MAT 324: Real Analysis, Fall 2017
Solutions to Problem Set 11

Problem 1 (8pts)

Let μ, ν_1, ν_2 be measures on a measurable space $(X, \mathcal{F})$. Show that

(a) *if $\nu_1, \nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*

(b) *if $\nu_1, \nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.*

(c) *if $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*

(a; **2pts**) Suppose $E \in \mathcal{F}$ and $\mu(E) = 0$. Then,

$$\{\nu_1 + \nu_2\}(E) = \nu_1(E) + \nu_2(E) = 0 + 0 = 0.$$

Thus, $\nu_1 + \nu_2 \ll \mu$.

(b; **4pts**) Let $A_1, A_2, B_1, B_2 \in \mathcal{F}$ be such that

$$A_1 \cap B_1, A_2 \cap B_2 = \emptyset, \qquad \mu(A_1^c), \mu(A_2^c), \nu_1(B_1^c), \nu_2(B_2^c) = 0.$$

Take $A = A_1 \cap A_2$ and $B = B_1 \cup B_2$, Then, $A \cap B = \emptyset$ and

$$\mu(A^c) = \mu(A_1^c \cup A_2^c) \le \mu(A_1^c) + \mu(A_2^c) = 0,$$
$$\{\nu_1 + \nu_2\}(B^c) = \{\nu_1 + \nu_2\}(B_1^c \cap B_2^c) \le \nu_1(B_1^c) + \nu_2(B_2^c) = 0.$$

Thus, $\nu_1 + \nu_2 \perp \mu$.

(c; **2pts**) Let $A, B \in \mathcal{F}$ be such that $A \cap B = \emptyset$ and $\mu(A^c), \nu_2(B^c) = 0$. Since $\nu_1 \ll \mu$, $\nu_1(A^c) = 0$. Since $A \cap B = \emptyset$ and $\nu_2(B^c) = 0$, this implies that $\nu_1 \perp \nu_2$.

Problem 2 (10pts)

Let μ, ν be measures on a measurable space $(X, \mathcal{F})$.

(a) *Suppose for every $\epsilon > 0$ there exists $\delta > 0$ such that $\nu(E) < \epsilon$ for every $E \in \mathcal{F}$ with $\mu(E) < \delta$. Show that $\nu \ll \mu$.*

(b) *Show that the converse is true if $\nu(X) < \infty$.*

(c) *Give an example showing that the converse can fail if $\mu(X) < \infty$ and $\nu(X) = \infty$.*

(a; **2pts**) Suppose $E \in \mathcal{F}$ and $\mu(E) = 0$. Since $\mu(E) < \delta$ for every $\delta > 0$, the assumption implies that $\nu(E) < \epsilon$ for every $\epsilon > 0$. Thus, $\nu(E) = 0$ and $\nu \ll \mu$.

(b; **4pts**) Suppose not. For some $\epsilon > 0$ and every $n \in \mathbb{Z}^+$, there then exists $E_n \in \mathcal{F}$ such that $\nu(E_n) \ge \epsilon$ and $\mu(E_n) < 2^{-n}$. Let

$$A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} E_n \,.$$

Thus, $A\in\mathcal{F}$ and

$$\mu(A) \le \mu\bigg(\bigcup_{n=k}^{\infty} E_n\bigg) \le \sum_{n=k}^{\infty} \mu(E_n) = 2^{-k+1} \quad \forall\, k\in\mathbb{Z}^+, \tag{1}$$

$$\nu\bigg(\bigcup_{n=k}^{\infty} E_n\bigg) \ge \epsilon \quad \forall\, k\in\mathbb{Z}^+, \qquad \nu(A) = \lim_{k\longrightarrow\infty} \nu\bigg(\bigcup_{n=k}^{\infty} E_n\bigg) \ge \epsilon; \tag{2}$$

the equality in the last statement above uses that

$$\nu\bigg(\bigcup_{n=1}^{\infty} E_n\bigg) \le \nu(X) < \infty.$$

By (1), $\mu(A)=0$. By the second statement in (2), $\nu(A)>0$. This contradicts $\nu\ll\mu$.

(c; **4pts**) Let $\nu=m_{\mathbb{R}^+}$ be the standard Lebesgue measure on $(\mathbb{R},\mathcal{M}_{\mathbb{R}^+})$ and

$$\mu\colon \mathcal{M}_{\mathbb{R}^+} \longrightarrow [0,\infty], \qquad \mu(E) = \sum_{n=1}^{\infty} 2^{-n} m\big(E\cap[n-1,n]\big).$$

If $\mu(E)=0$, then $m(E\cap[n-1,n])$ for every $n\in\mathbb{Z}^+$ and $\nu(E)=0$. Thus, $\nu\ll\mu$. On the other hand,

$$\mu([n-1,n]) = 2^{-n} \qquad \text{and} \qquad \nu([n-1,n]) = 1.$$

Thus, ν and μ do not satisfy the ϵ-δ condition in (a).

Problem 3 (8pts)

Let m be the standard Lebesgue measure on $(\mathbb{R},\mathcal{M})$ and μ be the counting measure on $(\mathbb{R},\mathcal{M})$. Show that $m\ll\mu$, but there exists no Lebesgue measurable function $g\colon\mathbb{R}\longrightarrow\mathbb{R}^{\ge0}$ such that $m=\mu_g$. Why doesn't this contradict the Radon-Nikodym theorem?

If $E\in\mathcal{M}$ and $\mu(E)=0$, then $E=\emptyset$ and so $m(E)=0$. Thus, $m\ll\mu$. If $g\colon\mathbb{R}\longrightarrow\mathbb{R}^{\ge0}$ is a Lebesgue measurable function and $m=\mu_g$, then

$$0 = m\big(\{x\}\big) = \int_{\{x\}} g\mathrm{d}\mu = g(x)\mu\big(\{x\}\big) = g(x) \qquad \forall\, x\in\mathbb{R}.$$

This implies that $g=0$. Since m is not the zero measure on $(\mathbb{R},\mathcal{M})$, we conclude that $m\neq\mu_g$ for any Lebesgue measurable function $g:\mathbb{R}\longrightarrow\mathbb{R}^{\ge0}$. This does not contradict the Radon-Nikodym theorem because the measure space $(\mathbb{R},\mathcal{M},\mu)$ is not σ-finite (countable union of subsets X_n of $\mathbb{R}$ with $\mu(X_n)<\infty$ is countable).

Problem 4 (14pts)

Let $\mathcal{F}=2^{\mathbb{Z}^+}$ be the σ-field of all subsets of $\mathbb{Z}^+$ and

$$\mu,\nu\colon \mathcal{F}\longrightarrow\mathbb{R}, \qquad \mu(E) = \sum_{n\in E} 2^{-n}, \quad \nu(E) = \sum_{n\in E} 3^{-n}.$$

(a) *Show that* $\nu \ll \mu$ *and* $\mu \ll \nu$.

(b) *Find the Radon-Nikodym derivatives* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ *and* $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

(c) *For* $k, a \in \mathbb{Z}^+$, *let*

$$E_{k;a} = \{n \in \mathbb{Z}^+ : n \equiv a \mod k\}.$$

Show that for each $k \in \mathbb{Z}^+$, *the collection*

$$\mathcal{P}_k \equiv \{E_{k;a} : a = 1, 2, \ldots, k\}$$

is a finite partition of $\mathbb{Z}^+$ *into measurable subsets.*

(d) *For each* $k \in \mathbb{Z}^+$, *find the associated function* $h_{\mathcal{P}_k} : \mathbb{Z}^+ \longrightarrow \mathbb{R}$ *as on page 191. Find the limit of this sequence of functions.*

(a; **2pts**) If $E \in \mathcal{F}$ and $\mu(E) = 0$, then $E = \emptyset$ and so $\nu(E) = 0$. Thus, $\nu \ll \mu$. The same applies with μ and ν interchanged.

(b; **4pts**) For any $g : \mathbb{Z}^+ \longrightarrow [0, \infty]$ and $E \subset \mathbb{Z}^+$,

$$\int_E g \mathrm{d}\mu = \sum_{n \in E} g(n) 2^{-n}, \qquad \int_E g \mathrm{d}\nu = \sum_{n \in E} g(n) 3^{-n}.$$

Thus,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} : \mathbb{Z}^+ \longrightarrow [0, \infty], \quad n \longrightarrow (2/3)^n, \qquad \frac{\mathrm{d}\mu}{\mathrm{d}\nu} : \mathbb{Z}^+ \longrightarrow [0, \infty], \quad n \longrightarrow (3/2)^n.$$

(c; **3pts**) The collection $\mathcal{P}_k$ is finite because it contains only finitely many subsets of $\mathbb{Z}^+$ (k, more precisely). Each element $E_{k;a}$ of $\mathcal{P}_k$ is also an element of $\mathcal{F}$ and so $\mathcal{P}_k$ is a collection of measurable subsets. For every $n \in \mathbb{Z}^+$, there exists a unique $a = 1, 2, \ldots, k$ such that $n - a$ is divisible k. For every $n \in \mathbb{Z}^+$, there thus exists a unique $a = 1, 2, \ldots, k$ such that $n \in E_{k;a}$. This means that

$$E_{k;a} \cap E_{k;a'} \ \forall\, a, a' = 1, 2, \ldots, k, \ a \neq a', \qquad \mathbb{Z}^+ = \bigcup_{a=1}^{a=k} E_{k;a}.$$

Thus, $\mathcal{P}_k$ is a partition of $\mathbb{Z}^+$.

(d; **5pts**) For $k \in \mathbb{Z}^+$ and $a = 1, 2, \ldots, k$,

$$\mu(E_{k;a}) = \sum_{n=0}^{\infty} 2^{-(a+kn)} = \frac{2^{-a}}{1 - 2^{-k}}, \qquad \nu(E_{k;a}) = \sum_{n=0}^{\infty} 3^{-(a+kn)} = \frac{3^{-a}}{1 - 3^{-k}}.$$

By the definition on page 191,

$$h_{\mathcal{P}_k} : \mathbb{Z}^+ \longrightarrow \mathbb{R}, \qquad h_{\mathcal{P}_k}(n) = \frac{\nu(E_{k;a})}{\mu(E_{k;a})} = \left(\frac{2}{3}\right)^a \frac{1 - 2^{-k}}{1 - 3^{-k}} \quad \forall\, n \in E_{k;a}.$$

In particular,

$$h_{\mathcal{P}_k}(n) = \left(\frac{2}{3}\right)^n \frac{1 - 2^{-k}}{1 - 3^{-k}} \quad \forall\, k \geq n.$$

Thus, $h_{\mathcal{P}_k}(n) \longrightarrow (2/3)^n$ as $k \longrightarrow$, i.e. $h_{\mathcal{P}_k}$ converges to the Radon-Nikodym derivatives $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ of ν with respect to ν.