MAT 401: Undergraduate Seminar
Introduction to Enumerative Geometry
Fall 2018

Homework Assignment IV

Written Assignment due on Tuesday, 10/23, at 1pm in ESS 181
(or by 10/23, noon, in Math 3-111)

Please do 5 of the following problems with Problem F counted as 2 problems and Problem I as 3 problems: Chapter 5 #1,2; Problems E-I below

Problem E

Let M be a smooth manifold of dimension m. Suppose X and Y are compact smooth manifolds of dimensions k and $m-k$, respectively, and $f: X \longrightarrow M$ and $g: Y \longrightarrow M$ are smooth maps that intersect transversally in M. Recall that the last condition means that for each point

$$(x_0, y_0) \in f \cap g \equiv \big\{(x,y)\in X\times Y:\ f(x)=g(y)\big\},$$

there exist an open neighborhood U_p of $p=f(x_0)=g(y_0)$ in M (i.e. U_p is an open subset of M and $p\in U_p$), a smooth chart

$$\varphi_p\colon U_p \longrightarrow \mathbb{R}^m,$$

and open neighborhoods V_{x_0} of x_0 in $f^{-1}(U_p)$ and W_{y_0} of y_0 in $g^{-1}(U_p)$ such that

$$\varphi_p\circ f\colon V_{x_0} \longrightarrow \mathbb{R}^k\times 0^{m-k} \qquad \text{and} \qquad \varphi_p\circ g\colon W_{y_0} \longrightarrow 0^k\times\mathbb{R}^{m-k}$$

are charts. In particular, φ_p is a homeomorphism. Show that

(a) $f\cap g$ is a compact subset of $X\times Y$;

(b) $f\cap g$ is finite.

Problem F

Let M be a smooth manifold of dimension m and $\mathbb{I}=[0,1]$. Suppose X and Y are smooth manifolds of dimensions k and $m-k$, respectively, and

$$F\colon \mathbb{I}\times X \longrightarrow M \qquad \text{and} \qquad g\colon Y\longrightarrow M$$

are smooth maps so that the map

$$f_t\colon X \longrightarrow M, \qquad f_t(x) = F(t,x),$$

is transverse to g for every $t\in\mathbb{I}$. Show that

(a) $F\cap g\subset\mathbb{I}\times X\times Y$ is a smooth one-dimensional submanifold and the restriction of the projection $\pi_{\mathbb{I}}: F\cap g \longrightarrow \mathbb{I}$ to each connected component of $F\cap g$ is a diffeomorphism onto an open subset of $\mathbb{I}$;

(b) if X and Y are compact, then $F\cap g\subset\mathbb{I}\times X\times Y$ is a compact subset and the restriction of the projection $\pi_{\mathbb{I}}\colon F\cap g\longrightarrow\mathbb{I}$ to each connected component of $F\cap g$ is a diffeomorphism onto $\mathbb{I}$;

(c) if X and Y are compact, then $f_t\cap g\subset X\times Y$ is a finite subset of cardinality independent of $t\in\mathbb{I}$.

Problem G

Let M_1 and M_2 be $k_1\times(n+1)$ and $k_2\times(n+1)$-matrices of full rank, with $k_1,k_2\leq n$. Thus,

$$\ker M_i \equiv \big\{X\in\mathbb{C}^{n+1}\colon M_iX=0\in\mathbb{C}^{k_i}\big\}$$

is a linear subspace of $\mathbb{C}^{n+1}$ of dimension $n+1-k_i$, while

$$\mathbb{P}(\ker M_i) \equiv \big\{[X]\in\mathbb{C}P^n\colon M_iX=0\in\mathbb{C}^{k_i}\big\}$$

is a "linear" subspace of $\mathbb{C}P^n$ which is isomorphic to $\mathbb{C}P^{n-k_i}$. Find the necessary and sufficient conditions on (M_1,M_2) so that $\mathbb{P}(\ker M_1)$ and $\mathbb{P}(\ker M_2)$ are transverse in $\mathbb{C}P^n$. Recall that the latter means that for every point $p\in\mathbb{P}(\ker M_1)\cap\mathbb{P}(\ker M_2)$ there exist an open neighborhood U_p of p in $\mathbb{C}P^{n-k_i}$ (see Problem E) and a holomorphic function

$$\varphi_p\colon U_p\longrightarrow\mathbb{C}^{k_1+k_2}\quad\text{s.t.}\quad \varphi_p^{-1}(\mathbb{C}^{k_2}\times 0^{k_1})=\mathbb{P}(\ker M_1)\cap U_p,\quad \varphi_p^{-1}(0^{k_2}\times\mathbb{C}^{k_1})=\mathbb{P}(\ker M_2)\cap U_p,$$

and the complex Jacobian of φ_p at p has full rank.

Problem H

Let $f\colon\mathbb{C}^n\longrightarrow\mathbb{C}^n$ be an analytic map; in particular, it is a smooth map $f\colon\mathbb{R}^{2n}\longrightarrow\mathbb{R}^{2n}$.
(a) Relate the complex and real Jacobians of f:

$$J_{\mathbb{C}}(f)=\left(\frac{\partial f_i}{\partial z_j}\right)_{i,j=1,\dots,n},\qquad J_{\mathbb{R}}(f)=\begin{pmatrix}\left(\frac{\partial g_i}{\partial x_j}\right)_{i,j=1,\dots,n} & \left(\frac{\partial g_i}{\partial y_j}\right)_{i,j=1,\dots,n}\\ \left(\frac{\partial h_i}{\partial x_j}\right)_{i,j=1,\dots,n} & \left(\frac{\partial h_i}{\partial y_j}\right)_{i,j=1,\dots,n}\end{pmatrix},$$

where $z_j=x_i+\mathfrak{i}y_j$, $f=(f_1,\dots,f_n)$, $f_i=g_i+\mathfrak{i}h_i$;
(b) Show that any biholomorphism (holomorphic diffeomorphism) between open subsets of $\mathbb{C}^n$ is orientation-preserving (the determinant of the real Jacobian is positive).

Problem I

Let M, X, and Y be smooth manifolds of dimensions m, k, and l, respectively, and $f\colon X\longrightarrow M$ and $g\colon Y\longrightarrow M$ be transverse maps. Recall that the last condition means that for every

$$(x_0,y_0)\in f\cap g\equiv\big\{(x,y)\in X\times Y\colon\ f(x)=g(y)\big\},$$

there exists a chart $\varphi\colon U_p\longrightarrow\mathbb{R}^m$ around $p=f(x_0)=g(y_0)$ such that the smooth map

$$\psi\colon f^{-1}(U_)\times g^{-1}(U_p)\longrightarrow\mathbb{R}^m,\qquad \phi(x,y)=\varphi(f(x))-\varphi(g(y)),$$

has full rank at (x_0,y_0) (i.e. the rank of Jacobian of ψ at (x_0,y_0) is m).

(a) Show that there exists a chart $\widetilde{\varphi}\colon\widetilde{U}\longrightarrow\mathbb{R}^{k+l-m}\times\mathbb{R}^m$ around (x_0,y_0) on $X\times Y$ such that the second component of $\widetilde{\varphi}$ is ψ.

(b) Thus, $\widetilde{\varphi}$ restricts to a homeomorphism $\phi\colon(f\cap g)\cap U\longrightarrow\mathbb{R}^{k+l-m}$. Show that any two homeomorphisms obtained in this way overlap smoothly (thus $f\cap g$ is a smooth manifold).

(c) If ϕ_1 and ϕ_2 are obtained in this way from pairs (φ_1,φ_2) and $(\widetilde{\varphi}_1,\widetilde{\varphi}_2)$ that overlap positively, so do ϕ_1 and ϕ_2 (i.e. the determinant of the Jacobian of $\phi_1^{-1}\circ\phi_2$ is everywhere positive).

It follows that $f\cap g$ is a smooth oriented manifold if M, X and Y are.

Discussion Topic

Grassmannians of 2-planes

Day 1: Topology and Intersection Theory of $G(2,4)$:

- Recall what $G(2,4)$ is and the two definitions of its topology in Problem B on PS3.
- Recall the charts on $G(2,4)$. What is its dimension?
- Describe cycles on $G(2,4)$, their intersections, and relations with the Young tableaux in Aaron's talk.
- Describe applications to counting lines in $\mathbb{C}^3$.

This is closely related to Example 4.22 and pp95-98, but you will need to fill in all the details. In particular, for the 3rd part above, you'll need to describe bordisms between different cycles.

Days 2,3: Topology and Intersection Theory of $G(2,n)$:

- Describe what $G(2,n)$ is and prove that the analogues of the two topologies of Problem B on PS3 are the same.
- Describe charts on $G(2,n)$ and show that they overlap analytically so that $G(2,n)$ is a complex manifold. What is its dimension?
- Describe cycles on $G(2,n)$ and relations with Young tableaux.
- Describe the intersections of cycles on $G(2,n)$ and relations with Young tableaux. In particular, verify the following formulas:
    - if $n, a_1, \ldots, a_k, b_1, \ldots, b_k$ are non-negative integers,
    $$\big\langle \sigma_{a_1 b_1} \cdot \ldots \cdot \sigma_{a_k b_k}, G(2,n) \big\rangle = \big\langle \sigma_{a_1 - b_1} \cdot \ldots \cdot \sigma_{a_k - b_k}, G(2, n - b_1 - \ldots - b_k) \big\rangle \,;$$
    - if n, a, b, a', b' are non-negative integers and $n-2 \geq a \geq b \geq 0$,
    $$\big\langle \sigma_{ab}\sigma_{a'b'}, G(2,n) \big\rangle = \begin{cases} 1, & \text{if } a' = n-2-b, \ b' = n-2-a; \\ 0, & \text{otherwise;} \end{cases}$$
    - if $n, a_1, a_2, a_3 \in \mathbb{Z}^+$ are such that $n-2 \geq a_1, a_2, a_3 \geq 0$,
    $$\big\langle \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}, G(2,n) \big\rangle = \begin{cases} 1, & \text{if } a_1 + a_2 + a_3 = 2n-4; \\ 0, & \text{otherwise;} \end{cases}$$
    - if $a_1, a_2 \geq 0$,
    $$\sigma_{a_1} \cdot \sigma_{a_2} = \sum_{c \geq a_1, a_2} \sigma_{c, a_1 + a_2 - c}.$$

    This is a special case of Pieri's formula for $G(2,n)$. In light of the previous identities, the full statement of Theorem 7.1 is not necessary for the purposes of computing intersection on Grassmannians of *two*-planes.

This is closely related to pp99-101, but you will need to give far more details. The supplementary notes may help with this.