Topology
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Problem Set I Solutions

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p39, #5:

Which of the following subsets of $\mathbb{R}^\omega$ can be expressed as the cartesian product of subsets of $\mathbb{R}$?

a $\{\mathbf{x}|x_i \text{ is an integer for all } i\}$

This is $\mathbb{Z}^\omega$.

b $\{\mathbf{x}|x_i \geq i \text{ for all } i.\}$

This is $\prod_{i\in\mathbb{Z}_+}\{x \in \mathbb{R}|x \geq i\}$.

c $\{\mathbf{x}|x_i \text{ is an integer for all } i \geq 100\}$

This is $\prod_{i=1}^{99}\mathbb{R}\prod_{i=100}^{\infty}\mathbb{Z}$

d $\{\mathbf{x}|x_2 = x_3\}$

If this were a cartesian product, then it would be the set of ω-tuples in $\mathbb{R}$ with $x_i \in X_i \subset \mathbb{R}$ for some set of subsets X_i. In particular, since x_2 and x_3 can take any value in $\mathbb{R}$, we would have $X_2 = X_3 = \mathbb{R}$. But then this cartesian product would contain tuples with differing x_2 and x_3 as well.

p83, #4:

a If $\{\mathscr{T}_\alpha\}$ is a family of topologies on X, show that $\cap\mathscr{T}_\alpha$ is a topology on X. Is $\cup\mathscr{T}_\alpha$ a topology on X?

We need to establish the three properties of the definition of a topology for $\cap\mathscr{T}_\alpha$.

For any α, since $\mathscr{T}_\alpha$ is a topology for X, it contains X and $\emptyset$. Then the intersection of all these also contains these.

Suppose $U_i \in \cap\mathscr{T}_\alpha$. Then $U_i \in \mathscr{T}_\alpha$ for each α so $\cup U_i \in \mathscr{T}_\alpha$. Then it is also in their intersection.

Suppose $U_1, \ldots, U_n \in \cap\mathscr{T}_\alpha$. Then $U_i \in \mathscr{T}_\alpha$ for each α so $\cap_{i=1}^n U_i \in \mathscr{T}_\alpha$. Then it is also in their intersection.

On the other hand, neither the second nor third properties will necessarily be satisfied for the case of a union. For example, let $X = \{a, b, c\}$. Then two topologies on X are given by $\{\emptyset, \{a, b\}, X\}$ and $\{\emptyset, \{a, c\}, X\}$, but their union fails to be closed under finite intersection.

b Let $\{\mathscr{T}_a\}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the $\mathscr{T}_a$ and a unique largest topology contained in all $\mathscr{T}_a$.

For the first of these, examine the topology generated by the subbasis $\cup\mathscr{T}_a$. This is certainly a topology on X containing each $\mathscr{T}_a$. On the other hand, any topology containing each of these topologies must contain unions of finite intersections of elements of the various topologies $\mathscr{T}_a$ and hence must contain the subbasis topology generated by them. So any topology containing all $\mathscr{T}_a$ must contain this one so it is the desired unique topology.

For the second, look at the intersection of the topologies, which is contained in each and which we have shown to be a topology. If $\mathscr{T}$ is a topology contained in every $\mathscr{T}_a$ and $U \in \mathscr{T}$ then $U \in \mathscr{T}_a$ for all a. So then each element of $\mathscr{T}$ is contained in every $\mathscr{T}_a$ so that it is contained in their intersection, meaning that $\mathscr{T}$ is contained in the intersection topology.

c if $X = \{a, b, c\}$, $\mathscr{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, and $\mathscr{T}_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$, find the two unique topologies from the last part of the exercise.

These are $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\{\emptyset, \{a\}, X\}$.

7: Consider the following topologies on $\mathbb{R}$:

- $\mathscr{T}_1$, the standard topology
- $\mathscr{T}_2$, the $\mathbb{R}_K$ topology
- $\mathscr{T}_3$, the finite complement topology
- $\mathscr{T}_4$, the upper limit topology with $(a, b]$ as basis
- $\mathscr{T}_5$, the topology having sets $(-\infty, a)$ as basis.

Determine the inclusions here.

We will show that three and five are contained in one, which is contained in two, which is contained in four, all inclusions strict, and that three and five are incomparable.

First, $(-\infty, a) = \cup_{b<a}(b, a)$ is open in the standard topology, and similarly (a, ∞).

$\mathscr{T}_3 \subset \mathscr{T}_1$:
A set with finite complement is a union of the standard open intervals between its complement points and the half-infinite open intervals at the ends (which we just showed to be standard open), so is in the standard topology. $(0, 1)$ is standard open but not finite complement.

$\mathscr{T}_5 \subset \mathscr{T}_1$:
Since the basis sets of $\mathscr{T}_5$ are in the standard topology, any unions are as well, so the whole topology is thus contained. $(0, 1)$ can't be written as a union of basis elements in this topology since such a union is unbounded below, but is standard open.

$\mathscr{T}_1 \subset \mathscr{T}_2$:
Again, since $\mathscr{T}_2$ contains the basis $\{(a, b)\}$ of $\mathscr{T}_1$, it contains the whole topology. Since $(-1, 1) - K$ is not open in the standard topology, the inclusion is strict.

$\mathscr{T}_2 \subset \mathscr{T}_4$:
A basis element of $\mathscr{T}_2$ is either an open interval (a, b) or a set of form $(a, b) - K$, that is, a union of open intervals like $(\frac{1}{n+1}, \frac{1}{n})$ and possibly an interval of the form $(a, 0]$. Clearly $(a, 0]$ is open in $\mathscr{T}_4$, and since $(a, b) = \cup_{a<c<b}(a, c]$, standard open intervals are open under this topology as well, which shows that a basis for $\mathscr{T}_2$ is contained in $\mathscr{T}_4$, which is all that we need. Since $(1, 2]$ is open in one but not the other the inclusion is strict.

$\mathscr{T}_3$ and $\mathscr{T}_5$ are incomparable:
$(-\infty, 0)$ is in $\mathscr{T}_5$ but does not have finite complement. On the other hand, a union of basis elements from $\mathscr{T}_5$ is of form $\cup(-\infty, a)$, so is of form $(-\infty, \sup\{a\})$, so is a single interval. Then $\mathbb{R}\backslash\{0\}$ is finite complement but not in $\mathscr{T}_5$.

8:

a Apply lemma 13.2 to show that the countable collection $\{(a, b)|a < b, a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$.

  Let $x \in (a, b) \subset \mathbb{R}$. Then $a < x < b$. Since these numbers differ, their decimal expansions differ at some point, so we can truncate decimal expansions to find rational numbers c, d between a and x and between x and b. Then $x \in (c, d) \subset (a, b)$, and (c, d) is in our countable collection, so by lemma 13.2 that collection generates the standard topology.

b Show that $\{[a, b)|a < b, a, b \in \mathbb{Q}\}$ generates a topology different from the lower limit topology.

  It is clear that this set satisfies the requirements to be a basis for some topology. Let x be irrational and negative. We will show that the topology generated by this set does not include $[x, 0)$. Suppose otherwise. For this interval to be in the topology, it would have to be a union of basis elements. Then one of the basis elements would contain x. Since the endpoints of the basis intervals are rational, we would have $a < x < b$ for some $[a, b)$ in the union, meaning the union would contain $a < x$ and so would not have lower endpoint x, a contradiction.

p92, #3: Consider $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. We decide which of the following are open in Y and in $\mathbb{R}$:

$A = (-1, -1/2) \cup (1/2, 1)$
$B = [-1, -1/2) \cup (1/2, 1]$
$C = (-1, -1/2] \cup [1/2, 1)$
$D = [-1, -1/2] \cup [1/2, 1]$
$E = (-1, 0) \cup (0, 1) \backslash K = (-1, 0) \cup \bigcup(\frac{1}{n+1}, \frac{1}{n})$

So, A and E are open in $\mathbb{R}$ and none of the others are, because they contain intervals containing their endpoints (in $\mathbb{R}$ under the standard topology, if an open set contains a point x it contains a basis element containing x, that is an interval around x).

Since A and E are open in $\mathbb{R}$ they are also open in Y. Since $B = ((-2, -1/2) \cup (1/2, 2)) \cap Y$ is the intersection of an open set in $\mathbb{R}$ with Y it is also open in Y.

For C and D, suppose either one was the intersection of Y with an open set U of $\mathbb{R}$. Then $1/2 \in U$ so U would contain an interval around $1/2$, meaning it would contain a point x less than $1/2$ and greater than 0. Then $x \in Y$ so $x \in U \cap Y$. Since neither C or D contain such a point they cannot be open in Y.

4: A map $f : X \to Y$ is open if for every open U of X, $f(U)$ is open in Y. Show that $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open.

The arguments are identical so we will only do the first one. Let U be open in the product. Then it is a union of basis elements $V_i \times W_i$, where V_i, W_i are nonempty open sets of X, Y respectively. So $\pi_1(U) = \pi_1(\cup V_i \times W_i) = \cup \pi_1(V_i \times W_i) = \cup V_i$ which is open in X.

8: If L is a straight line in the plane, describe the topology it inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

A basis for the topology of the first product is given by $[a, b) \times (c, d)$. A line which is not vertical intersects the vertical side of a rectangle at every point, so half-closed intervals are open for such a line and it inherits the lower limit topology. A vertical line inherits the standard topology.

For the second product, if the slope of the line is nonnegative or undefined, the line inherits the lower limit topology by essentially the same argument. If the slope is negative, then an open rectangle $[a, b) \times [c, d)$ can be chosen to intersect the line at only one point and the topology is therefore discrete.

10: Let $I = [0, 1]$. Compare the product topology on $I \times I$, the dictionary order topology, and the subspace topology from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$.

The dictionary order topology and product topologies are contained strictly in the subspace topology from the dictionary order on the plane, and are incomparable with one another.

If $((a, b), (c, d))$ is open in the dictionary topology, then it can be written, if $a = c$, as $((a, b), (a, d))$. Otherwise it is $((a, b), (a, 1)) \cup \bigcup_{a<x<c}((x, 0), (x, 1)) \cup ((c, 0), (c, d))$. Both ways of writing this are open in the subspace topology, so we have a topology inclusion. Example 3 on page 90 shows it to be strict.

A basis set for the product topology is a rectangle $U \times V$, where these are intervals open in I. This can be written $\bigcup_{x \in U} x \times V$, as a union of sets open in the subspace topology. On the other hand, the ray $1/2 \times (1/3, 2/3)$ is open in the subspace topology but not the product topology, so this is strict.

The same ray shows that the product topology does not contain the order topology. On the other hand, the rectangle $I \times (0, 1]$ is open in the product topology, but not in the order topology because an open set containing 0×1 would have to contain everything up to (a, b) for some $(a, b), a > 0$, and that would contain $(a, 0)$.

p100, #7: Criticize this proof: If $\{A_\alpha\}$ is a collection of sets in X ond if $x \in \overline{\cup A_\alpha}$ then every neighborhood U of x intersects $\cup A_\alpha$. Thus U intersects some A_α so x belongs to the closure of some A_α.

The conclusion that U intersects some A_α is true but the particular α may vary with choice of U, so there may be, for each α, a U disjoint from A_α. For instance, let $X = \mathbb{R},\ A_n = [\frac{1}{n+1}, \frac{1}{n}]$. Then $\cup A_\alpha = (0, 1]$ with closure $[0, 1]$. Let $x = 0$. Then $U_n = (-1/n, 1/n)$ is a neighborhood of x disjoint from every A_m for $m < n$. If you try to pick an A_m to have x in its closure you run into trouble because of U_{n+1}.

9: Let $A \subset X, B \subset Y$. Show that in $X \times Y, \overline{A \times B} = \overline{A} \times \overline{B}$.

First we show that $\overline{A \times B} \subset \overline{A} \times \overline{B}$.

Now, $(a, b) \in A \times B$ implies $a \in A, b \in B$, which in turn implies $a \in \overline{A}, b \in \overline{B}$, which in turn implies $(a, b) \in \overline{A} \times \overline{B}$.

$\overline{A}$ and $\overline{B}$ are closed in X and Y respectively, so $\overline{A} \times \overline{B}$ is closed in $X \times Y$. This is because $\overline{A} \times \overline{B} = \overline{A} \times Y \cap X \times \overline{B}$. (The complements of the two sets in this intersection, namely, $(X - \overline{A}) \times Y$ and $X \times (Y - \overline{B})$ are the products of open sets, hence open. So the two sets in the intersection are closed, so their intersection is closed.)

Then $\overline{A} \times \overline{B}$ is a closed set which contains $A \times B$, so it contains the intersection of all closed sets containing it, namely $\overline{A \times B}$.

Now we show that $\overline{A \times B} \supset \overline{A} \times \overline{B}$.
Say $(a, b) \in \overline{A} \times \overline{B}$, so that $a \in \overline{A} = A \cup A', b \in \overline{B} = B \cup B'$.

If $a \in A, b \in B$, then $(a, b) \in A \times B \subset \overline{A \times B}$ and we're done. Otherwise, examine a neighborhood U of $(a, b) \in X \times Y$. This neighborhood is the union of basis sets of form $U_X \times U_Y$ with U_X, U_Y open in X, Y, respectively.

Then for some $U_X, U_Y, (a, b) \in U_X \times U_Y$. By definition, this means that $a \in U_X, b \in U_Y$. Returning to a particular case, if $a \in A', b \in B$, then U_X contains a point $a' \in A - \{a\}$ and as above, U_Y contains b. Then the arbitrary neighborhood of (a, b) containing some $U_X \times U_Y$ must contain a point of $A \times B$, namely (a', b). So $(a, b) \in (A \times B)'$.

The case where $a \in A, b \in B'$ is equivalent by symmetry. Finally, if $a \in A', b \in B'$ then U_X contains an a' as above and U_Y contains a corresponding $b' \in B$ because b is a limit point of B. Then $U \supset U_X \times U_Y \ni a' \times b' \in A \times B$ and once again, $(a, b) \in (A \times B)'$. In all of the cases, $(a, b) \in (A \times B) \cup (A \times B)' = \overline{A \times B}$ and the desired result holds.

11: Show that the product of two Hausdorff spaces is Hausdorff.

Let X, Y be Hausdorff, and $(a, b), (c, d)$ be disjoint points of $X \times Y$.
If $a = c$ then $b \neq d$ so there exist disjoint open U, V in Y containing them, respectively. Then $X \times U, X \times V$ are disjoint open sets of the product containing the original two points.
If $a \neq c$ then there are disjoint open U, V in X containing them, and $U \times Y, V \times Y$ are disjoint open sets of the product containing the original two points.

21: Done by Zinger

p111, #2: Suppose that $f : X \to Y$ is continuous. If x is a limit point of $A \subset X$ is $f(x)$ necessarily a limit point of $f(A)$?

No. Let f be a constant map. Then for any A, $f(A)$ is a single point, which cannot be its own limit point, so cannot have the limit point $f(x)$.

3: Let X and X' denote a single set in the two topologies $\mathscr{T}$ and $\mathscr{T}'$ respectively. Let $i : X' \to X$ be the identity.

a Show i is continuous if and only if $\mathscr{T}$ is coarser.

   For every open set U in $X, i^{-1}(U) = U$ is open in X'. So $\mathscr{T} \subset \mathscr{T}'$, and thus $\mathscr{T}'$ is finer than $\mathscr{T}$.

   On the other hand, say $\mathscr{T}'$ is finer than $\mathscr{T}$ so that $\mathscr{T} \subset \mathscr{T}'$. Then every set U in $\mathscr{T}$ has preimage $i^{-1}(U) = U$ which must be in the superset $\mathscr{T}'$. This satisfies the requirements for continuity.

b Show i is a homeomorphism if and only if the topologies are equal.

   If i is a homeomorphism, then i is continuous and i^{-1}, the identity from X to X', is continuous. Then part a gives us that each of $\mathscr{T}$ and $\mathscr{T}'$ includes the other. Therefore they are equal.

   On the other hand, if the topologies are equal, then U is open in X' if and only if $U = i(U)$ is open in X, which makes i a homeomorphism.

6: Find a function $\mathbb{R} \to \mathbb{R}$ that is continuous at only one point.

Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

We will show that f is continuous only at 0.
Say x is rational, $x \neq 0$. Then $f(x) = x$ and $f(x)$ is in the neighborhood $(x/2, 2x)$ if $x > 0$, $(2x, x/2)$ if $x < 0$. But neither of these neighborhoods include 0, so their preimages under f do not include any irrational number. Since every neighborhood of x must include irrational numbers, the preimage of a neighborhood of $f(x)$ does not include any neighborhood of x and therefore f is not continuous at x.

Now say x is irrational. Then $f(x) = 0$ is in the neighborhood $(-|x/2|, |x/2|)$. Any neighborhood of x must contain rational numbers arbitrarily close to x, but the closest rational numbers in the preimage of the given neighborhood are over $|x/2|$ away. So there is no neighborhood of x contained in the preimage of this neighborhood and so f is again not continuous at x.

Finally consider a neighborhood V of $0 = f(0)$. This neighborhood must contain $(-\epsilon, \epsilon)$ for some $\epsilon > 0$.

Then $(-\epsilon, \epsilon)$ is a neighborhood of 0 and $f(U)$, which consists of 0 and the rationals with absolute value at most ϵ, is contained in V. The criterion for continuity at 0 is thus satisfied.

12: Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by $\frac{xy}{x^2+y^2}$ except that $F((0,0)) = 0$.

a Show that F is continuous in each variable separately.

  This function is symmetric in x and y so we will only treat x. There are two cases.
  If $y = 0$ the function is identically zero, and a constant function is continuous.
  If $y \neq 0$ the function is $\frac{xy}{x^2+y^2}$, the quotient with nonzero denominator of (continuous) polynomials.

b We compute $i : \mathbb{R} \to \mathbb{R}$ given by $g(x) = F((x,x))$.

  If $x = 0$ then $g(x) = 0$. Otherwise it is $\frac{x^2}{2x^2} = 1/2$.

c We show F is not continuous.

  g is not continuous: The preimage of the open set $(-1/2, 1/2)$ is the single point 0, which is not open. Then we can write $g = F \circ \Delta$, where Δ is the diagonal map $x \mapsto (x, x)$, which is easily seen to be continuous. The composition of continuous maps is continuous, so since g is not, f is not.