MAT 324: Real Analysis, Fall 2017
Solutions to Bonus Problem Set

Problem 1 (10pts)

Complete the following definitions.

(a) *An* equivalence relation *on a set X is a subset $A\subset X^2$ such that*
(i) $(x,x)\in A\ \forall\, x\in X$ (ii) if $(x,y)\in A$, then $(y,x)\in A$ (iii) if $(x,y),(y,z)\in A$, then $(x,z)\in A$.

(b) *A* metric space *is a pair (X,d), where X is a set and $d\colon X^2\longrightarrow\mathbb{R}^{\ge 0}$ is a function such that*
(i) $d(x,y)=0$ iff $x=y$ (ii) $d(x,y)=d(y,x)$ (iii) $d(x,z)\le d(x,y)+d(y,z)$.

(c) *A map $f\colon (X,d)\longrightarrow(X',d')$ between two metric spaces is an* isometry *if*
f is a bijection and $d'(f(x),f(y))=d(x,y)\ \forall\, x,y\in X$.

Let (X,d) be a metric space.

(d) *A sequence of points $x_n\in X$* converges *to a point $x\in X$ if* $\lim\limits_{n\longrightarrow\infty} d(x,x_n)=0$.

(e) *The* closure *of a subset $Y\subset X$ is the subset $\overline{Y}\subset X$ consisting of the points $x\in X$ such that* there exists a sequence $x_n\in Y$ converging to x.

(f) *A subset $Y\subset X$ is* dense *if* $\overline{Y}=X$.

(g) *A sequence of points $x_n\in X$ is* Cauchy *if* $\lim\limits_{m,n\longrightarrow\infty} d(x_m,x_n)=0$.

(h) *The metric space (X,d) is* complete *if* every Cauchy sequence $x_n\in X$ converges to some $x\in X$.

(i) *A* completion *of (X,d) is a complete metric space $(\overline{X},\overline{d})$ such that*
$X\subset\overline{X}$, X is dense in $\overline{X}$, and $\overline{d}|_{X^2}=d$.

(j) *Two completions $(\overline{X},\overline{d})$ and $(\overline{X}',\overline{d}')$ of (X,d) are* equivalent *if*
there exists an isometry $f\colon(\overline{X},\overline{d})\longrightarrow(\overline{X}',\overline{d}')$ s.t. $f(x)=x\ \forall\, x\in X$.

Problem 2 (10pts)

Show that every metric space (X,d) admits a unique completion, up to equivalence.

Construct a completion $(\overline{X},\overline{d})$ of (X,d) as follows. We declare two Cauchy sequences $\mathbf{x}\equiv(x_n)_{n\in\mathbb{Z}^+}$ and $\mathbf{y}\equiv(y_n)_{n\in\mathbb{Z}^+}$ in (X,d) to be equivalent if

$$\lim_{n\longrightarrow\infty} d(x_n,y_n)=0.$$

This is an equivalence relation. Let $\overline{X}$ be the set of equivalence classes of Cauchy sequences in (X,d) with respect to this equivalence relation. Define

$$\overline{d}\colon\overline{X}^2\longrightarrow\mathbb{R}^{\ge 0},\qquad \overline{d}\big([(x_n)_{n\in\mathbb{Z}^+}],[(y_n)_{n\in\mathbb{Z}^+}]\big)=\lim_{n\longrightarrow\infty} d(x_n,y_n).$$

Since $(x_n)_{n\in\mathbb{Z}^+}$ and $(y_n)_{n\in\mathbb{Z}^+}$ are Cauchy in (X,d), the sequence $d(x_n,y_n)$ is Cauchy in $\mathbb{R}$ and so the limit above exists. If $[(x_n)_{n\in\mathbb{Z}^+}]=[(x'_n)_{n\in\mathbb{Z}^+}]$ and $[(y_n)_{n\in\mathbb{Z}^+}]=[(y'_n)_{n\in\mathbb{Z}^+}]$, then

$$\big|d(x_n,y_n)-d(x'_n,y'_n)\big|\le d(x_n,x'_n)+d(y_n,y'_n)\longrightarrow 0\quad\text{as}\quad n\longrightarrow\infty$$

and thus the function $\overline{d}$ is well-defined.

Since $d(x,x)=0$ for all $x\in X$, $d([\mathbf{x}],[\mathbf{x}])=0$ for all $\mathbf{x}\in\overline{X}$. By the definition of the equivalence and $\overline{d}$, $[\mathbf{x}]=[\mathbf{y}]$ if $d([\mathbf{x}],[\mathbf{y}])=0$. Since d satisfies (ii) and (iii) in Problem 1(a), do does $\overline{d}$. Thus, $(\overline{X},\overline{d})$ is a metric space. We identify X with the subset of $\overline{X}$ consisting of the equivalence classes $[(x_n\!=\!x)_{n\in\mathbb{Z}^+}]$; all such classes are distinct. Furthermore,

$$\overline{d}\big(\big[(x_n\!=\!x)_{n\in\mathbb{Z}^+}\big],\big[(y_n\!=\!y)_{n\in\mathbb{Z}^+}\big]\big)=\lim_{n\longrightarrow\infty}d(x,y)=d(x,y)\quad\forall\,x,y\in X$$

and so $\overline{d}|_{X^2}=d$ under this identification. If $\mathbf{x}\equiv[(x_n)_{n\in\mathbb{Z}^+}]\in\overline{X}$,

$$\lim_{m\longrightarrow\infty}\overline{d}\big([\mathbf{x}],\big[(x_{m;n}\!=\!x_m)_{n\in\mathbb{Z}^+}\big]\big)\equiv\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}d(x_n,x_m)=0,$$

since $(x_n)_{n\in\mathbb{Z}^+}$ is Cauchy in (X,d), i.e. the sequence $[\mathbf{x}_m]\in\overline{X}$, where $\mathbf{x}_m\equiv(x_{m;n}=x_m)_{n\in\mathbb{Z}^+}$, converges to $[\mathbf{x}]$ in $(\overline{X},\overline{d})$. Thus, X is dense in $(\overline{X},\overline{d})$.

Suppose $[\mathbf{x}_m]$ is a Cauchy sequence in $(\overline{X},\overline{d})$ with $\mathbf{x}_m=(x_{m;n})_{n\in\mathbb{Z}^+}$, i.e.

$$\lim_{n,n'\longrightarrow\infty}d\big(x_{m;n},x_{m;n'}\big)=0\ \ \forall\,m,\qquad\lim_{m,m'\longrightarrow\infty}\overline{d}\big([\mathbf{x}_m],[\mathbf{x}_{m'}]\big)\equiv\lim_{m,m'\longrightarrow\infty}\lim_{n\longrightarrow\infty}d\big(x_{m;n},x_{m';n}\big)=0.$$

By the first condition, for each m there exists $N_m\in\mathbb{Z}^+$ such that $d(x_{m;N_m},x_{m;n})\le 1/m$ for all $n\ge N_m$. Let $\mathbf{x}=(x_{n;N_n})_{n\in\mathbb{Z}^+}$. Thus,

$$\begin{aligned}\lim_{m\longrightarrow\infty}\overline{d}\big([\mathbf{x}],[\mathbf{x}_m]\big)&\equiv\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}d\big(x_{n;N_n},x_{m;n}\big)\\&\le\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}\lim_{n'\longrightarrow\infty}\Big(d\big(x_{n;N_n},x_{n;n'}\big)+d\big(x_{n;n'},x_{m;n'}\big)+d\big(x_{m;n'},x_{m;n}\big)\Big)\\&\le\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}\big(1/n\big)+\lim_{m,n\longrightarrow\infty}\lim_{n'\longrightarrow\infty}d\big(x_{m;n'},x_{m;n}\big)+\lim_{m\longrightarrow\infty}\lim_{n,n'\longrightarrow\infty}d\big(x_{m;n'},x_{m;n}\big)=0,\end{aligned}$$

i.e. the sequence $[\mathbf{x}_m]$ converges to $[\mathbf{x}]$ in $(\overline{X},\overline{d})$. Thus, the metric space $(\overline{X},\overline{d})$ is complete.

Let $(\overline{X},\overline{d})$ and $(\overline{X}',\overline{d}')$ be completions of (X,d). Construct an isometry $f:(\overline{X},\overline{d})\longrightarrow(\overline{X}',\overline{d}')$ s.t. $f(x)=x\ \forall\,x\in X$ as follows. Since X is dense in $(\overline{X},\overline{d})$, for each $x\in\overline{X}$ there exists a sequence $x_n\in X$ converging to x. Since x_n is Cauchy in $(\overline{X},\overline{d})$ and $\overline{d}'|_{X^2}=d=\overline{d}|_{X^2}$, x_n is Cauchy in $(\overline{X}',\overline{d}')$ and so converges to some $f(x)\in\overline{X}$. If $x_n'\in X$ is another sequence converging to x in $(\overline{X},\overline{d})$, then

$$\overline{d}'(x_n,x_n')=d(x_n,x_n')=\overline{d}(x_n,x_n')\longrightarrow 0\quad\text{as}\quad n\longrightarrow\infty$$

and thus x_n' also converges to $f(x)$ in $(\overline{X},\overline{d})$. If $y_n\in X$ is a sequence converging to y in $(\overline{X},\overline{d})$, then

$$\overline{d}'\big(f(x),f(y)\big)=\lim_{n\longrightarrow\infty}\overline{d}'\big(x_n,y_n\big)=\lim_{n\longrightarrow\infty}d\big(x_n,y_n\big)=\lim_{n\longrightarrow\infty}\overline{d}\big(x_n,y_n\big)=\overline{d}(x,y).$$

If $x\in X$, we can take $x_n=x$ and so $f(x)=x$ for all $x\in X$.

In the same way, we can construct $g\colon\overline{X}'\longrightarrow\overline{X}$ such that

$$\overline{d}(g(x'),g(y'))=\overline{d}'(x',y')\ \ \forall\,x',y'\in\overline{X}',\qquad g(x)=x\ \ \forall\,x\in X.$$

The functions $F=g\circ f\colon\overline{X}\longrightarrow\overline{X}$ and $G=f\circ g\colon\overline{X}'\longrightarrow\overline{X}'$ satisfy

$$\begin{aligned}\overline{d}\big(F(x),F(y)\big)&=\overline{d}(x,y)\ \ \forall x,y\in\overline{X},&\qquad F(x)&=x\ \ \forall\,x\in X,\\\overline{d}'\big(G(x'),G(y')\big)&=\overline{d}'(x',y')\ \ \forall x',y'\in\overline{X}',&\qquad G(x)&=x\ \ \forall\,x\in X.\end{aligned}$$

Since X is dense in $(\overline{X},\overline{d})$, the first line above implies that $F(x)=x$ for all $x\in\overline{X}$. Since X is dense in $(\overline{X}',\overline{d}')$, the second line implies that $G(x')=x'$ for all $x'\in\overline{X}'$. Thus, f is a bijection (with inverse g).