MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2008

Homework Assignment IV

Reminder: Your grade in this class will be based on class participation and the problem sets. Therefore, you are expected to regularly attend classes and contribute to the discussion (with presentations, questions, and comments), as well as to submit solutions to the written assignments and study the discussion problems ahead of time even if you won't be presenting them. Written solutions must be turned in by the beginning of class on the due date (typed solutions can also be e-mailed by the same time). **Late problem sets will not be accepted**.

Written Assignment due on Tuesday, 10/28, at 11:20am in Physics P-117
(or by 10/28, 11am, in Math 3-111)

Please do 5 of the following problems with Problem G counted as 3 problems and Problem H as 2 problems: Chapter 5 #1,2; Problems D-H below

Problem D

Let M be a smooth manifold of dimension m. Suppose X and Y are compact smooth manifolds of dimensions k and $m-k$, respectively, and $f: X \longrightarrow M$ and $g: Y \longrightarrow M$ are smooth maps that intersect transversally in M. Recall that the last condition means that for each point

$$(x_0, y_0) \in f \cap g \equiv \big\{(x,y) \in X \times Y:\ f(x) = g(y)\big\},$$

there exist an open neighborhood U_p of $p = f(x_0) = g(y_0)$ in M (i.e. U_p is an open subset of M and $p \in U_p$), a smooth chart

$$\varphi_p: U_p \longrightarrow \mathbb{R}^m,$$

and open neighborhoods V_{x_0} of x_0 in $f^{-1}(U_p)$ and W_{y_0} of y_0 in $g^{-1}(U_p)$ such that

$$\varphi_p \circ f: V_{x_0} \longrightarrow \mathbb{R}^k \times 0^{m-k} \qquad \text{and} \qquad \varphi_p \circ g: W_{y_0} \longrightarrow 0^k \times \mathbb{R}^{m-k}$$

are charts. In particular, φ_p is a homeomorphism. Show that
(a) $f \cap g$ is a compact subset of $X \times Y$;
(b) $f \cap g$ is finite.

Problem E

Let M_1 and M_2 be $k_1 \times (n+1)$ and $k_2 \times (n+1)$-matrices of full rank, with $k_1, k_2 \le n$. Thus,

$$\ker M_i \equiv \big\{X \in \mathbb{C}^{n+1}: M_i X = 0 \in \mathbb{C}^{k_i}\big\}$$

is a linear subspace of $\mathbb{C}^{n+1}$ of dimension $n+1-k_i$, while

$$\mathbb{P}(\ker M_i) \equiv \big\{[X] \in \mathbb{C}P^n: M_i X = 0 \in \mathbb{C}^{k_i}\big\}$$

is a "linear" subspace of $\mathbb{C}P^n$ which is isomorphic to $\mathbb{C}P^{n-k_i}$. Find the necessary and sufficient conditions on (M_1, M_2) so that $\mathbb{P}(\ker M_1)$ and $\mathbb{P}(\ker M_2)$ are transverse in $\mathbb{C}P^n$. Recall that the latter means that for every point $p\in\mathbb{P}(\ker M_1)\cap\mathbb{P}(\ker M_2)$ there exist an open neighborhood U_p of p in M (see Problem D) and an analytic function

$$\varphi_p\colon U_p \longrightarrow \mathbb{C}^{k_1+k_2} \qquad \text{s.t.} \quad \varphi_p^{-1}(\mathbb{C}^{k_2}\times 0^{k_1}) = X\cap U_p, \quad \varphi_p^{-1}(0^{k_2}\times\mathbb{C}^{k_1}) = Y\cap U_p$$

and the complex Jacobian of φ_p at p has full rank.

Problem F

Let $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be an analytic map; in particular, it is a smooth map $f\colon \mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n}$.
(a) Relate the complex and real jacobians of f:

$$J_{\mathbb{C}}(f) = \left(\frac{\partial f_i}{\partial z_j}\right)_{i,j=1,\ldots,n}, \qquad J_{\mathbb{R}}(f) = \begin{pmatrix} \left(\frac{\partial g_i}{\partial x_j}\right)_{i,j=1,\ldots,n} & \left(\frac{\partial g_i}{\partial y_j}\right)_{i,j=1,\ldots,n} \\ \left(\frac{\partial h_i}{\partial x_j}\right)_{i,j=1,\ldots,n} & \left(\frac{\partial h_i}{\partial y_j}\right)_{i,j=1,\ldots,n} \end{pmatrix},$$

where $z_j = x_i + \mathfrak{i} y_j$, $f=(f_1,\ldots,f_n)$, $f_i = g_i + \mathfrak{i} h_i$;
(b) Show that any biholomorphism (analytic diffeomorphism) between open subsets of $\mathbb{C}^n$ is orientation-preserving (the determinant of the real jacobian is positive).

Problem G

One would expect a typical system of n (polynomial) equations in $n-1$ variables to have no solutions. The space of all such systems forms a complex vector space; this problem will show that the "atypical" systems are contained in the zero set of a non-zero analytic function on this vector space. Following a principle introduced in Chapter 1, we approach this problem by studying systems of homogeneous polynomials of the same degree in n variables.

If V is any complex vector space and $d\in\mathbb{Z}^+$, let

$$HP_d(V) \equiv \big\{ f\colon V\longrightarrow\mathbb{C} \,|\, f(\lambda v) = \lambda^d v \ \forall\, v\in V,\, \lambda\in\mathbb{C} \big\}$$

be the space of homogeneous polynomials of degree d on V. For any $n\in\mathbb{Z}^+$, denote by nV the direct sum of n copies of V, e.g.

$$3V = V\oplus V\oplus V,$$
$$a\cdot(u_1,u_2,u_3) + b\cdot(v_1,v_2,v_3) = (au_1+bv_1, au_2+bv_2, au_3+bv_3) \quad \forall\, a,b\in\mathbb{C},\, u_i,v_i\in V.$$

Show that there exists an n-linear map (map linear in each of the n-inputs)

$$P\colon n\,HP_d(\mathbb{C}^n) \longrightarrow HP_{dn}(n\mathbb{C}^n)$$

such that $P(f_1,\ldots,f_n)=0$ if and only if the system

$$\begin{cases} f_1(v) = 0, \\ \vdots \\ f_n(v) = 0, \end{cases} \tag{*}$$

has a solution $v\in\mathbb{C}^n-0$. The following are suggested steps, which you do not need to follow:
(a) if $\alpha_1,\ldots,\alpha_n\colon\mathbb{C}^n\longrightarrow\mathbb{C}$ are linear maps, let

$$D(\alpha_1,\ldots,\alpha_n)=\alpha_1\wedge\ldots\wedge\alpha_n\in HP_n(n\mathbb{C}^n)$$

be the "determinant" function:

$$\big\{D(\alpha_1,\ldots,\alpha_n)\big\}(v_1,\ldots,v_n)=\det\big(\alpha_i(v_j)\big)_{i,j=1,\ldots,n}\qquad\forall\ v_1,\ldots,v_n\in\mathbb{C}^n.$$

If $M\in GL_n(\mathbb{C})$, show that

$$\big\{D(\alpha_1,\ldots,\alpha_n)\big\}(Mv_1,\ldots,Mv_n)=(\det M)\cdot D(\alpha_1,\ldots,\alpha_n)$$

and we can take $P=D$ if $d=1$;
(b) Let $f_1=\alpha_{1,1}\ldots\alpha_{1,d}$, $\ldots$, $f_n=\alpha_{n,1}\ldots\alpha_{n,d}$ be degree d homogeneous polynomials on $\mathbb{C}^n$ written out as products of linear factors (most polynomials are *not* products of linear factors). Set

$$D_d(f_1,\ldots,f_n)=\sum_{\tau_2,\ldots,\tau_n\in S_d}\prod_{r=1}^{r=d}D(\alpha_{1,r},\alpha_{2,\tau_2(r)},\ldots,\alpha_{n,\tau_n(n)}),$$

where S_d is the group of permutations of d elements. If $f_1^{(1)},\ldots,f_1^{(m)}$ are products of d linear factors, show that

$$f_1^{(1)}+\ldots+f_1^{(m)}=0\qquad\Longrightarrow\qquad\sum_{s=1}^{m}D_d(f_1^{(s)},f_2,\ldots,f_n)=0.$$

Conclude that D_d induces a well-defined multi-linear map $n\,HP_d(\mathbb{C}^n)\longrightarrow HP_{dn}(n\mathbb{C}^n)$.
(c) If $M\in GL_n(\mathbb{C})$, show that

$$\big\{D_d(f_1,\ldots,f_n)\big\}(Mv_1,\ldots,Mv_n)=(\det M)^d\cdot D(f_1,\ldots,f_n)$$

and that D_d has the desired property.

Thus, n-tuples $(f_1,\ldots,f_n)$ of homogeneous polynomials of degree d on $\mathbb{C}^n$ for which the system (*) has a nonzero solution lie in the zero set of a nonzero holomorphic function P on the vector space of all such polynomials. Formulate and prove the analogous statement for a system of n polynomials (not necessary homogeneous) on $\mathbb{C}^{n-1}$.

Problem H

Let M, X, and Y be smooth manifolds of dimensions m, k, and l, respectively, and $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ transverse maps. Recall that the last condition means that for every

$$(x_0,y_0)\in f\cap g\equiv\big\{(x,y)\in X\times Y:\ f(x)=g(y)\big\},$$

there exists a chart $\varphi\colon U_p\longrightarrow\mathbb{R}^m$ around $p=f(x_0)=g(y_0)$ such that the smooth map

$$\psi\colon f^{-1}(U)\times g^{-1}(U)\longrightarrow\mathbb{R}^m,\qquad (x,y)\longrightarrow\varphi(f(x))-\varphi(g(y)),$$

has full rank at (x_0,y_0) (i.e. the rank of Jacobian of ψ at (x_0,y_0) is m).
(a) Show that there exists a chart $\tilde\varphi\colon\tilde U\longrightarrow\mathbb{R}^{k+l-m}\times\mathbb{R}^m$ around (x_0,y_0) on $X\times Y$ such that the second component of $\tilde\varphi$ is ψ.
(b) Thus, $\tilde\varphi$ restricts to a homeomorphism $\phi\colon(f\cap g)\cap U\longrightarrow\mathbb{R}^{k+l-m}$. Show that any two homeomorphisms obtained in this way overlap smoothly (thus $f\cap g$ is a smooth manifold).
(c) If ϕ_1 and ϕ_2 are obtained in this way from pairs (φ_1,φ_2) and $(\tilde\varphi_1,\tilde\varphi_2)$ that overlap positively, so do ϕ_1 and ϕ_2 (i.e. the determinant of the Jacobian of $\phi_1^{-1}\circ\phi_2$ is everywhere positive). It follows that $f\cap g$ is a smooth oriented manifold if M, X and Y are.

Discussion Topic

Grassmannians of 2-planes

Day 1: Topology and Intersection Theory of $G(2,4)$:

- Describe what $G(2,4)$ is and prove that the two topologies of Problem A on PS3 are the same.
- Describe charts on $G(2,4)$ and show that they overlap analytically so that $G(2,4)$ is a complex manifold. What is its dimension?
- Describe cycles on $G(2,4)$ and their intersections.
- Describe applications to counting lines in $\mathbb{C}^3$.

This is closely related to Example 4.22 and pp95-98, but you will need to fill in all the details. In particular, for the 3rd part above, you'll need to describe bordisms between different cycles.

Day 2: Topology and Intersection Theory of $G(2,n)$:

- Describe what $G(2,n)$ is and prove that the analogues of the two topologies of Problem A on PS3 are the same.
- Describe charts on $G(2,n)$ and show that they overlap analytically so that $G(2,n)$ is a complex manifold. What is its dimension?
- Describe cycles on $G(2,n)$ and their intersections. In particular, verify the following formulas:
  - if $n, a_1, \ldots, a_k, b_1, \ldots, b_k$ are non-negative integers,
  $$\big\langle \sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}, G(2,n) \big\rangle = \big\langle \sigma_{a_1 - b_1} \cdot \ldots \cdot \sigma_{a_k - b_k}, G(2, n - b_1 - \ldots - b_k) \big\rangle\,;$$
  - if n, a, b, a', b' are non-negative integers and $n-2 \ge a \ge b \ge 0$,
  $$\big\langle \sigma_{ab}\sigma_{a'b'}, G(2,n) \big\rangle = \begin{cases} 1, & \text{if } a' = n-2-b,\ b' = n-2-a; \\ 0, & \text{otherwise;} \end{cases}$$
  - if $n, a_1, a_2, a_3 \in \mathbb{Z}^+$ are such that $n-2 \ge a_1, a_2, a_3 \ge 0$,
  $$\big\langle \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}, G(2,n) \big\rangle = \begin{cases} 1, & \text{if } a_1 + a_2 + a_3 = 2n-4; \\ 0, & \text{otherwise;} \end{cases}$$
  - if $a_1, a_2 \ge 0$,
  $$\sigma_{a_1} \cdot \sigma_{a_2} = \sum_{c \ge a_1, a_2} \sigma_{c, a_1 + a_2 - c}.$$

  This is a special case of Pieri's formula for $G(2,n)$. In light of the previous identities, the full statement of Theorem 7.1 is not necessary for the purpose of computing intersection on Grassmannians of *two*-planes.

This is closely related to pp99-101, but you will need to give far more details.