by ι_x is again not trivial. This implies that $M \times N$ is not orientable. Given $x \in M$, let

$$f_x: N \longrightarrow M, \quad f_x(y) = x,$$

be the constant map sending N to x . Since $\pi_1 \circ \iota_x = f_x$,

$$\iota_x^* \pi_1^*(\Lambda^{\text{top}}TM) = \{\pi_1 \circ \iota_x\}^*(\Lambda^{\text{top}}TM) = f_x^*(\Lambda^{\text{top}}TM) = N \times (\Lambda^{\text{top}}TM)_x$$

is the trivial line bundle. Thus,

$$\begin{aligned} \iota_x^*(\pi_1^*(\Lambda^{\text{top}}TM) \otimes \pi_2^*(\Lambda^{\text{top}}TN)) &= \iota_x^*(\pi_1^*(\Lambda^{\text{top}}TM)) \otimes \iota_x^*(\pi_2^*(\Lambda^{\text{top}}TN)) \\ &= (N \times \mathbb{R}) \otimes \Lambda^{\text{top}}TN \approx \Lambda^{\text{top}}TN. \end{aligned}$$

Since N is not orientable, the line bundle $\Lambda^{\text{top}}TN$ is not trivial as needed.

Alternatively, we can assume that M is connected. Let

$$p: \tilde{M} \longrightarrow M$$

be the orientable (connected) double cover of M similarly to part (b). Then,

$$p \times \text{id}: \tilde{M} \times N \longrightarrow M \times N$$

is a covering projection. Since $\tilde{M} \times N$ is not orientable by part (b), $M \times N$ is not orientable either (a nowhere-vanishing top form on the base induces a nowhere-vanishing top form on the total space of a covering map (but not conversely)).