MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 9

Solution to Problem p353, #4

Suppose you are given the fact that for every n there is no retraction $r\colon B^{n+1}\longrightarrow S^n$. Show that
(a) The identity map $i\colon S^n\longrightarrow S^n$ is not null-homotopic.
(b) The inclusion map $j\colon S^n\longrightarrow\mathbb{R}^{n+1}-0$ is not null-homotopic.
(c) If v is a nonvanishing vector field on B^{n+1}, then
(c-i) $v(x)=ax$ for some $a\in\mathbb{R}^-$ and $x\in S^n$;
(c-ii) $v(x)=ax$ for some $a\in\mathbb{R}^+$ and $x\in S^n$.
(d) Every continuous map $f\colon B^{n+1}\longrightarrow B^{n+1}$ has a fixed point.
(e) Every $(n+1)\times(n+1)$-matrix A of positive reals has a positive eigenvalue.
(f) If $h\colon S^n\longrightarrow S^n$ is null-homotopic, then $h(x)=x$ for some $x\in S^n$ and $h(x)=-x$ for some $x\in S^n$.

Remark: For same reason as for π_1, if $r\colon X\longrightarrow A$ is a retraction, then the induced homomorphism

$$r_*\colon \pi_n(X,a_0)\longrightarrow\pi_n(A,a_0)$$

between the nth homotopy groups is surjective. As $\pi_n(B^{2n+1},a_0)$ is trivial, while $\pi_n(S^n,a_0)\approx\mathbb{Z}$, there exists no retraction from $X=\mathcal{B}^{n+1}$ to $A=S^2$.

The solution is essentially Section 55, with S^1 and B^2 replaced everywhere S^n and B^{n+1}. The only difference is that part (c) of Lemma 55.3 has to be omitted. The correct replacement is that h_* is the trivial homomorphism on π_n.

(a) If $f\colon S^n\longrightarrow S^n$ is null-homotopic, there exist $c\in S^n$ and a continuous map

$$F\colon S^n\times I\longrightarrow S^n,\qquad F(x,0)=f(x)\ \ \forall\,x\in S^n,\quad\text{and}\quad F(x,1)=c\ \ \forall\,x\in S^n.$$

Since F is constant on $S^n\times 1$, it induces a map from the quotient space

$$\bar{F}\colon X=(S^n\times I)/\sim\ \longrightarrow S^n,\qquad\text{where}\qquad (x,1)\sim(x',1)\ \forall\,x\in S^n.$$

Since F is continuous, $\bar{F}$ is continuous in the quotient topology on X. With this topology, X is homeomorphic to B^{n+1} by the map

$$[x,t]\longrightarrow(1-t)x\in\mathbb{R}^{n=1}.$$

Thus, if $f\colon S^n\longrightarrow S^n$ is null-homotopic, it extends to a continuous map $g\colon B^{n+1}\longrightarrow S^n$. If $f=i$, such an extension would be a retraction. Since no retraction of B^{n+1} onto S^n, the identity map

$i\colon S^n \longrightarrow S^n$ is not null-homotopic.

(b) Let $r\colon \mathbb{R}^{n+1}-0 \longrightarrow S^n$ be the natural retraction given by $r(x)=x/|x|$. Then,

$$r\circ j = i\colon S^n \longrightarrow S^n.$$

Since i is not null-homotopic by part (a), neither is j (nor r).

(c-i) Suppose there exists no $x\in S^n$ and $a\in\mathbb{R}^-$ such that $v(x)=ax$. Define

$$F\colon S^n\times I \longrightarrow \mathbb{R}^{n+1}-0 \qquad \text{by} \qquad F(x,t) = (1-t)v(x) + tx \in \mathbb{R}^{n+1}.$$

Note that if $F(x,t)=0$, then either $t=1$ and $v(x)=0$ or $v(x) = -(t/(1-t))x$. Neither is the case by our assumptions. Thus, F is homotopy between

$$v,j\colon S^n \longrightarrow \mathbb{R}^{n+1}-0.$$

Since j is not null-homotopic by part (b), neither is v. Thus, by the proof of by part (a), v cannot extend to a continuous map $B^{n+1}\longrightarrow S^n$, contrary to our assumptions.

(c-ii) This follows from (c-i) applied to $-v$.

(d) Let $v(x)=f(x)-x$. If $f(x)\neq x$ for all $x\in B^{n+1}$, then

$$v\colon B^{n+1} \longrightarrow \mathbb{R}^{n+1}-0$$

is a continuous map. Thus, by (c-ii), for some $x\in S^n$ and $a\in\mathbb{R}^+$

$$v(x) = ax \qquad \Longrightarrow \qquad f(x) = ax+x = (a+1)x \qquad \Longrightarrow \qquad |f(x)| = |a+1||x| = |a+1| > 1.$$

This is impossible, since $f(x)\in B^{n+1}$ for all $x\in S^n$.

(e) Let

$$B = \big\{(x_1,\ldots,x_n)\colon x_1,\ldots,x_n>0,\ x_1^2+\ldots x_n^2=1\big\}.$$

Since all entries of the vector A are positive, all entries of the vector Ax, for $x\in B$, are nonnegative and at least one is positive. Thus, we can define

$$f\colon B \longrightarrow B \qquad \text{by} \qquad f(x) = Bx\big/|Bx|.$$

Since f is continuous and B is homeomorphic to B^n, $f(x)=x$ for some $x\in B$ by part (d). Then,

$$Bx = |Bx|f(x) = |Bx|x.$$

In other words, x is an eigenvector of A with (positive) eigenvalue $|Bx|$.

(f) Since the map $h\colon S^n \longrightarrow S^n$ is null-homotopic, by the proof of part (a) it extends to a continuous map

$$v\colon B^{n+1} \longrightarrow S^n \subset \mathbb{R}^{n+1}-0.$$

By (c), there exist $x_+, x_- \in S^n$, $a_+ \in R^+$, and $a^- \in \mathbb{R}^-$ such that

$$v(x_+) = a_+ x_+ \qquad \text{and} \qquad v(x_-) = a_- x_-.$$

Since $|v(x)|=1$ for all $x \in \mathcal{B}^{n+1}$,

$$|a_+| = |a_-| = 1 \qquad \Longrightarrow \qquad a_\pm = \pm 1 \qquad \Longrightarrow \qquad v(x_\pm) = \pm x_\pm.$$

Solution to Problem p359, #4

Suppose you are given the fact that for every n no continuous antipode-preserving $h\colon S^n \longrightarrow S^n$ is null-homotopic. Show that:
(a) There is no retraction $r\colon B^{n+1} \longrightarrow S^n$.
(b) There is no continuous antipode-preserving map $g\colon S^{n+1} \longrightarrow S^n$.
(c) If $f\colon S^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is a continuous map, $f(x)=f(-x)$ for some $x \in S^{n+1}$.
(d) If $A_1, \dots, A_{n+1}$ are bounded measurable sets in $\mathbb{R}^{n+1}$, there exists n-plane that bisects each of them.

(a) If such a retraction exists, the identity map $S^n \longrightarrow S^n$ is null-homotopic. However, since the identity map is antipode-preserving, it is not null-homotopic.

(b) There restriction of such a map to the upper-hemisphere B_+^{n+1} would give a retraction onto S^n. Since (B_+^{n+1}, S^n) is homeomorphic to (B^{n+1}, S^n) (by dropping the last coordinate), no such retraction exists by part (a).

(c) Let $h(x)=f(x)-f(-x)$. If $f(x)\neq f(-x)$ for all $x \in S^{n+1}$, then we can define

$$g\colon S^{n+1} \longrightarrow S^n \qquad \text{by} \qquad g(x) = h(x)/|h(x)|.$$

Since $h(-x)=-h(x)$, $g(-x)=-g(x)$, i.e. g is antipode-preserving. However, no such map exists by part (b).

(d) For each $x\in S^{n+1}$, let $\mathcal{H}_x$ be the hyperplane in $\mathbb{R}^{n+2}$ which is orthogonal to the unit vector x and passes through the point

$$p=(0,\ldots,0,1).$$

In particular, $\mathcal{H}_p=\mathcal{H}_{-p}$. If $x=\pm p$, $\mathcal{H}_p$ is parallel to $\mathbb{R}^{n+1}\times 0$; otherwise, $\mathcal{H}_p\cap\mathbb{R}^{n+1}$ is a hyperplane in $\mathbb{R}^{n+1}$. For each $k=1,\ldots,n+1$, let $f_k(x)\in\bar{\mathbb{R}}^+$ be the measure of the portion of A_k that lies on the side of $\mathcal{H}_x$ corresponding to x. In particular,

$$f_k(x)+f_k(-x)=\text{Area}\,A_k \qquad \text{and} \qquad f_k(p)=0.$$

The function $f_k\colon S^{n+1}\longrightarrow\mathbb{R}$ is a continuous, and so is the function

$$f=\big(f_1,\ldots,f_{n+1}\big)\colon S^{n+1}\longrightarrow\mathbb{R}^{n+1}.$$

Thus, by part (c),

$$f_k(x)=f_k(-x)=\frac{1}{2}\text{Area}\,A_k$$

for some $x\in S^{n+1}$.

Solution to Problem p366, #9

If $h: S^1\longrightarrow S^1$ is a continuous map and $x_0\in S^1$, choose a path $\alpha: I\longrightarrow S^1$ from x_0 to $h(x_0)$. Define

$$\deg h\in\mathbb{Z} \qquad \text{by} \qquad h_*=(\deg h)\cdot\hat{\alpha}\colon \pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,h(x_0)).$$

(a) *Show that the* $\deg h$ *is independent of the choice of* α *and* x_0.
(b) *Show that if* $h,k\colon S^1\longrightarrow S^1$ *are homotopic, then* $\deg h=\deg k$.
(c) *Show that* $\deg(h\circ k)=(\deg h)(\deg k)$.
(d) *Compute the degree of a constant map, the identity map, the reflection map ($\rho(x,y)=\rho(x,-y)$), and the map $h(z)=z^n$.*
(e) *Show that if* $h,k\colon S^1\longrightarrow S^1$ *have the same degree, then they are homotopic.*

(a) Suppose $\beta\colon I\longrightarrow S^1$ is another path from x_0 to $h(x_0)$, then the isomorphisms

$$\hat{\alpha},\hat{\beta}\colon \pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,h(x_0))$$

are the same because $\pi_1(S^1,x_0)\approx\mathbb{Z}$ is abelian. Thus,

$$\hat{\alpha}^{-1}\circ h_*=\hat{\beta}^{-1}\circ h_*\colon \pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,x_0)\approx\mathbb{Z}$$

are the multiplication by the same number, which is denoted by $\deg h$. Suppose $x_0'\in S^1$ and $\gamma\colon I\longrightarrow S^1$ is a path from x_0 to x_0'. Then,

$$\beta\equiv\bar{\gamma}*\alpha*(h\circ\gamma)\colon I\longrightarrow S^1$$

is a path from x_0' to $h(x_0')$. Furthermore,

$$\hat{\beta}^{-1}\circ h_* = \big(\widehat{h\circ\gamma}\circ\hat{\alpha}\circ\hat{\bar{\gamma}}\big)^{-1}\circ h_* = \hat{\gamma}\circ\hat{\alpha}^{-1}\circ\widehat{h\circ\gamma}^{-1}\circ h_*$$
$$= \hat{\gamma}\circ(\hat{\alpha}^{-1}\circ h_*)\circ\hat{\gamma}^{-1}\colon \pi_1(S^1,x_0')\longrightarrow\pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,x_0').$$

Thus, if $\hat{\alpha}^{-1}\circ h_*$ is the multiplication by $\deg h$, then so is $\hat{\beta}^{-1}\circ h_*$. It follows that $\deg h$ is independent of the choice of x_0 and α.

(b) Since h and k are homotopic, there exists a path $\beta\colon I\longrightarrow S^1$ from $h(x_0)$ to $k(x_0)$ such that

$$k_* = \hat{\beta}\circ h_*\colon \pi_1(X,x_0)\longrightarrow\pi_1(X,h(x_0))\longrightarrow\pi_1(X,k(x_0)).$$

If $\alpha\colon I\longrightarrow S^1$ is a path from x_0 to $h(x_0)$, then $\alpha*\beta$ is a path from x_0 to $h(x_0)$. Furthermore,

$$k_* = \hat{\beta}\circ h_* = \hat{\beta}\circ\big((\deg h)\hat{\alpha}\big) = (\deg h)\hat{\beta}\circ\hat{\alpha} = (\deg h)\widehat{\alpha*\beta}.$$

Since by definition $k_*=(\deg h)\widehat{\alpha*\beta}$ and $\widehat{\alpha*\beta}$ is an isomorphism, we conclude that $\deg k=\deg h$.

(c) Let $\alpha,\beta\colon I\longrightarrow S^1$ be paths from x_0 to $k(x_0)$ and from $k(x_0)$ to $h(k(x_0))$, respectively. Then, $\alpha*\beta$ is a path from x_0 to $h(k(x_0))$. Furthermore, by definition of the degree,

$$k_* = (\deg k)\hat{\alpha},\quad h_* = (\deg h)\hat{\beta}\qquad\Longrightarrow$$
$$(h\circ k)_* = h_*\circ k_* = \big((\deg h)\hat{\beta}\big)\circ\big((\deg k)\hat{\alpha}\big) = (\deg h)(\deg k)\hat{\beta}\circ\hat{\alpha} = (\deg h)(\deg k)\widehat{\alpha*\beta}.$$

Since by definition $(h\circ k)_* = (\deg(h\circ k))\widehat{\alpha*\beta}$ and $\widehat{\alpha*\beta}$ is an isomorphism, we conclude that $\deg(h\circ k)=(\deg h)(\deg k)$.

(d) If $h\colon S^1\longrightarrow S^1$ is a constant map, the homomorphism $h_*\colon\pi_1(S^1,x_0)\longrightarrow\pi_1(S^1,h(x_0))$ is trivial and thus $\deg h=0$. If $h\colon S^1\longrightarrow S^1$ is the identity map, the homomorphism h is the identity and thus $\deg h=1$. If $h(z)=z^n$, for some $n\in\mathbb{Z}^+$,

$$h_*\colon\pi_1(S^1,1)\longrightarrow\pi_1(S^1,1)$$

is the multiplication by n as computed on the previous problems (the natural loop generating $\pi_1(S^1,1)$ is taken to n times itself). Thus, $\deg h=n$. The reflection map is the $n=-1$ case of this.

(e) By (b) and (c), we can assume that $h(1)=k(1)=1$. We define the loop at 1 in S^1 by

$$q\colon I\longrightarrow S^1\qquad\text{and}\qquad q(t)=e^{2\pi it}.$$

Since $\deg h=\deg k$,

$$h_*=k_*\colon\pi_1(S^1,1)\longrightarrow\pi_1(S^1,1).$$

In particular, $[h\circ q]=[k\circ q]$. Let

$$p\colon \mathbb{R} \longrightarrow S^1, \qquad q(t) = e^{2\pi i t},$$

be the standard covering map. Let

$$\tilde{q}_h, \tilde{q}_k\colon (I,0) \longrightarrow (\mathbb{R},0)$$

be the lifts of $h\circ q, k\circ q\colon (I,0)\longrightarrow(S^1,1)$. Since $h\circ q$ is path homotopic to $k\circ q$,

$$\tilde{q}_k(1) = \tilde{q}_k(1) \in \mathbb{Z}=p^{-1}(1) \subset \mathbb{R}.$$

Let

$$\tilde{F}\colon \big(I\times I, 0\times I, 1\times I\big) \longrightarrow \big(\mathbb{R}, 0, \tilde{q}_h(1)\big)$$

be a path homotopy between $\tilde{q}_h$ and $\tilde{q}_k$ in $\mathbb{R}$. Then,

$$p\circ\tilde{F}\colon \big(I\times I, 0\times I, 1\times I\big) \longrightarrow \big(S^1, 1, 1\big)$$

is a path-homotopy between the loops $h\circ q$ and $k\circ q$. It descends to a map on the quotient

$$F\colon X = (I\times I)\big/\!\sim \longrightarrow S^1, \qquad \text{where} \qquad (0,t) \sim (1,t) \;\; \forall\, t\in I.$$

This map is continuous in the quotient topology. With this topology, X is homeomorphic to $S^1\times I$. The quotient project map is

$$q\times\mathrm{id}\colon I\times I \longrightarrow S^1\times I.$$

(we have simply identified the two vertical edges of the square $I\times I$). Thus, we have found a continuous map

$$F\colon S^1\times I \longrightarrow S^1 \qquad \text{s.t.} \qquad p\circ\tilde{F} = F\circ q, \quad \tilde{F}|_{0\times I} = \tilde{q}_h, \quad \tilde{F}|_{1\times I} = \tilde{q}_k$$
$$\Longrightarrow \qquad F\circ q|_{0\times I} = p\circ\tilde{F}|_{0\times I} = p\circ\tilde{q}_h = h\circ q, \quad F\circ q|_{1\times I} = p\circ\tilde{F}|_{1\times I} = p\circ\tilde{q}_k = k\circ q$$
$$\Longrightarrow \qquad F|_{0\times I} = h, \quad F|_{1\times I} = k.$$

We conclude F is a homotopy between h and k.