MAT 530: Topology&Geometry, I
Fall 2005

Problem Set 10

Solution to Problem p483, #2

(a) *Show that every continuous map* $f: \mathbb{R}P^2 \longrightarrow S^1$ *is null-homotopic.*
(b) *Find a continuous map from the* 2*-dimensional torus to* S^1 *which is not null-homotopic.*

(a) Let $q: \mathbb{R} \longrightarrow S^1$, $q(s) = e^{2\pi i s}$, be the standard covering map. Fix $x_0 \in \mathbb{R}P^2$. Let $b_0 = f(x_0)$. Choose $e_0 \in q^{-1}(b_0)$. Since

$$\pi_1(\mathbb{R}P^2, x_0) = \mathbb{Z}_2 \equiv \mathbb{Z}/2\mathbb{Z} \qquad \text{and} \qquad \pi_1(S^1, b_0) = \mathbb{Z},$$

the homomorphism $f_*: \pi_1(\mathbb{R}P^2, x_0) \longrightarrow \pi_1(S^1, b_0)$ must be trivial. Thus,

$$f_*\pi_1(\mathbb{R}P^2, x_0) \subset q_*\pi_1(\mathbb{R}, e_0) \subset \pi_1(S^1, b_0)$$

and by the General Lifting Lemma the map $f: (\mathbb{R}P^2, x_0) \longrightarrow (S^1, b_0)$ lifts to a continuous map

$$\tilde{f}: (\mathbb{R}P^2, x_0) \longrightarrow (\mathbb{R}, e_0),$$

i.e. $f = \tilde{f} \circ q$ as indicated in Figure 1. Since $\mathbb{R}$ is contractible, $\tilde{f}$ is null-homotopic. If $\tilde{H}$ is a homotopy from $\tilde{f}$ to the map sending $\mathbb{R}P^2$ to some $e \in \mathbb{R}$, then $H \equiv q \circ \tilde{H}$ is a homotopy from f to the map sending $\mathbb{R}P^2$ to $q(e) \in S^1$. Thus, f is null-homotopic.

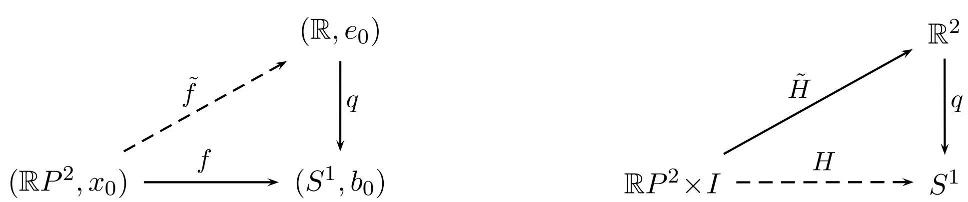

Figure 1: Diagrams for p483, #2

Remark: This argument depends only on the facts that the homomorphism f_* between the fundamental groups of the domain and the target of f is trivial and that the universal cover of the target is contractible.

(b) The 2-dimensional torus T is homeomorphic to $S^1 \times S^1$. Let

$$\pi_1: (T, x_0 \times y_0) \longrightarrow (S^1, x_0)$$

be the projection onto the first component. Since the homomorphism

$$\pi_{1*}: \pi_1(T, x_0 \times y_0) \longrightarrow \pi_1(S^1, x_0) \approx \mathbb{Z}$$

is surjective, it is not trivial. Thus, π_1 is not null-homotopic.

Solution to Problem p483, #5

Suppose $T = S^1 \times S^1$, $x_0 \in S^1$, *and* $x_0 = b_0 \times b_0$.
(a) *Show that every isomorphism of* $\pi_1(T, x_0)$ *with itself is induced by a homeomorphism of* (T, x_0) *with itself.*
(b) *If* E *is a covering space of* T, *then* E *is homeomorphic to* $\mathbb{R}^2$, $S^1 \times \mathbb{R}$, *or* $S^1 \times S^1$.

(a) Let

$$q\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad q(u,v) = \big(e^{2\pi i u}, e^{2\pi i v}\big),$$

be the standard covering map. We assume that $x_0 = q(0)$. Let

$$\tilde{\alpha}, \tilde{\beta}\colon I \longrightarrow \mathbb{R}^2, \qquad \tilde{\alpha}(s) = (s, 0) \quad \text{and} \quad \tilde{\beta}(s) = (0, s),$$

be the horizontal path running from 0 to $(1,0)$ and the vertical path running from 0 to $(0,1)$. The group $\pi_1(T, x_0)$ is the free abelian group generated by the loops

$$\alpha \equiv q \circ \tilde{\alpha} \quad \text{and} \quad \beta \equiv q \circ \tilde{\beta}, \qquad \text{i.e.} \qquad \pi_1(T, x_0) = \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta].$$

Under this isomorphism, the class $m\alpha + n\beta$ corresponds to the equivalence class of the loop $\gamma \equiv q \circ \tilde{\gamma}$ in T, where $\tilde{\gamma}$ is any path in X from 0 to $(m,n) \in \mathbb{Z}^2 \subset \mathbb{R}^2$. Via this identification, an isomorphism T of $\pi_1(T, x_0)$ with itself corresponds to a 2×2 integer matrix

$$A\colon \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta] \longrightarrow \mathbb{Z}[\alpha] \oplus \mathbb{Z}[\beta]$$

which has an integer matrix inverse B. Define

$$\tilde{h}_A, \tilde{h}_B\colon (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0) \qquad \text{by} \qquad \tilde{h}_A(v) = Av \quad \text{and} \quad \tilde{h}_B(v) = Bv.$$

Since the maps

$$q \circ \tilde{h}_A, q \circ \tilde{h}_B\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \longrightarrow T$$

are constant along the fibers of quotient projection map $q\colon \mathbb{R}^2 \longrightarrow T$, they induce maps

$$h_A, h_B\colon T \longrightarrow T \qquad \text{s.t} \qquad q \circ \tilde{h}_A = h_A \circ q \quad \text{and} \quad q \circ \tilde{h}_B = h_B \circ q;$$

see Figure 2.

$$\begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{\tilde{h}_A} & (\mathbb{R}^2, 0) \\ \downarrow q & & \downarrow q \\ (\mathbb{R}^2, 0) & \overset{h_A}{\dashrightarrow} & (\mathbb{R}^2, 0) \end{array} \qquad\qquad \begin{array}{ccc} (\mathbb{R}^2, 0) & \xrightarrow{\tilde{h}_B} & (\mathbb{R}^2, 0) \\ \downarrow q & & \downarrow q \\ (\mathbb{R}^2, 0) & \overset{h_B}{\dashrightarrow} & (\mathbb{R}^2, 0) \end{array}$$

Figure 2: Diagrams for p483, #5a

Since q (on the left of each diagram) is a quotient map and the maps $q \circ \tilde{h}_A$ and $q \circ \tilde{h}_B$ are continuous (and open), so are the maps h_A and h_B. Since

$$\tilde{h}_A \circ \tilde{h}_B = AB = \mathrm{Id} = \mathrm{id}_{\mathbb{R}^2} \qquad \text{and} \qquad \tilde{h}_B \circ \tilde{h}_A = BA = \mathrm{Id} = \mathrm{id}_{\mathbb{R}^2},$$

$h_A \circ h_B = \mathrm{id}_T$ and $h_B \circ h_A = \mathrm{id}_T$. In particular, $h_A\colon (T,x_0) \longrightarrow (T,x_0)$ is a homeomorphism. Since

$$\tilde{h}_A(0)=0 \qquad q\circ\tilde{h}_A\circ\tilde{\alpha}=h_A\circ q\circ\tilde{\alpha}, \qquad q\circ\tilde{h}_A\circ\tilde{\beta}=h_A\circ q\circ\tilde{\beta},$$
$$\{\tilde{h}_A\circ\tilde{\alpha}\}(1)=A\tilde{\alpha}(1)=A(1,0)^t, \qquad \{\tilde{h}_A\circ\tilde{\beta}\}(1)=A\tilde{\beta}(1)=A(0,1)^t,$$

it follows that

$$h_{A*}[\alpha]=[q\circ\tilde{h}_A\circ\tilde{\alpha}]=A(1,0)^t=T\alpha \qquad \text{and} \qquad h_{A*}[\beta]=[q\circ\tilde{h}_A\circ\tilde{\beta}]=A(0,1)^t=T\beta.$$

Thus, $h_* = T$ as needed.

(b) If $p\colon E \longrightarrow T$ is covering map, choose $e_0 \in p^{-1}(x_0)$. Let

$$H_{e_0}=p_*\pi_1(E,e_0)\subset\pi_1(T,x_0).$$

Since $\pi_1(T,x_0)$ is a free abelian group of rank 2, there exists a basis $\{e_1,e_2\}$ for $\pi_1(T,x_0)$ such that
(i) $\{me_1, ne_2\}$ is a basis for H_{e_0} for some $m,n\in\mathbb{Z}^+$, or
(ii) $\{me_1\}$ is a basis for H_{e_0} for some $m\in\mathbb{Z}^+$, or
(iii) $H_{e_0}=\{0\}$.
By part (a), there exists a homeomorphism $h\colon (T,x_0)\longrightarrow (T,x_0)$ such that

$$h_*e_1=[\alpha] \qquad \text{and} \qquad h_*e_2=[\beta].$$

Then, $p' \equiv h\circ p\colon (E,e_0)\longrightarrow (T,x_0)$ is a covering map and
(i) $\{m\alpha, n\beta\}$ is a basis for $p'_*\pi_1(E,e_0)$ for some $m,n\in\mathbb{Z}^+$, or
(ii) $\{m\alpha\}$ is a basis for $p'_*\pi_1(E,e_0)$ for some $m\in\mathbb{Z}^+$, or
(iii) $p'_*\pi_1(E,e_0)=\{0\}$.
Let $\bar{p}\colon (\bar{E},\bar{e}_0)\longrightarrow (T,x_0)$, be the covering map given by
(i) $\tilde{E}=S^1\times S^1$, $\tilde{p}(w,z)=(w^m,z^n) \implies \bar{p}_*\pi_1(\bar{E},\bar{e}_0)=\mathbb{Z}\{m\alpha,n\beta\}$;
(ii) $\tilde{E}=S^1\times\mathbb{R}$, $\tilde{p}(w,v)=(w^m,e^{2\pi iv}) \implies \bar{p}_*\pi_1(\bar{E},\bar{e}_0)=\mathbb{Z}\{m\alpha\}$;
(iii) $\tilde{E}=\mathbb{R}\times\mathbb{R}$, $\tilde{p}(u,v)=(e^{2\pi iu},e^{2\pi iv}) \implies \bar{p}_*\pi_1(\bar{E},\bar{e}_0)=\{0\}$.
Since $p'_*\pi_1(E,e_0)=\bar{p}_*\pi_1(\bar{E},\bar{e}_0)\subset\pi_1(T,x_0)$, the covering maps (E,p',T) and $(\bar{E},\bar{p},T)$ are equivalent. In particular, there exists a homeomorphism $g\colon E\longrightarrow\bar{E}$. Thus, E is homeomorphic to $S^1\times S^1$, $S^1\times\mathbb{R}$, or $\mathbb{R}^2$ depending on the case.

Solution to Problem p493, #5

If n and k are relatively prime positive numbers, let h be the map of $S^3\subset\mathbb{C}^2$ to itself given by

$$h\colon S^3\longrightarrow S^3, \qquad h(z_1,z_2)=\big(e^{2\pi i/n}z_1, e^{2\pi ik/n}z_2\big).$$

(a) *Show that h generates a subgroup G of the homeomorphism group of S^3 and that the action of G is fixed-point free.*
(b) *The lens space $L(n,k)$ is the quotient space S^3/G. Show that if $L(n,k)$ is homeomorphic to $L(n',k')$, then $n=n'$.*
(c) *Show that $L(n,k)$ is a (smooth) compact 3-manifold.*

(a) If $m \in \mathbb{Z}^+$, then the map h^m is given by

$$h^m \colon S^3 \longrightarrow S^3, \qquad h(z_1, z_2) = \big(e^{2\pi i m/n} z_1, e^{2\pi i k m/n} z_2\big).$$

In particular, h^n is the identity map on S^3 and thus $(h^m)^{-1} = h^{n-m}$. Since h is continuous, so are its composites h^m. Thus, all maps h^m are homeomorphisms, and h generates a subgroup G of the homeomorphism group of S^3. Furthermore,

$$\begin{aligned} h^m(z_1, z_2) = (z_1, z_2) \ \ \forall\, (z_1, z_2) \in S^3 \quad &\Longleftrightarrow \quad e^{2\pi i m/n} z_1 = z_1, \ e^{2\pi i k m/n} z_2 = z_2 \ \ \forall\, (z_1, z_2) \in S^3 \\ &\Longleftrightarrow \quad e^{2\pi i m/n} = 1, \ e^{2\pi i k m/n} = 1. \end{aligned}$$

Thus, $h^m = \mathrm{id}$ if and only if m is divisible by n, i.e. G is a cyclic group of order n. Finally, if $(z_1, z_2) \in S^3$,

$$h^m(z_1, z_2) = (z_1, z_2) \qquad \Longleftrightarrow \qquad e^{2\pi i m/n} z_1 = z_1, \ e^{2\pi i k m/n} z_2 = z_2.$$

Since either $z_1 \neq 0$ or $z_2 \neq 0$, this implies that either $e^{2\pi i m/n} = 1$ or $e^{2\pi i k m/n} = 1$. Since k and n are relatively prime, the two conditions are equivalent to m being divisible by n, i.e. $h^m = \mathrm{id}$. So, the action of G is fixed point-free.

(b) Since the action of the *finite* group G on the *Hausdorff* space S^3 is fixed-point free, by p493, #4 this action is properly discontinuous as well. Thus, the quotient map

$$q \colon S^3 \longrightarrow L(n,k) = S^3/G$$

is a covering map. Since S^3 is simply connected,

$$\pi_1\big(L(n,k), x_0\big) = G \approx \mathbb{Z}_n \equiv Z/n\mathbb{Z}.$$

If $L(n,k)$ is homeomorphic to $L(n',k')$, $\pi_1(L(n,k), x_0)$ and $\pi_1(L(n',k'), x_0')$ must be isomorphic and in particular must have the same cardinality, i.e. $n = n'$.

(c) Since the map q in part (b) is a quotient map and S^3 is compact, so is $L(n,k)$. Since q is a covering map and S^3 is Hausdorff, so is $L(n,k)$. If $V \subset L(n,k)$ is an open set evenly covered by q, V is homeomorphic to a proper open subset U of S^3. Since $S^3 - \{pt\}$ is homeomorphic to $\mathbb{R}^3$, it follows that every point in $L(n,k)$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^3$. We conclude that S^3 is a compact *topological* 3-manifold.

Remark: In contrast to higher-dimensional manifolds, every 3-dimensional topological manifold admits a unique smooth structure (this was proved by Munkres). So, $L(n,k)$ with its unique smooth structure is a compact smooth 3-manifold. In fact, h is a diffeomorphism of S^3. Thus, the group G is a subgroup of the diffeomorphism group of S^3 and the smooth structure on S^3 induces a smooth structure on $L(n,k)$.