MAT 324: Real Analysis, Fall 2017
Homework Assignment 9

Please read carefully Sections 5.2 and 6.1-6.3 in the textbook and pp160-164 in Rudin's book (see website), prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below.

Problem Set 9 (**due in class on Thursday, 11/16**): Problems 1-3 below

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose there exist $A,B\in\mathcal{F}$ such that $A\cap B=\emptyset$ and $\mu(A),\mu(B)\in\mathbb{R}^+$. Show that the norm $\|\cdot\|_p$ on $L^p(X)$ is not induced by an inner-product on $L^p(X)$ for any $p\in[1,\infty]-\{2\}$.

Problem 2

Let $C^{0,2}(\mathbb{R})\subset L^2(\mathbb{R})$ denote the subspace of continuous square-integrable functions. Define

$$L^2_0(\mathbb{R})=\big\{f\in L^2(\mathbb{R})\colon f=0 \text{ a.e. on } [0,1]\big\},\qquad C^{0,2}_0(\mathbb{R})=L^2_0(\mathbb{R})\cap C^{0,2}(\mathbb{R}).$$

(a) Let $f\in C^{0,2}(\mathbb{R})$ be such that $f(x)\neq0$ for some $x\in\mathbb{R}-[0,1]$. Show that there exists $g\in C^{0,2}_0(\mathbb{R})$ such that $\langle\!\langle f,g\rangle\!\rangle_2\neq0$. Conclude that $f\in C^{0,2}(\mathbb{R})$ has a projection to $C^{0,2}_0(\mathbb{R})$ if and only if $f(0)=f(1)=0$.

(b) Let $f\in L^2(\mathbb{R})$. Determine the projection of f to $L^2_0(\mathbb{R})$.

Problem 3

(a) Let X_1,X_2 be sets, $\sigma(\mathcal{S}_1)$ be the σ-field on X_1 generated by a collection $\mathcal{S}_1\subset2^{X_1}$ of subsets of X_1, and $\sigma(\mathcal{S}_2)$ be the σ-field on X_2 generated by a collection $\mathcal{S}_2\subset2^{X_2}$ of subsets of X_2. Show that the σ-fields $\sigma(S_1\times S_2)$ and $\sigma(\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2))$ on $X_1\times X_2$ generated by the collections

$$\mathcal{S}_1\times\mathcal{S}_2=\big\{A\times B\colon A\in\mathcal{S}_1,\ B\in\mathcal{S}_2\big\}\qquad\text{and}$$
$$\sigma(\mathcal{S}_1)\times\sigma(\mathcal{S}_2)=\big\{A\times B\colon A\in\sigma(\mathcal{S}_1),\ B\in\sigma(\mathcal{S}_2)\big\},$$

respectively, are the same.

(b) For $n\in\mathbb{Z}^+$, let $\mathcal{M}_n\subset2^{\mathbb{R}^n}$ be the collection of Lebesgue measurable subsets as described in Section 6.1. Show that

$$\sigma\big(\mathcal{M}_{n_1}\times\mathcal{M}_{n_2}\big)\subsetneq\mathcal{M}_{n_1+n_2}\qquad\forall\ n_1,n_2\in\mathbb{Z}^+;$$

note the inequality above.

Hint: proof of Proposition 6.3 might be helpful for (a); the description of $\mathcal{M}_n$ in Section 6.1 and the statement of Theorem 6.4 might be helpful for (b).