MAT 531: Topology&Geometry, II
Spring 2010

Problem Set 3
Due on Thursday, 2/18, in class

1. (a) For what values of $t\in\mathbb{R}$, is the subspace
$$\big\{(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}: x_1^2+\ldots+x_n^2-x_{n+1}^2=t\big\}$$
a smooth embedded submanifold of $\mathbb{R}^{n+1}$?
(b) For such values of t, determine the diffeomorphism type of this submanifold (i.e. show that it is diffeomorphic to something rather standard).
Hint: Draw some pictures.

2. Show that the special unitary group
$$SU_n=\big\{A\in\mathrm{Mat}_n\mathbb{C}\colon \bar{A}^tA=\mathbb{I}_n,\ \det A=1\big\}$$
is a smooth compact manifold. What is its dimension?

3. (a) Suppose $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ are smooth maps that are transverse to each other:
$$T_{f(x)}M=\mathrm{Im}\,\mathrm{d}_xf+\mathrm{Im}\,\mathrm{d}_yg \qquad \forall\ (x,y)\in X\times Y \text{ s.t } f(x)=g(y).$$
Show that
$$X\times_M Y\equiv\big\{(x,y)\in X\times Y:\ f(x)=g(y)\big\}$$
is a smooth (embedded) submanifold of $X\times Y$.
(b) Suppose that $f\colon X\longrightarrow M$ is a smooth map and $\pi\colon V\longrightarrow M$ is a smooth vector bundle. The pullback of V by f, $\pi_1\colon f^*V\longrightarrow X$, is the vector bundle defined by taking
$$f^*V=\big\{(x,v)\in X\times V\colon f(x)=\pi(v)\big\}\subset X\times V.$$
In particular, f^*V is supposed to be a smooth manifold. Show that f^*V is in fact a smooth submanifold of $X\times V$.

4. Chapter 1, #22 (p51)

5. Chapter 1, #17 (p51)

6. Let V be the vector field on $\mathbb{R}^3$ given by
$$V(x,y,z)=y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$
Explicitly describe and sketch the flow of V.