MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 3

Solution to Problem p158, #12

Let S_Ω be the minimal uncountable well-ordered set, as in Section 10. Denote by L the ordered set $S_\Omega \times [0,1)$ in the dictionary order with its smallest element deleted. The topological space L, with the order topology, is called the long line.

Theorem: The long line L is path-connected and locally homeomorphic to $\mathbb{R}$, but it cannot be embedded in $\mathbb{R}$.

(a) Suppose X is an ordered set, $a, b, c \in X$, and $a < b < c$. Show that $[a,c)$ has the order type of $[0,1)$ if and only if both $[a,b)$ and $[b,c)$ do.

(b) Suppose X is an ordered set, $x_0, x_1, \ldots \in X$ is a strictly increasing sequence in X, and $b = \sup\{x_i\}$. Show that $[x_0, b)$ has the order type of $[0,1)$ if and only if every interval $[x_i, x_{i+1})$ with $i \geq 0$ does.

(c) Let a_0 be the smallest interval of S_Ω. Show that for every $a \in S_\Omega - \{a_0\}$, the interval

$$\big[a_0 \times 0, a \times 0\big) \subset S_\Omega \times [0,1)$$

has the order type of $[0,1)$.

(d) Show that L is path connected.

(e) Show that every point in L has a neighborhood homeomorphic to an open interval in $\mathbb{R}$.

(f) Show that L cannot be embedded into in $\mathbb{R}^n$ for any n.

Recall that two ordered sets $(X, <)$ and $(Y, <)$ are said to have *the same order type* if there exists a surjective map

$$f\colon X \longrightarrow Y \qquad \text{s.t.} \qquad x_1, x_2 \in X, \;\; x_1 < x_2 \implies f(x_1) < f(x_2).$$

Such a map f is necessarily bijective and is a homeomorphism with respect to the order topologies on X and Y.

(a) Suppose $f\colon [a,c) \longrightarrow [0,1)$ is an order-preserving bijection as above. Then, so are

$$\begin{aligned} g\colon [a,b) &\longrightarrow [0,1), \quad g(x) = f(x)/f(b), \qquad \text{and} \\ h\colon [b,c) &\longrightarrow [0,1), \quad g(x) = \big(f(x) - f(b)\big)/\big(1 - f(b)\big). \end{aligned}$$

Conversely, if $g\colon [a,b) \longrightarrow [0,1)$ and $h\colon [b,c) \longrightarrow [0,1)$ are order-preserving bijections, then so is

$$f\colon [a,c) \longrightarrow [0,1), \qquad f(x) = \begin{cases} g(x)/2, & \text{if } x \in [a,b); \\ 1/2 + h(x)/2, & \text{if } x \in [b,c). \end{cases}$$

(b) Suppose $f\colon [x_0,b)\longrightarrow[0,1)$ is an order-preserving bijection. Then, so is

$$f_i\colon [x_i,x_{i+1})\longrightarrow[0,1), \qquad f_i(x)=\big(f(x)-f(x_i)\big)/\big(f(x_{i+1})-f(x_i)\big), \qquad i=0,1,\ldots$$

since $f(x_{i+1})-f(x_i)>0$. Conversely, if $f_i: [x_i,x_{i+1})\longrightarrow[0,1)$ is an order-preserving bijection for each $i=0,1,\ldots$, then so is

$$f\colon [x_0,b)\longrightarrow[0,1), \qquad f(x)=\begin{cases}(1-2^{-i})+2^{-(i+1)}f_i(x), & \text{if } x\in[x_i,x_{i+1});\\ 1, & \text{if } x=b.\end{cases}$$

This map is injective and order-preserving on each interval $[x_i,x_{i+1})$, since f_i is. Furthermore, if $y\in[x_i,x_{i+1})$ and $z\in[x_j,x_{j+1})$ with $i<j$, then

$$f(x)<1-2^{-(i+1)}\le 1-2^{-j}\le f(y).$$

Thus, f is injective and order-preserving on the entire interval $[x_0,b)$, and the image of f is $[0,1)$.

(c) Let

$$A=\big\{a\in S_\Omega-\{a_0\}\colon [a_0\times0,a\times0) \text{ has order type of } [0,1)\big\}.$$

We will show that if $c\in S_\Omega-\{a_0\}$ and $a\in A$ for all $a\in S_\Omega-\{a_0\}$ such that $a<c$, then $c\in A$. By transfinite induction, this implies that $A=S_\Omega-\{a_0\}$; see Exercise 7 on p67.

Suppose first that c has an immediate predecessor, $b<c$. Then, $[a_0\times0,b\times0)$ has the order type of $[0,1)$ and so does

$$\big[b\times0,c\times0\big)=b\times[0,1).$$

Thus, by part (a), $[a_0\times0,c\times0)$ has the order type of $[0,1)$, and $c\in A$.

Suppose c has no immediate predecessor. The set

$$S_c\equiv\big\{a\in S_\Omega\colon a<c\big\}$$

is countable and must be infinite, since c has no immediate predecessor. Thus, there exists a strictly increasing sequence $a_i\in S_\Omega-\{a_0\}$ that converges to c. Since $[a_0\times0,a_{i+1}\times0)$ has the order type of $[0,1)$, so does $[a_i\times0,a_{i+1}\times0)$ by part (a). Since c has no immediate predecessor, the sequence $x_i=a_i\times0$ converges to $c\times0$. Since $[a_i\times0,a_{i+1}\times0)$ has the order type of $[0,1)$ and $a_i\times0$ converges to $c\times0$, $[a_0\times0,c\times0)$ has the order type of $[0,1)$ by part (b). Thus, $c\in A$.

(d) Suppose $x,y\in L$ and $x<y$. Since S_Ω has no largest element, there exists $b\in S_\Omega$ such that $y<b\times0$ and thus $x,y\in[a_0\times0,b\times0)$. Since $[a_0\times0,b\times0)$ has the order type of $[0,1)$, there exists an order-preserving bijection

$$f\colon[0,1)\longrightarrow[a_0\times0,b\times0).$$

This map is continuous (in fact, a homeomorphism), and so is its restriction

$$f\colon\big[f^{-1}(x),f^{-1}(y)\big]\longrightarrow[x,y]\subset L.$$

This is a path from x to y in L.

(e) Given $x\in L$, choose $b\in S_\Omega$ such that $x<b\times 0$; see part (d). Let

$$f\colon [0,1) \longrightarrow [a_0\times 0, b\times 0)$$

be a homeomorphism as above. Then,

$$f\colon (0,1) \longrightarrow (a_0\times 0, b\times 0)$$

is a homeomorphism between $(0,1)$ and a neighborhood of x in L.

(f) The space $\mathbb{R}^n$ is second-countable, i.e. has a countable basis for its standard topology. So, does every subspace Y of $\mathbb{R}^n$; a countable basis for the subspace topology on Y can be obtained by intersecting Y with the elements of a countable basis for $\mathbb{R}^n$. If $f\colon L\longrightarrow Y$ is a homeomorphism and Y is second-countable, then so is L. On the other hand,

$$\big\{a\times(1/4,3/4)\colon a\in S_\Omega\big\}$$

is an uncountable collection of disjoint nonempty open subsets of L. Thus, L is not second-countable.

Solution to Problem p178, #5

Suppose that X is a compact Hausdorff space and $\{A_n\}_{n\in\mathbb{Z}^+}$ is a countable collection of closed subsets of X such that $\mathrm{Int}\,A_n=\emptyset$ *for all $n\in\mathbb{Z}^+$. Show that* $\mathrm{Int}\,\bigcup_{n=1}^{\infty}A_n=\emptyset$.

We first note that if X is a compact Hausdorff space and $\mathcal{U}\subset X$ is a nonempty open subset, then there exists a nonempty open subset $V\subset X$ such that $\bar{V}\subset\mathcal{U}$. Indeed, choose $p\in\mathcal{U}$. Since $X-\mathcal{U}$ is a closed, and thus compact, subset of X and does not contain p, by Lemma 26.4 there exist open sets $V,W\subset X$ such that

$$x\in V,\quad X-\mathcal{U}\subset W,\quad V\cap W=\emptyset \qquad\Longrightarrow$$
$$V\neq\emptyset,\quad \bar{V}\cap W=\emptyset \qquad\Longrightarrow\qquad \bar{V}\cap(X-\mathcal{U})=\emptyset \qquad\Longrightarrow\qquad \bar{V}\subset\mathcal{U},$$

as needed.

Suppose $V_0=\mathrm{Int}\,\bigcup_{n=1}^{\infty}A_n$ is nonempty. Recall that this is the largest open set contained in $\bigcup_{n=1}^{\infty}A_n$. Suppose $n\le 1$ and for all $i<n$ we have constructed a nonempty open subset $V_i\subset X$ such that

$$V_i\supset V_j \;\;\forall\, i<j<n \qquad\text{and}\qquad \bar{V}_i\cap A_i=\emptyset \;\;\forall\, i<n.$$

Since $\mathrm{Int}\,A_n=\emptyset$, $V_{n-1}\not\subset A_n$ and thus $V_{n-1}-A_n$ is an nonempty open subset of X. By the previous paragraph, we can choose a nonempty open subset V_n of X such that

$$\bar{V}_n\subset V_{n-1}-A_n \qquad\Longrightarrow\qquad \bar{V}_n\cap A_n=\emptyset.$$

The inductive assumptions are satisfied. Thus, we can find nonempty open subsets $\{V_i\}_{i\in Z^+}$ of X such that

$$V_i \supset V_j \ \ \forall\, i<j \qquad \text{and} \qquad \bar{V}_i \cap A_i = \emptyset \ \ \forall\, i.$$

Since

$$\bigcap_{i=1}^{i=n} \bar{V}_i = \bar{V}_n \neq \emptyset,$$

the collection $\{V_i\}_{i\in\mathbb{Z}^+}$ satisfies the finite-intersection property. Since X is compact,

$$\bigcap_{i=1}^{i=\infty} \bar{V}_i \neq \emptyset.$$

However,

$$\bigcap_{i=1}^{i=\infty} \bar{V}_i \subset V_0 \equiv \mathrm{Int} \bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \bigcap_{i=1}^{i=\infty} \bar{V}_i \cap A_n \subset \bar{V}_n \cap A_n = \emptyset.$$

The last two statements are contradictory.