MAT 324: Real Analysis, Fall 2017
Homework Assignment 7

Please read carefully Sections 5.1 and 5.3 in the textbook, prove all propositions and do all exercises you encounter along the way, and write up clear solutions to the written assignment below. The exams in this class will be based to a large extent on these propositions, exercises, and assignments.

Problem Set 7 (**due in class on Thursday, 11/2**): Problems 1-4 on the next page

Please write your solutions legibly; the TA may disregard solutions that are not readily readable. All solutions must be stapled (no paper clips) and have your name (first name first) and HW number in the upper-right corner of the first page.

Problem 1

Let $(X,\mathcal{F},\mu)$ be a measure space and $p,q,r\in[1,\infty]$ be such that $1/p+1/q=1/r$. Show that

$$\big\|fg\big\|_r \le \|f\|_p\|g\|_q$$

for all measurable functions $f,g\colon X\longrightarrow\mathbb{R}$ (cases with $p,q,r=\infty$ may require separate treatment).

Problem 2

Find a function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ which is in $L^2(\mathbb{R}^+)$, but not in $L^p(\mathbb{R}^+)$ for any $p\in[1,\infty]-\{2\}$. Justify your answer.

Problem 3

For each of the following sequences of measurable functions $f_1,f_2,\ldots\colon\mathbb{R}^+\longrightarrow\mathbb{R}$, determine whether it (the sequence) lies in $L^1(\mathbb{R}^+)$, $L^2(\mathbb{R}^+)$, and if so whether it is Cauchy in there (this is between 2 and 4 questions for each sequence below). Justify your answers.

(a) $f_n=\mathbb{1}_{(0,n)}/\sqrt{x}$ (b) $f_n=\mathbb{1}_{(0,n)}/(x+1)$

Problem 4

Let $p,q\in[1,\infty]$ with $1/p+1/q=1$. For a differentiable function $f\colon\mathbb{R}\longrightarrow\mathbb{R}$, define

$$\|f\|_{p,1}=\|f\|_p+\|f'\|_p\,.$$

(a) Show that this defines a norm on the vector space

$$C^{1,p}(\mathbb{R})\equiv\big\{f\in C^1(\mathbb{R})\colon\|f\|_{p,1}<\infty\big\}.$$

Do not forget to justify why $C^{1,p}(\mathbb{R})$ is a vector space ($C^1(\mathbb{R})$ is the space of differentiable functions $f\colon\mathbb{R}\longrightarrow\mathbb{R}$).

(b) Show that

$$\big|f(x)-f(y)\big|\le\|f'\|_p|x-y|^{1/q},\qquad\left|f(x)-\frac{1}{2}\int_{x-1}^{x+1}f(y)\mathrm{d}y\right|\le\|f'\|_p,\qquad\|f\|_\infty\le\|f\|_{p,1}$$

for all $f\in C^{1,p}(\mathbb{R})$ and $x,y\in\mathbb{R}$ (cases with $p,q=\infty$ may require separate treatment).

(c) Let $f_1,f_2,\ldots\in C^{1,p}(\mathbb{R})$ be a Cauchy sequence with respect to the norm $\|\cdot\|_{p,1}$. Show that it converges uniformly to a bounded continuous function $f\colon\mathbb{R}\longrightarrow\mathbb{R}$.

Hint: Fundamental Theorem of Calculus and Hölder's Inequality might be helpful for (b).

Note: the above problem establishes the $k,n=1$, $\ell=0$ case of the main part of Sobolev's Embedding Theorem:

$$L^p_k(\mathbb{R}^n)\subset C^\ell(\mathbb{R}^n)\qquad\text{if}\quad k-\frac{p}{n}>\ell\,, \tag{1}$$

where $C^\ell(\mathbb{R}^n)$ is the space of functions on $\mathbb{R}^n$ with continuous derivatives of order up to ℓ (including ℓ) and $L^p_k(\mathbb{R}^n)$ is the completion of $C^k(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{p,k}$ consisting of the $\|\cdot\|_p$-norms of all derivatives of order up to k. In the $n=1$ case, the strict inequality $>$ in (1) can be replaced by $\ge$.