MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 2

Problem 1 (5pts)

Suppose $A\subset B\subset C\subset\mathbb{R}$ are such that $A,C\in\mathcal{M}$ and $m(A)=m(C)<\infty$. Show that $B\in\mathcal{M}$ and $m(A)=m(B)=m(C)$.

Since $A,C\in\mathcal{M}$, $A\subset C$, and $m(A)<\infty$,

$$m(C-A) = m(C) - m(A) = 0.$$

Thus, $C-A$ is a null set. Since $B-A\subset C-A$, $B-A$ is also a null set and thus $B-A\in\mathcal{M}$. Since A and $B-A$ are disjoint measurable sets,

$$B = A\cup(B-A) \in \mathcal{M} \qquad \text{and} \qquad m(B) = m(A) + m(B-A).$$

Problem 2 (10pts)

Let $E_1,E_2,\ldots$ be disjoint measurable sets and $A\subset\mathbb{R}$ be any subset. Show that

$$m^*\Big(A\cap\bigcup_{n=1}^{\infty}E_n\Big) = \sum_{n=1}^{\infty}m^*\big(A\cap E_n\big).$$

By the countable subaddivity

$$m^*\Big(A\cap\bigcup_{n=1}^{\infty}E_n\Big) = m^*\Big(\bigcup_{n=1}^{\infty}(A\cap E_n)\Big) \le \sum_{n=1}^{\infty}m^*\big(A\cap E_n\big).$$

Thus, only the opposite inequality needs to be shown. For each $k\in\mathbb{Z}^{\ge0}\sqcup\{\infty\}$, let

$$E^k = \bigcup_{n=1}^{k}E_n \in \mathcal{M}, \qquad A_k = A\cap E^k \subset \mathbb{R}.$$

In particular,

$$A_{k-1}\cap E_n = A_k\cap E_n \;\forall\, n<k, \qquad A_k\cap E^{k-1} = A_{k-1}, \quad A_k\cap(E^{k-1})^c = A_k\cap E_k \qquad \forall\; k\in\mathbb{Z}^+;$$

the last equality holds because E_k is disjoint from E^{k-1} and thus $E_k\subset(E^{k-1})^c$.

We show by induction that

$$m^*(A_k) = \sum_{n=1}^{k}m^*\big(A_k\cap E_n\big) = \sum_{n=1}^{k}m^*\big(A_\infty\cap E_n\big) \qquad \forall\; k\in\mathbb{Z}^{\ge0}; \tag{1}$$

only the first equality needs a proof. This statement is true for $k=0$. Suppose $k\in\mathbb{Z}^+$ and (1) is true for $k-1$, i.e.

$$m^*\big(A_{k-1}\big) = \sum_{n=1}^{k-1} m^*\big(A_{k-1}\cap E_n\big) = \sum_{n=1}^{k-1} m^*\big(A_k\cap E_n\big). \tag{2}$$

Since $E^{k-1}\in\mathcal{M}$,

$$m^*(A_k) = m^*\big(A_k\cap E^{k-1}\big) + m^*\big(A_k\cap(E^{k-1})^c\big) = m^*\big(A_{k-1}\big) + m^*\big(A_k\cap E_k\big).$$

Combining this with (2), we obtain the first equality in (1). Since $A_1\subset A_2\subset\ldots\subset A_\infty$,

$$m^*\big(A_\infty\big) \ge \lim_{k\longrightarrow\infty} m^*(A_k) = \lim_{k\longrightarrow\infty}\sum_{n=1}^{k} m^*\big(A_\infty\cap E_n\big) = \sum_{n=1}^{\infty} m^*\big(A\cap E_n\big);$$

the first equality above is (1). This establishes the desired inequality.

Problem 3 (5pts)

Let $E_1, E_2, \ldots, E_{20}\subset[0,1]$ be measurable subsets. Show that

$$m\bigg(\bigcap_{n=1}^{20} E_n\bigg) \ge \sum_{n=1}^{20} m(E_n) - 19.$$

Since E_n's, their intersection $E^\cap$, and $\mathbb{I}=[0,1]$ are measurable and their measures are finite,

$$\begin{aligned} m(E^\cap) &= m(\mathbb{I}) - m\big(\mathbb{I}-E^\cap\big) = 1 - m\bigg(\bigcup_{n=1}^{20}(\mathbb{I}-E_n)\bigg) \ge 1 - \sum_{n=1}^{20} m(\mathbb{I}-E_n) = 1 - \sum_{n=1}^{20}\big(m(\mathbb{I})-m(E_n)\big) \\ &= 1 - \bigg(20 - \sum_{n=1}^{20} m(E_n)\bigg) \ge \sum_{n=1}^{20} m(E_n) - 19. \end{aligned}$$

Problem 4 (10pts)

Show that there exist $A, B\subset\mathbb{R}$ such that

$$A\cap B = \emptyset \qquad \text{and} \qquad m^*(A\cup B) < m^*(A)+m^*(B).$$

By p302, there exists a non-measurable subset $E\subset\mathbb{R}$. Thus, there exists $F\subset\mathbb{R}$ such that

$$m^*(F) < m^*\big(F\cap E\big) + m^*\big(F\cap E^c\big).$$

We can thus take $A=F\cap E$ and $B=F\cap E^c$.

Here is a more explicit example. Let $E_1, E_2, \ldots \subset [-1,2]$ be the translates of the non-measurable set $E \subset [0,1]$ constructed on p302. Since the outer measure m^* is translation-invariant, there exists $\delta \in [0,1]$ such that $m^*(E_n) = \delta$ for all $n \in \mathbb{Z}^+$. Since all null sets are measurable, $\delta > 0$. Since

$$\bigcup_{n=1}^{\infty} E_n \subset [-1,2] \qquad \text{and} \qquad m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le 3,$$

there exists $k \in \mathbb{Z}^+$ such that

$$m^*\left(\bigcup_{n=1}^{k} E_n\right) < k\delta = \sum_{n=1}^{k} m^*(E_n).$$

Take the smallest such k; then $k \ge 2$. Let

$$A = \bigcup_{n=1}^{k-1} E_n, \qquad B = E_k.$$

Since all E_n's are pairwise disjoint, $A \cap B = \emptyset$. By the choice of k,

$$m^*(A) = m^*\left(\bigcup_{n=1}^{k-1} E_n\right) = (k-1)\delta, \qquad m(B) = m(E_k) = \delta,$$

$$m^*(A \cup B) = m^*\left(\bigcup_{n=1}^{k} E_n\right) < k\delta = m^*(A) + m^*(B).$$

So, A and B are as needed.

Problem 5 (8pts)

Let $\ell_1, \ell_2, \ldots \in (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} 2^{n-1}\ell_n < 1$. Starting with $C_0 \equiv [0,1]$, let $C_n \subset [0,1]$ for $n \in \mathbb{Z}^+$ be the subset obtained from C_{n-1} by removing the open middle interval of length ℓ_n from each of the 2^{n-1} disjoint closed intervals making up C_{n-1}. Show that

$$C \equiv \bigcap_{n=1}^{\infty} C_n \subset [0,1]$$

is a closed Borel subset. Find its measure.

The set C is closed because it is an intersection of closed sets. It is a Borel set because every closed set is Borel. Since $C_n \subset [0,1]$ for $n \in \mathbb{Z}^+$ is obtained from C_{n-1} by removing 2^{n-1} disjoint intervals of length ℓ_n,

$$m(C_n) = m\big([0,1]\big) - \sum_{k=1}^{n} 2^{k-1}\ell_k\,.$$

Since $C_0 \supset C_1 \supset \ldots$ and $m(C_0) < \infty$,

$$m(C) = \left(\bigcap_{n=0}^{\infty} C_n\right) = \lim_{n \longrightarrow \infty} m(C_n) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k\,.$$

Problem 6 (12pts + bonus 5pts)

For $X\subset\mathbb{R}$, let

$$\mathcal{M}_X=\big\{E\cap X\colon E\in\mathcal{M}\big\}, \qquad \mu_X=m^*\big|_{\mathcal{M}_X}.$$

(a) *Show that $(X,\mathcal{M}_X,\mu_X)$ is a complete measure space if $X\subset\mathbb{R}$ is measurable.*

(b) *Which properties of a complete measure space $(X,\mathcal{M}_X,\mu_X)$ may not satisfy if X is not assumed to be measurable? Give an example.*

(a) Since $\mathbb{R}\in\mathcal{M}$, $X=\mathbb{R}\cap X\in\mathcal{M}_X$. If $A\in\mathcal{M}_X$, then $A=E\cap X$ for some $E\in\mathcal{M}$ and

$$X-A=(\mathbb{R}-E)\cap X\in\mathcal{M}_X$$

because $\mathbb{R}-E\in\mathcal{M}$. If $A_1,A_2,\ldots\in\mathcal{M}_X$, then $A_n=E_n\cap X$ for some $E_n\in\mathcal{M}$ and

$$\bigcup_{n=1}^{\infty}A_n=\bigcup_{n=1}^{\infty}(E_n\cap X)=\bigg(\bigcup_{n=1}^{\infty}E_n\bigg)\cap X\in\mathcal{M}_X$$

because the union of E_n's belongs to $\mathcal{M}$. Thus, $\mathcal{M}_X$ is a σ-field on X. If $A\subset B$ for some $B\in\mathcal{M}_X$ with $\mu_X(B)=m^*(B)=0$, then B is a null set and thus so is A. Since $\mathcal{M}$ contains all null sets, $A\in\mathcal{M}$ and so $A=A\cap X\in\mathcal{M}_X$. Thus, the σ-field $\mathcal{M}_X$ is complete with respect to μ_X.

Since $X\in\mathcal{M}$ and $\mathcal{M}$ is closed under intersections, $\mathcal{M}_X\subset\mathcal{M}$. If $A_1,A_2,\ldots\in\mathcal{M}_X$ are such that $A_n\cap A_{n'}=\emptyset$ for all $n\neq n'$,

$$\mu_X\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)=m\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)=\sum_{n=1}^{\infty}m(A_n)=\sum_{n=1}^{\infty}\mu_X(A_n).$$

Thus, μ_X is a measure.

(b) The only part of the argument in (a) that depends on X being measurable is that μ_X is countably additive. Thus, $(X,\mathcal{M}_X,\mu_X)$ satisfies all properties of a complete measure space with the possible exception of the countable additivity for μ_X.

Bonus: In fact, μ_X is countably additive even if X is not measurable. Suppose $A_1,A_2,\ldots\in\mathcal{M}_X$ are such that $A_n\cap A_{n'}=\emptyset$ for all $n\neq n'$ and $E_n\in\mathcal{M}$ are such that $A_n=E_n\cap X$. Let

$$F_n=E_n-E_1\cup\ldots\cup E_{n-1}\in\mathcal{M}.$$

Thus,

$$F_n\cap F_n'=\emptyset\ \forall\ n\neq n',\quad \bigcup_{n=1}^{\infty}A_n=\bigg(\bigcup_{n=1}^{\infty}E_n\bigg)\cap X=\bigg(\bigcup_{n=1}^{\infty}F_n\bigg)\cap X,$$

$$F_n\cap X=\big(E_n-(E_1\cap E_n)\cup\ldots\cup(E_{n-1}\cap E_n)\big)\cap X=A_n-(A_1\cap A_n)\cup\ldots\cup(A_{n-1}\cap A_n)=A_n.$$

From Problem 2 with $A=X$ and E_n replaced by F_n, we then obtain

$$\mu_X\bigg(\bigcup_{n=1}^{\infty}A_n\bigg)=m^*\bigg(X\cap\bigcup_{n=1}^{\infty}F_n\bigg)=\sum_{n=1}^{\infty}m^*(X\cap F_n)=\sum_{n=1}^{\infty}\mu_X(A_n).$$

Thus, μ_X is countably additive.