MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 8

Problem 1 (10pts)

Let $(V,\langle\cdot,\cdot\rangle)$ be an inner-product space over $\mathbb{C}$ and $e_1,e_2,\ldots\in V$ be a sequence of orthonormal vectors, i.e.

$$\langle e_i,e_j\rangle=\begin{cases}1, & \text{if } i=j;\\ 0, & \text{if } i\neq j.\end{cases} \tag{1}$$

Let $|\cdot|$ be the norm on V determined by $\langle\cdot,\cdot\rangle$. Show that

(a) *the collection $\{e_i: i\in\mathbb{Z}^+\}$ is linearly independent (no non-trivial finite linear combination of e_i's with complex coefficients adds up to 0);*

(b) $v-\sum\limits_{i=1}^{i=k}\langle v,e_i\rangle e_i$ *is orthogonal to $e_1,\ldots,e_k$ for all $v\in V$;*

(c) $\sum\limits_{i=1}^{i=k}\big|\langle v,e_i\rangle\big|^2\le|v|^2$ *for all $v\in V$ and $k\in\mathbb{Z}$;*

(d) *the sequence $v_k\equiv\sum\limits_{i=1}^{i=k}\langle v,e_i\rangle e_i$ is Cauchy in $(V,\langle\cdot,\cdot\rangle)$.*

(a; **2pts**) Suppose $c_1,\ldots,c_k\in\mathbb{C}$ are such that $\sum\limits_{i=1}^{i=k}c_ie_i=0$. Taking inner-product of this equality with e_j for $j=1,\ldots,k$ and using (1), we obtain

$$c_j=\sum_{i=1}^{i=k}c_i\langle e_i,e_j\rangle=0\,.$$

Thus, $c_j=0$ for all $j=1,\ldots,k$.

(b; **2pts**) Let $j=1,\ldots,k$. By (1),

$$\Big\langle v-\sum_{i=1}^{i=k}\langle v,e_i\rangle e_i,e_j\Big\rangle=\langle v,e_j\rangle-\sum_{i=1}^{i=k}\langle v,e_i\rangle\langle e_i,e_j\rangle=\langle v,e_j\rangle-\langle v,e_j\rangle=0.$$

This establishes the claim.

(c; **3pts**) Let v_k denote the sum in (b) and v_k^c the difference. By (1),

$$|v_k|_2^2=\sum_{i=1}^{i=k}\big|\langle v,e_i\rangle\big|^2|e_i|^2+\sum_{i\neq j}\langle v,e_i\rangle\overline{\langle v,e_j\rangle}\langle e_i,e_j\rangle=\sum_{i=1}^{i=k}\big|\langle v,e_i\rangle\big|^2\,. \tag{2}$$

By (b), $\langle v_k^c,v_k\rangle=0$. Since $v=v_k+v_k^c$,

$$|v|^2=|v_k|^2+|v_k^c|^2+2\,\mathrm{Re}\langle v_k^c,v_k\rangle=|v_k|^2+|v_k^c|^2\,.$$

Along with (2), this establishes the claim.

(d; **3pts**) If $k\le\ell$, then

$$\big|v_k-v_\ell\big|^2=\bigg|\sum_{i=k+1}^{i=\ell}\langle v,e_i\rangle e_i\bigg|^2=\sum_{i=k+1}^{i=\ell}\big|\langle v,e_i\rangle\big|^2.$$

By (c), the right-hand side above approaches 0 as $k,\ell\longrightarrow\infty$. Thus, the sequence v_k is Cauchy in $(V,\langle\cdot,\cdot\rangle)$.

Problem 2 (20pts)

Let $\mathbb{I}=[0,1]$, $L^2(\mathbb{I};\mathbb{C})$ be the L^2-space of $\mathbb{C}$-valued functions on $\mathbb{I}$, and $f\in L^2(\mathbb{I};\mathbb{C})$. For each $n\in\mathbb{Z}$, define

$$\psi_n\colon\mathbb{I}\longrightarrow\mathbb{C},\quad \psi_n(x)=\mathrm{e}^{2\pi\mathfrak{i}\,nx},\qquad c_n(f)=\langle\!\langle f,\psi_n\rangle\!\rangle_2\equiv\int_{\mathbb{I}}f\overline{\psi}_n\mathrm{d}m\in\mathbb{C}. \tag{3}$$

(a) *Show that the collection $\{\psi_n\colon n\in\mathbb{Z}^+\}$ consists of orthonormal elements of $L^2(\mathbb{I};\mathbb{C})$.*

(b) *Show that the sum*

$$\sum_{n\in\mathbb{Z}}c_n(f)\psi_n\equiv\lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m}c_n(f)\psi_n$$

converges with respect to the L^2-norm to some $h_f\in L^2(\mathbb{I};\mathbb{C})$ so that $\|h_f\|_2\le\|f\|_2$ and $\langle\!\langle f-h_f,\psi_n\rangle\!\rangle_2=0$ for all $n\in\mathbb{Z}$.

(c) *Suppose f is twice continuously differentiable, $f(0)=f(1)$, and $f'(0)=f'(1)$. Show that*

$$c_n(f)=\frac{1}{2\pi\mathfrak{i}\,n}c_n(f')=-\frac{1}{4\pi^2\,n^2}c_n(f'')\qquad\forall\ n\neq0.$$

(d) *Under the assumptions on f in (c), show that the sum in (b) converges uniformly to a continuous function $h_f\colon\mathbb{I}\longrightarrow\mathbb{C}$.*

(a; **2pts**) For $n,n'\in\mathbb{Z}$,

$$\langle\!\langle\psi_n,\psi_{n'}\rangle\!\rangle_2=\int_0^1\psi_n\overline{\psi_{n'}}\mathrm{d}x=\int_0^1\mathrm{e}^{2\pi\mathfrak{i}(n-n')x}\mathrm{d}x=\begin{cases}1, & \text{if } n=n';\\ 0, & \text{if } n\neq n'.\end{cases}$$

This establishes the claim.

(b; **6pts**) By Problem 1(d) with $v=f$, the sequence

$$f_{k,m}\equiv\sum_{n=-k}^{n=m}c_n(f)\psi_n$$

is Cauchy with respect to the L^2-norm $\|\cdot\|_2\|$ for any identification of $\mathbb{Z}^2$ with $\mathbb{Z}^+$. Since $L^2(\mathbb{I};\mathbb{C})$ is complete with respect to $\|\cdot\|_2\|$, this sequence converges to some $h_f\in L^2(\mathbb{I};\mathbb{C})$ with respect to $\|\cdot\|_2\|$. Furthermore,

$$\|h_f\|_2=\lim_{k,m\longrightarrow\infty}\bigg\|\sum_{n=-k}^{n=m}c_n(f)\psi_n\bigg\|_2=\lim_{k,m\longrightarrow\infty}\sum_{n=-k}^{n=m}\big|c_n(f)|^2\le\|f\|_2^2;$$

the second equality above holds by (a), while the last inequality holds by Problem 1(c).

By Problem 1(c), $\langle\!\langle f-f_{k,m},\psi_j\rangle\!\rangle_2=0$ for all $j,k,m\in\mathbb{Z}$. Thus,

$$\big|\langle\!\langle f-h_f,\psi_j\rangle\!\rangle_2\big|\le\big|\langle\!\langle f-f_{k,m},\psi_j\rangle\!\rangle_2\big|+\big|\langle\!\langle f_{k,m}-h_f,\psi_j\rangle\!\rangle_2\big|\le 0+\big\|f_{k,m}-h_f\big\|_2\|\psi_j\|_2=\big\|f_{k,m}-h_f\big\|_2;$$

the inequality above holds by Cauchy-Schwartz. Since $f_{k,m}\longrightarrow h_f$ as $k,m\longrightarrow h_f$, the right-hand side above tends to 0 as $k,m\longrightarrow h_f$. This establishes the last claim.

(c; **6pts**) By integration by parts,

$$\begin{aligned}c_n(f)=\int_0^1 f\mathrm{e}^{-2\pi\mathfrak{i}\,nx}\mathrm{d}x=\frac{\mathfrak{i}}{2\pi n}\int_0^1 f\mathrm{d}\mathrm{e}^{-2\pi\mathfrak{i}\,nx}&=\frac{\mathfrak{i}}{2\pi n}\bigg(f\mathrm{e}^{-2\pi\mathfrak{i}\,nx}\Big|_0^1-\int_0^1 f\mathrm{e}^{-2\pi\mathfrak{i}\,nx}\mathrm{d}x\bigg)\\&=\frac{1}{2\pi\mathfrak{i}n}\big(-0+c_n(f')\big);\end{aligned}$$

the last equality holds because $f(0)=f(1)$. Thus,

$$c_n(f)=\frac{1}{2\pi\mathfrak{i}n}c_n(f')=\frac{1}{2\pi\mathfrak{i}n}\cdot\frac{1}{2\pi\mathfrak{i}n}c_n(f'').$$

This establishes the claim.

(d; **6pts**) By (c) and Problem 1(b),

$$|c_n(f)|=\frac{|c_n(f'')|}{4\pi^2n^2}\le\frac{\|f''\|_2}{4\pi^2}n^{-2}\qquad\forall\,n\neq0.$$

By (c),

$$\begin{aligned}\bigg|\sum_{n=k}^{\infty}c_{-n}(f)\psi_n(x)+\sum_{n=\ell}^{\infty}c_n(f)\psi_n(x)\bigg|&\le\sum_{n=k}^{\infty}\big|c_{-n}(f)\big|\big|\psi_n(x)\big|+\sum_{n=\ell}^{\infty}\big|c_n(f)\big|\big|\psi_n(x)\big|\\&\le\frac{1}{4\pi^2}\bigg(\sum_{n=k}^{\infty}\frac{1}{n^2}+\sum_{n=\ell}^{\infty}\frac{1}{n^2}\bigg)\qquad\forall\,x\in\mathbb{I}.\end{aligned}$$

Since $\sum\limits_{n=1}^{\infty}n^{-2}$ converges, it follows that the sequence $f_{k,m}$ for any identification of $\mathbb{Z}^2$ with $\mathbb{Z}^+$ converges uniformly to a function $h_f\colon\mathbb{I}\longrightarrow\mathbb{C}$. Since $f_{k,m}$ is a sequence of continuous functions, so is h_f.

Problem 3 (20pts)

Let $\mathbb{I}=[0,1]$, $L^2(\mathbb{I};\mathbb{C})$ be the L^2-space of $\mathbb{C}$-valued functions on $\mathbb{I}$, $f\in L^2(\mathbb{I};\mathbb{C})$, and ψ_n and $c_n(f)$ be as in (3).

(a) *Suppose f is continuous and $f(0)=f(1)$. Show that for every $\epsilon>0$ there exist $N\in\mathbb{Z}^+$ and $a_n\in\mathbb{C}$ with $n\in\mathbb{Z}$ such that*

$$\bigg|f(x)-\sum_{n=-N}^{n=N}a_n\psi_n(x)\bigg|\le\epsilon\qquad\forall\,x\in\mathbb{I}.$$

(b) *Suppose f is twice continuously differentiable, $f(0)=f(1)$, and $f'(0)=f'(1)$. Show that*

$$f = \sum_{n\in\mathbb{Z}} c_n(f)\mathrm{e}^{2\pi\mathfrak{i}\,nx} \tag{4}$$

with the sum converging uniformly on $\mathbb{I}$.

(a; **12pts**) Weierstrass Approximation Theorem or WAT (MAT 320). Let $a,b\in\mathbb{R}$ and $f\colon[a,b]\longrightarrow\mathbb{C}$ be a continuous function. For every $\epsilon>0$, there exist $N\in\mathbb{Z}^+$ and $a_n\in\mathbb{C}$ with $n\in\mathbb{Z}^{\ge0}$ such that

$$\bigg|f(x)-\sum_{n=0}^{n=N}a_nx^n\bigg|\le\epsilon \qquad \forall\, x\in[a,b].$$

A continuous function $f\colon[0,1]\longrightarrow\mathbb{C}$ such that $f(0)=f(1)$ corresponds to a continuous function $S^1\longrightarrow\mathbb{C}$. This is a natural perspective for this question because the functions ψ_n are also natural functions on the circle. If S^1 is identified with the unit circle, then

$$\psi_n(z)=\begin{cases} z^n\equiv(x+\mathfrak{i}y)^n, & \text{if } n\in\mathbb{Z}^{\ge0};\\ \overline{z}^{-n}\equiv(x-\mathfrak{i}y)^{-n}, & \text{if } n\in\mathbb{Z}^{\le0}.\end{cases}$$

WAT and the claim of (a) for S^1 are special cases of the Stone-Weierstrass Theorem for metric (and more general topological) spaces. On the other hand, one could expect the S^1 statement to be directly deducible from the statement of WAT. This is done below.

WAT II. Let $a,b,c,d\in\mathbb{R}$ and $f\colon[a,b]\times[c,d]\longrightarrow\mathbb{C}$ be a continuous function. For every $\epsilon>0$, there exist $N\in\mathbb{Z}^+$ and $a_{m,n}\in\mathbb{C}$ with $m,n\in\mathbb{Z}^{\ge0}$ such that

$$\bigg|f(x,y)-\sum_{m=0}^{N}\sum_{n=0}^{N}a_{m,n}x^my^n\bigg|\le\epsilon \qquad \forall\,(x,y)\in[a,b]\times[c,d]. \tag{5}$$

Proof. We can assume that $a<b$ and $c<d$; otherwise, the statement reduces to WAT itself. Since the rectangle $[a,b]\times[c,d]\subset\mathbb{R}^2$ is closed and bounded, it is compact. Since f is continuous, it follows that it is uniformly continuous. Thus, there exists $\delta\in\mathbb{R}^+$ such that

$$\big|f(x,y)-f(x',y')\big|\le\frac{\epsilon}{4} \qquad \forall\,(x,y),(x',y')\in[a,b]\times[c,d]\ \text{s.t.}\ |x-x'|,|y-y'|\le\delta.$$

Choose $N'\in\mathbb{Z}^+$ so that $N'\ge(b-a)/\delta$ and let

$$x_i=a+\frac{i}{N'}(b-a), \qquad \forall\, i=0,1,\ldots,N'.$$

By the above assumption on δ,

$$\big|f(x,y)-f(x_{i-1},y)\big|,\big|f(x,y)-f(x_i,y)\big|\le\frac{\epsilon}{4} \quad \forall\,[x,y]\in[x_{i-1},x_i]\times[c,d],\ i=1,\ldots,N'.$$

Thus,

$$\begin{split}&\Big|f\big((1-t)x_{i-1}+tx_i,y\big)-\big((1-t)f(x_{i-1},y)+tf(x_i,y)\big)\Big|\\ &\quad\le(1-t)\big|f(x,y)-f(x_{i-1},y)\big|+t\big|f(x,y)-f(x_i,y)\big|\le\frac{\epsilon}{4} \quad \forall\,t\in[0,1],\ y\in[a,b],\ i=1,\ldots,N'.\end{split} \tag{6}$$

By WAT, for every $i=0,1,\ldots,N'$ there exist $N_i\in\mathbb{Z}^+$ and $a_{i;n}\in\mathbb{C}$ with $n\in\mathbb{Z}^{\ge 0}$ such that

$$\left|f(x_i,y)-\sum_{n=0}^{N_i}a_{i;n}y^n\right|\le\frac{\epsilon}{4}\qquad\forall\ y\in[c,d].$$

Let N_2^* be the largest of the numbers $N_0,N_1,\ldots,N_{N'}$. By replacing every N_i by N_2^* and taking $a_{i;n}=0$ for $n>N_i$, we can assume that

$$\left|f(x_i,y)-\sum_{n=0}^{N_2^*}a_{i;n}y^n\right|\le\frac{\epsilon}{4}\qquad\forall\ y\in[c,d],\ i=0,1,\ldots,N'. \tag{7}$$

For every $n=0,1,\ldots,N_2^*$, let $f_n\colon[a,b]\longrightarrow\mathbb{C}$ be the piecewise linear function satisfying

$$f_n\big((1-t)x_{i-1}+tx_i\big)=(1-t)a_{i-1;n}+ta_{i;n}\qquad\forall\,t\in[0,1],\ i=1,\ldots,N'.$$

By (6) and (7),

$$\begin{aligned}&\left|f\big((1-t)x_{i-1}+tx_i,y\big)-\sum_{n=0}^{N_2^*}f_n\big((1-t)x_{i-1}+tx_i\big)y^n\right|\\&\quad\le\left|f\big((1-t)x_{i-1}+tx_i,y\big)-\big((1-t)f(x_{i-1},y)+tf(x_i,y)\big)\right|\\&\qquad+(1-t)\left|f(x_{i-1},y)-\sum_{n=0}^{N_2^*}a_{i-1;n}y^n\right|+t\left|f(x_i,y)-\sum_{n=0}^{N_2^*}a_{i;n}y^n\right|\le\frac{\epsilon}{2}\end{aligned}$$

for all $t\in[0,1]$, $y\in[c,d]$, and $i=1,\ldots,N'$. Thus,

$$\left|f(x,y)-\sum_{n=0}^{N_2^*}f_n(x)y^n\right|\le\frac{\epsilon}{2}\qquad\forall\ (x,y)\in[a,b]\times[c,d]. \tag{8}$$

By WAT, for every $n=0,1,\ldots,N_2^*$ there exist $N_n\in\mathbb{Z}^+$ and $a_{m;n}\in\mathbb{C}$ with $m\in\mathbb{Z}^{\ge 0}$ such that

$$\left|f_n(x)-\sum_{m=0}^{N_i}a_{m;n}x^m\right|\le\frac{\epsilon}{2(1+N_2^*)(1+|c|+|d|)^{N_2^*}}\qquad\forall\ x\in[a,b].$$

Let N_1^* be the largest of the numbers $N_0,N_1,\ldots,N_{N_2^*}$. By replacing every N_n by N_1^* and taking $a_{m;n}=0$ for $m>N_n$, we can assume that

$$\left|f_n(x)-\sum_{m=0}^{N_1^*}a_{m;n}x^m\right|\le\frac{\epsilon}{2(1+N_2^*)(1+|c|+|d|)^{N_2^*}}\qquad\forall\ x\in[a,b].$$

Combining this with (8), we obtain

$$\left|\sum_{n=0}^{N_2^*}f_n(x)y^n-\sum_{n=0}^{N_2^*}\sum_{m=0}^{N_1^*}a_{m;n}x^my^n\right|\le\epsilon\qquad\forall\ (x,y)\in[a,b]\times[c,d].$$

Along with (8), this implies (5). $\square$

We now return to the question in (a). Choose a continuous function

$$F\colon [-1,1]\times[-1,1] \longrightarrow \mathbb{C} \qquad \text{s.t.} \qquad F\big(\mathrm{e}^{2\pi\mathrm{i}\theta}\big) = f(\theta) \quad \forall\,\theta\in\mathbb{I}. \tag{9}$$

Since $f(0)=f(1)$, the condition on $F(1,0)$ above is well-defined. For example,

$$F\big(r\mathrm{e}^{2\pi\mathrm{i}\theta}\big) = rf(\theta) \qquad \forall\ r\in\mathbb{R}^{\ge0},\ \theta\in\mathbb{I},$$

would do. By WAT II, there exist $N\in\mathbb{Z}^+$ and $a_{m,n}\in\mathbb{C}$ with $m,n\in\mathbb{Z}^{\ge0}$ such that

$$\bigg|F(x,y) - \sum_{m=0}^{N}\sum_{n=0}^{N} a_{m,n}x^m y^n\bigg| \le \epsilon \qquad \forall\ x,y\in[-1,1].$$

Thus,

$$\bigg|f(\theta) - \sum_{m=0}^{N}\sum_{n=0}^{N} a_{m,n}\big(\cos(2\pi\theta)\big)^m\big(\sin(2\pi\theta)\big)^n\bigg| \le \epsilon \qquad \forall\ \theta\in\mathbb{I}. \tag{10}$$

Since

$$\cos(2\pi\theta) = \frac{\psi_1(\theta)+\psi_{-1}(\theta)}{2}, \qquad \sin(2\pi\theta) = \frac{\psi_1(\theta)-\psi_{-1}(\theta)}{2\mathfrak{i}}, \qquad \text{and} \qquad \psi_{\pm1}^n = \psi_{\pm n}\,,$$

the claim follows from (10).

(b; **8pts**) By Problem 2, the sum on the right-hand side of (4) converges uniformly to a continuous function $h_f\colon\mathbb{I}\longrightarrow\mathbb{C}$ such that $\langle\!\langle f-h_f,\psi_n\rangle\!\rangle_2=0$ for all $n\in\mathbb{Z}$. Since $f(0)=f(1)$ and $\psi_n(0)=\psi_n(1)$ for all $n\in\mathbb{Z}$, the continuous function $f-h_f$ has the same property and thus satisfies the conclusion in (a). Let $\epsilon\in\mathbb{R}^+$. By (a), there exist $N\in\mathbb{Z}^+$ and $a_n\in\mathbb{C}$ with $n\in\mathbb{Z}$ such that

$$\bigg|\big(f(x)-h_f(x)\big) - \sum_{n=-N}^{n=N} a_n\psi_n(x)\bigg| \le \epsilon \qquad \forall\ x\in\mathbb{I}.$$

Since $\langle\!\langle f-h_f,\psi_n\rangle\!\rangle_2=0$ for all $n\in\mathbb{Z}$, it follows that

$$\begin{aligned}\big\|f-h_f\big\|_2^2 &= \langle\!\langle f-h_f,f-h_f\rangle\!\rangle_2 - \sum_{n=-N}^{n=N}\langle\!\langle f-h_f,a_n\psi_n\rangle\!\rangle_2 = \bigg\langle\!\!\bigg\langle f-h_f,(f-h_f)-\sum_{n=-N}^{n=N}a_n\psi_n\bigg\rangle\!\!\bigg\rangle\\ &\le \big\|f-h_f\big\|_2\bigg\|_2\bigg\|\big(f-h_f\big) - \sum_{n=-N}^{n=N}a_n\psi_n\bigg\|_2 \le \big\|f-h_f\big\|_2\epsilon\cdot 1^{1/2}\,;\end{aligned}$$

the first inequality above follows from Cauchy-Schwartz. Since the above holds for every $\epsilon>0$, it follows that $\|f-h_f\|_2=0$ and thus $f-h_f=0$ almost everywhere on $\mathbb{I}$. Since this function is continuous, this implies that $f=h_f$.