MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 3

Problem 1 (25pts)

Let $(X,\mathcal{F},\mu)$ be a measure space and $\overline{\mathcal{F}}$ be the completion of $\mathcal{F}$ with respect to μ as defined in Section 2.5. Show that

(a) $\overline{\mathcal{F}} = \big\{E\cup A\colon E\in\mathcal{F},\ A\subset F \text{ for some } F\in\mathcal{F} \text{ with } \mu(F)=0\big\}$;

(b) *there exists a unique measure* $\overline{\mu}\colon \overline{\mathcal{F}}\longrightarrow[0,\infty]$ *so that* $\overline{\mu}|_{\mathcal{F}}=\mu$;

(c) *the measure space* $(X,\overline{\mathcal{F}},\overline{\mu})$ *is complete.*

(a) We need to show that the collection on RHS is the smallest σ-field containing $\mathcal{F}$ and all subsets $A\subset F$ with $F\in\mathcal{F}$ and $\mu(F)=0$. This collection contains every element $E=E\cup\emptyset$ with $E\in\mathcal{F}$ because $\emptyset\subset\emptyset$, $\emptyset\in\mathcal{F}$, and $\mu(\emptyset)=0$. It contains every element $A=\emptyset\cup A$ with $A\subset F$ for some $F\in\mathcal{F}$ with $\mu(F)=0$ because $\emptyset\in\mathcal{F}$. Furthermore, every collection of subsets of X containing $\mathcal{F}$ and all subsets $A\subset F$ with $F\in\mathcal{F}$ and $\mu(F)=0$ which is closed under even finite unions must contain the collection on RHS.

It remains to show that this collection is a σ-field on X. It contains the entire space X because $\mathcal{F}$ does and this collection contains all of $\mathcal{F}$. If $E\in\mathcal{F}$ and $A\subset F$ for some $F\in\mathcal{F}$ and $\mu(F)=0$, then

$$(E\cup A)^c \equiv X-(E\cup A) = \big(X-(E\cup F)\big)\cup(F-E\cup A) \equiv (E\cup F)^c\cup(F-E\cup A)\,.$$

Since $\mathcal{F}$ is closed under taking complements, $(E\cup F)^c\in\mathcal{F}$. Since $F-E\cup A\subset F$ and $F\in\mathcal{F}$ with $\mu(F)=0$, we conclude that $(E\cup A)^c$ belongs to the collection on RHS.

Suppose

$$E_1,F_1,E_2,F_2,\ldots\in\mathcal{F},\qquad \mu(F_1),\mu(F_2),\ldots=0,\qquad A_1\subset F_1,\ A_2\subset F_2,\ \ldots. \tag{1}$$

Then

$$\bigcup_{n=1}^{\infty}(E_n\cup A_n) = \bigg(\bigcup_{n=1}^{\infty}E_n\bigg)\cup\bigg(\bigcup_{n=1}^{\infty}A_n\bigg) \equiv E\cup A. \tag{2}$$

The countable union E of E_n's belongs to $\mathcal{F}$ because $\mathcal{F}$ is closed under countable unions. The countable union A of A_n's is contained in the set

$$F \equiv \bigcup_{n=1}^{\infty}F_n\,. \tag{3}$$

By the countably subadditivity of μ,

$$\mu(F) \le \sum_{n=1}^{\infty}\mu(F_n) = 0.$$

Thus, $E\cup A$ belongs to the collection on RHS and this collection is closed under countable unions. We have thus checked that it satisfies the three required properties for being a σ-field on X.

(b) We define $\overline{\mu}\colon \overline{\mathcal{F}}\longrightarrow[0,\infty]$ by

$$\overline{\mu}(E\cup A)=\mu(E) \qquad \text{if} \quad E\in\mathcal{F},\ A\subset F \text{ for some } F\in\mathcal{F} \text{ with } \mu(F)=0. \tag{4}$$

We need to check that $\overline{\mu}$ is well-defined. Suppose $E\cup A=E'\cup A'$ with E,A as above, $E'\in\mathcal{F}$, and $A'\subset F'$ for some $F'\in\mathcal{F}$ with $\mu(F')=0$. Thus,

$$E'-E\subset A\subset F, \qquad E-E'\subset A'\subset F'.$$

Since $E'-E, E-E, F, F'\in\mathcal{F}$ and $\mu(F),\mu(F')=0$, these statements imply that

$$\mu(E'-E),\mu(E-E')=0,$$
$$\mu(E)=\mu(E\cap E')+\mu(E-E')=\mu\big(E'-(E'-E)\big)+0=\mu(E')-\mu(E'-E)=\mu(E').$$

Thus, the value of $\overline{\mu}$ on $E\cup A=E'\cup A'$ is well-defined (independent of how this set is split into the two types of subsets). Since

$$\overline{\mu}(E)=\overline{\mu}(E\cup\emptyset)=\mu(E) \qquad \forall\ E\in\mathcal{F},$$

we obtain that $\overline{\mu}|_{\mathcal{F}}=\mu$.

Suppose $E_1,E_2,\ldots,E\in\mathcal{F}$ and $A_1,A_2,\ldots,A\subset X$ are as in (1)-(3) and $(E_n\cup A_n)\cap(E_{n'}\cup A_{n'})=\emptyset$ for all $n\neq n'$. The last condition implies that $E_n\cap E_{n'}=\emptyset$ for all $n\neq n'$. By the countable additivity of μ,

$$\overline{\mu}\bigg(\bigcup_{n=1}^{\infty}(E_n\cup A_n)\bigg)=\overline{\mu}(E\cup A)=\mu(E)=\mu\bigg(\bigcup_{n=1}^{\infty}E_n\bigg)=\sum_{n=1}^{\infty}\mu(E_n)=\sum_{n=1}^{\infty}\overline{\mu}(E_n\cup A_n).$$

Thus, $(X,\overline{\mathcal{F}},\overline{\mu})$ is a measure on X.

Suppose $\overline{\mu}'\colon\overline{\mathcal{F}}\longrightarrow[0,\infty]$ is any measure on X such that $\overline{\mu}'|_{\mathcal{F}}=\mu$. Let E,A,F be as in (4). By the countable subadditivity of $\overline{\mu}'$ and μ,

$$\mu(E)=\overline{\mu}'(E)\le\overline{\mu}'(E\cup A)\le\overline{\mu}'(E\cup F)=\mu(E\cup F)\le\mu(E)+\mu(F)=\mu(E).$$

Thus, $\overline{\mu}'(E\cup A)=\mu(E)=\overline{\mu}(E\cup A)$ and $\overline{\mu}'=\overline{\mu}$.

(c) Suppose $A'\subset F'$ for some $F'\in\overline{\mathcal{F}}$ with $\overline{\mu}(F')=0$. By (a), there exist $E\in\mathcal{F}$ and $A\subset F$ with $F\in\mathcal{F}$ and $\mu(F)=0$ such that $F'=E\cup A$. Thus,

$$A'\subset E\cup F, \quad E\cup F\in\mathcal{F}, \quad \mu(E\cup F)\le\mu(E)+\mu(F)=\mu(E)=\overline{\mu}(E)\le\overline{\mu}(E\cup A)=0.$$

Thus, $A'\in\overline{\mathcal{F}}$ and so the measure space $(X,\overline{\mathcal{F}},\overline{\mu})$ is complete.

Problem 2 (5pts)

Give an example of a non-measurable function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that $f^2:\mathbb{R}\longrightarrow\mathbb{R}$ is measurable.

Let $A\subset\mathbb{R}$ be any non-measurable subset. The function

$$f\equiv \mathbb{1}_A-\mathbb{1}_{A^c}:\mathbb{R}\longrightarrow\mathbb{R}, \qquad f(x)=\begin{cases}1, & \text{if } x\in A;\\ -1, & \text{if } x\in A^c;\end{cases}$$

is non-measurable because $f^{-1}(1)=A$ is a non-measurable subset. On the other hand, the function $f^2=\mathbb{1}_{\mathbb{R}}$ is measurable because it is constant.

Problem 3 (10pts)

Let $(X,\mathcal{F},\mu)$ be a measure space. A function $f:X\longrightarrow\mathbb{R}$ is called measurable if $f^{-1}(I)\in\mathcal{F}$ for every interval $I\subset\mathbb{R}$. Show that this condition is equivalent to each of the four conditions (ii)-(v) in Theorem 3.3 in the book with $a\in\mathbb{Q}$ separately.
Note: Theorem 3.3 establishes the equivalence for $(X,\mathcal{F},\mu)=(\mathbb{R},\mathcal{M},m)$ with each the four conditions (ii)-(v) with $a\in\mathbb{R}$. The proof for an arbitrary measure space $(X,\mathcal{F},\mu)$ and $a\in\mathbb{R}$ is identical (replace $\mathcal{M}$ by $\mathcal{F}$ everywhere). So, you only need to check that it is enough to take $a\in\mathbb{Q}$.

We need to show that (i) in Theorem 3.3 is equivalent to each of the following
(ii′) $f^{-1}((a,\infty))\in\mathcal{F}$ for every $a\in\mathbb{Q}$; (iii′) $f^{-1}([a,\infty))\in\mathcal{F}$ for every $a\in\mathbb{Q}$;
(iv′) $f^{-1}((-\infty,a))\in\mathcal{F}$ for every $a\in\mathbb{Q}$; (v′) $f^{-1}((-\infty,a])\in\mathcal{F}$ for every $a\in\mathbb{Q}$.
Condition (i) implies each of the above conditions because the former means that $f^{-1}(I)\in\mathcal{F}$ for *every* interval $I\subset\mathbb{R}$. Conditions (ii′) and (v′) are equivalent because

$$f^{-1}\big((a,\infty)\big)\in\mathcal{F} \iff f^{-1}\big((-\infty,a]\big)=f^{-1}\big(\mathbb{R}-(a,\infty)\big)=X-f^{-1}\big((a,\infty)\big)\in\mathcal{F}.$$

By the same reasoning, conditions (iii′) and (iv′) are equivalent.

Suppose $a\in\mathbb{R}$. Let $a_n\in\mathbb{Q}\cap(a,\infty)$ be any sequence converging to a. Then,

$$f^{-1}\big((a,\infty)\big)=f^{-1}\bigg(\bigcup_{n=1}^{\infty}(a_n,\infty)\bigg)=\bigcup_{n=1}^{\infty}f^{-1}\big((a_n,\infty)\big),$$
$$f^{-1}\big((a,\infty)\big)=f^{-1}\bigg(\bigcup_{n=1}^{\infty}[a_n,\infty)\bigg)=\bigcup_{n=1}^{\infty}f^{-1}\big([a_n,\infty)\big).$$

By the first line above, condition (ii′) implies condition (ii) in Theorem 3.3. By the second line, condition (iii′) implies condition (ii). Since condition (ii) in Theorem 3.3 has been shown to imply that f is measurable, we conclude that each of the conditions (ii′)-(v′) *separately* implies that f is measurable.

Problem 4 (10pts)

Let $A\subset\mathbb{R}$. Show that the function

$$f:\mathbb{R}\longrightarrow\mathbb{R},\qquad f(x)=\begin{cases}x, & \text{if } x\in A;\\ -x, & \text{if } x\notin A;\end{cases}$$

is measurable if and only if $A\in\mathcal{M}$.

Since the two subsets $\mathbb{R}^{\pm}\subset\mathbb{R}$ (meaning $\mathbb{R}^+$ or $\mathbb{R}^-$) are measurable, so are the indicator functions

$$\mathbb{1}_{\mathbb{R}^\pm}:\mathbb{R}\longrightarrow\mathbb{R},\qquad \mathbb{1}_{\mathbb{R}^\pm}(x)=\begin{cases}x, & \text{if } x\in\mathbb{R}^\pm;\\ 0, & \text{if } x\notin\mathbb{R}^\pm.\end{cases}$$

Let

$$g_\pm=\mathbb{1}_{\mathbb{R}^\pm}\cdot f:\mathbb{R}\longrightarrow\mathbb{R},\qquad g_\pm(x)=\begin{cases}x, & \text{if } x\in A\cap\mathbb{R}^\pm;\\ -x, & \text{if } x\in A^c\cap\mathbb{R}^\pm;\\ 0, & \text{otherwise.}\end{cases}$$

If the function f is measurable, then so are both functions $g_\pm$ (since they are then products of measurable functions). If both functions $g_\pm$ are measurable, then so is $f=g_++g_-$. Thus, it is sufficient to show that the two functions $g_\pm$ are measurable if and only if A is measurable.

If the two functions $g_\pm$ are measurable, then the set

$$A=\big(A\cap\mathbb{R}^+\big)\cup\big(A\cap\mathbb{R}^-\big)\cup\big(A\cap\{0\}\big)=g_+^{-1}(\mathbb{R}^+)\cup g_-^{-1}(\mathbb{R}^-)\cup\big(A\cap\{0\}\big)$$

is measurable because it is a union of three measurable sets (the first two sets above are measurable because they are preimages of intervals by measurable functions, and the last set is measurable because it is either $\emptyset$ or $\{0\}$).

If $I\subset\mathbb{R}$, then

$$g_\pm^{-1}(I)=\big(A\cap\mathbb{R}^\pm\cap I\big)\cup\big(A^c\cap\mathbb{R}^\pm\cap(-I)\big)\cup\begin{cases}\emptyset, & \text{if } 0\notin I;\\ \mathbb{R}^\mp\cup\{0\}, & \text{if } 0\in I.\end{cases}$$

The last set on RHS above is always measurable. If I is an interval, then so is $-I$. If A is measurable, then so is A^c. If I is an interval and A is measurable, the first two sets on RHS above are thus measurable and so is $g_\pm^{-1}(I)$. We conclude that the functions $g_\pm$ are measurable if the set A is measurable.

Problem 5 (10pts)

Let $(X,\mathcal{F},\mu)$ be a measure space.

(a) Suppose (Y,d) is a metric space, $g: Y \longrightarrow \mathbb{R}$ is a continuous function, and $h: X \longrightarrow Y$ is a function such that $h^{-1}(\mathcal{O}) \in \mathcal{F}$ for every open subset $\mathcal{O} \subset Y$. Show that the function $g\circ h\colon X \longrightarrow \mathbb{R}$ is measurable.

(b) Suppose $h_1,\ldots,h_k\colon X \longrightarrow \mathbb{R}$ are measurable functions and $h=(h_1,\ldots,h_k)\colon X \longrightarrow \mathbb{R}^k$. Show that $h^{-1}(\mathcal{O})\in\mathcal{F}$ for every open subset $\mathcal{O}\subset\mathbb{R}^k$.

(c) Show that a function $f\colon X \longrightarrow \mathbb{R}$ is measurable if and only if the function f^3 is measurable.

(a) It is sufficient to show that the subset

$$\{g\circ h\}^{-1}\big((a,\infty)\big) = h^{-1}\big(g^{-1}\big((a,\infty)\big)\big) \subset X$$

belongs to $\mathcal{F}$ for every $a\in\mathbb{R}$. Since g is continuous and $(a,-\infty)\subset\mathbb{R}$ is an open subset,

$$\mathcal{O} \equiv g^{-1}\big((a,\infty)\big) \subset Y$$

is also an open subset. Thus,

$$\{g\circ h\}^{-1}\big((a,\infty)\big) = h^{-1}(\mathcal{O}) \in \mathcal{F}$$

by the assumption on h.

(b) Every open subset $\mathcal{O}$ of $\mathbb{R}^k$ is a countable union of the open k-cubes

$$C = (a_1,b_1)\times(a_2,b_2)\times\ldots\times(a_k,b_k) \qquad \text{with} \quad a_1,b_1,a_2,b_2,\ldots,a_k,b_k\in\mathbb{R};$$

we can even take $a_1,b_1,a_2,b_2,\ldots,a_k,b_k\in\mathbb{Q}$ (in which case, there are only countably many of such cubes). For such a cube,

$$h^{-1}(C) = \big\{x\in X\colon h_i(x)\in(a_i,b_i)\ \forall\, i=1,\ldots,k\big\} = \bigcap_{i=1}^{k} h_i^{-1}\big((a_i,b_i)\big) \in \mathcal{F}$$

because each $h_i^{-1}((a_i,b_i))\in\mathcal{F}$. If $\mathcal{O}$ is a countable union of the open k-cubes C_n, then

$$h^{-1}(\mathcal{O}) = h^{-1}\bigg(\bigcup_{n=1}^{\infty} C_n\bigg) = \bigcup_{n=1}^{\infty} h^{-1}(C_n) \in \mathcal{F}$$

because each $h^{-1}(C_n)\in\mathcal{F}$.

(c) If f is measurable, then so is $f^3=f\cdot f\cdot f$ because a product of $\mathbb{R}$-valued measurable functions is measurable. Let

$$g\colon Y=\mathbb{R} \longrightarrow \mathbb{R}, \qquad g(y) = y^{1/3}\,.$$

Suppose the function $h=f^3$ is measurable. By (b), this implies that $h^{-1}(\mathcal{O})\in\mathcal{M}$ for every open subset $\mathcal{O}\subset\mathbb{R}$. Since g is a continuous function, (a) then implies that the function

$$f=g\circ h\colon \mathbb{R} \longrightarrow \mathbb{R}$$

is measurable.